A Wigner-type theorem
on symmetry transformations in Banach spaces

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Abstract. We obtain an analogue of Wigner's classical theorem on symmetries for Banach spaces. The proof is based on a result from the theory of linear preservers. Moreover, we present two other Wigner-type results for finite dimensional linear spaces over general fields.

Wigner's theorem on symmetry transformations (sometimes called unitary-antiunitary theorem) plays fundamental role in quantum mechanics. This result can be formulated in several ways. For example, in [2, Theorem 3.1] the statement reads as follows. In the sequel $P_1(H)$ denotes the set of all rank-one (orthogonal) projections (or, in the language of quantum mechanics, the set of all pure states) on the Hilbert space H. We let tr stand for the usual trace-functional.

Wigner's theorem. *Let H be a complex Hilbert space and let $\phi : P_1(H) \rightarrow P_1(H)$ be a bijective function for which*

$$(1) \qquad \operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in P_1(H)).$$

Then there exists an either unitary or antiunitary operator U on H such that ϕ is of the form

$$\phi(P) = UPU^* \qquad (P \in P_1(H)).$$

This formulation of Wigner's theorem makes us possible to formulate an analogous theorem in the more general setting of Banach spaces, which we state below. Other Wigner-type theorems for Hilbert modules over matrix algebras or for indefinite inner product spaces or for type II factors can be found in our recent papers [6]–[8], respectively.

If X is a Banach space, then X' denotes the (topological) dual of X. The set of all rank-one idempotents on X (which are the natural Banach space analogues of the projections) is denoted by $I_1(X)$.

Now, our first result reads as follows.

Theorem 1. *Let X be a (real or complex) Banach space. Let $\phi : I_1(X) \to I_1(X)$ be a bijective function for which*

$$(2) \qquad \operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in I_1(X)).$$

Then either there exists a bijective bounded linear operator $A : X \to X$ such that

$$\phi(P) = APA^{-1} \qquad (P \in I_1(X))$$

or there exists a bijective bounded linear operator $C : X' \to X$ such that

$$\phi(P) = CP'C^{-1} \qquad (P \in I_1(X)).$$

Before the proof we need some additional notation and definitions. If X is a Banach space, then $B(X)$ stands for the algebra of all bounded linear operators on X and $F(X)$ denotes the ideal of all finite rank elements in $B(X)$. The Banach space adjoint of the operator $A \in B(X)$ is denoted by A'. If $x \in X$, $f \in X'$, then $x \otimes f$ is the operator defined by

$$(x \otimes f)(y) = f(y)x \qquad (y \in X).$$

It is easy to see that $x \otimes f \in I_1(X)$ if and only if $f(x) = 1$. Clearly, every operator of the form $\sum_{i=1}^{n} x_i \otimes f_i$ belongs to $F(X)$, and, conversely, every finite rank operator $A \in F(X)$ can be written in the form

$$A = \sum_{i=1}^{n} x_i \otimes f_i$$

with some $x_1, \dots, x_n \in X$ and $f_1, \dots, f_n \in X'$. On the elements of $F(X)$ we define the trace-functional by

$$\operatorname{tr} \sum_{i=1}^{n} x_i \otimes f_i = \sum_{i=1}^{n} f_i(x_i).$$

One can see that this functional is well-defined and, in the Hilbert space case, it gives us the usual trace. It is not hard to see that tr is a linear functional on $F(X)$ with the property that

$$\operatorname{tr} TA = \operatorname{tr} AT$$

for every $A \in F(X)$ and $T \in B(X)$ (cf. [10, Section B.1]). In particular, this shows that any map ϕ of any of the forms appearing in our statement above necessarily satisfies (2).

PROOF of Theorem 1. The idea of the proof is very simple. We first extend ϕ from $I_1(X)$ to a linear transformation on the operator algebra $F(X)$ and then apply a result of Omladič and Šemrl on linear preservers. Notice that our approach is completely different from the usual proofs of Wigner's theorem and, in particular, this is the case also with the proof presented in [2].

First suppose that X is a complex Banach space. We define

$$\Phi\left(\sum_{i=1}^{n} \lambda_i P_i\right) = \sum_{i=1}^{n} \lambda_i \phi(P_i)$$

whenever $P_1, \dots, P_n$ are rank-one idempotents and $\lambda_1, \dots, \lambda_n$ are scalars. We assert that Φ is well-defined. To see this, we prove that

(3)
$$\sum_{i=1}^{n} \lambda_i P_i = 0$$

implies

$$\sum_{i=1}^{n} \lambda_i \phi(P_i) = 0.$$

Indeed, from (3) we obtain that

$$\sum_{i=1}^{n} \lambda_i P_i Q = 0 \qquad (Q \in I_1(X)).$$

By the linearity of the trace and the equality (2), it follows that

$$\mathrm{tr}\left(\sum_{i=1}^{n}\lambda_i\phi(P_i)\phi(Q)\right)=0 \qquad (Q\in I_1(X)).$$

Since ϕ maps onto $I_1(X)$, we can infer that

$$\sum_{i=1}^{n}\lambda_i\phi(P_i)=0.$$

So Φ is well-defined. Clearly, Φ is a linear transformation on $F(X)$. Since every matrix $A\in M_n(\mathbb{C})$ is a linear combination of rank-one idempotents, it follows that every finite-rank operator belongs to the linear span of $I_1(X)$. This yields that Φ is defined on the whole $F(X)$ and maps onto $F(X)$ (in fact, one can prove that it is injective as well).

So we have a surjective linear transformation Φ on $F(X)$ which preserves the rank-one idempotents. Now, we can apply a result of Omladič and Šemrl describing the form of all such maps. In view of [9, Main Theorem], we distinguish two cases. First suppose that X is infinite-dimensional. Then for the form of Φ, and hence for the form of ϕ, we have the following two possibilities:

(i) There exists a bijective bounded linear operator $A:X\to X$ such that

$$\phi(P)=APA^{-1} \qquad (P\in I_1(X)).$$

(ii) There exists a bijective bounded linear operator $C:X'\to X$ such that

$$\phi(P)=CP'C^{-1} \qquad (P\in I_1(X)).$$

Recall that our map Φ is linear. This is the reason that the two remaining possibilities in [9, Main Theorem] do not appear here. As for the finite dimensional case, the form of all surjective linear maps on $M_n(\mathbb{C})$ preserving rank-one idempotents is given in [9, Theorem 4.5]. Since every linear transformation on $M_n(\mathbb{C})$ is continuous and the only continuous ring automorphisms of $\mathbb{C}$ are the identity and the conjugation, we obtain from that result that in this case our map ϕ is either of the form $P\mapsto APA^{-1}$ or of the form $P\mapsto AP^tA^{-1}$ (^{t} stands for the transpose) with some non-singular matrix $A\in M_n(\mathbb{C})$. But under the identification of matrices and operators, we obtain the forms appearing in (i) and (ii) once again.

If X is a real Banach space, then one can argue in the same way but since in that case [9, Main Theorem] holds without any assumption on the dimension of X, there is no need to refer to [9, Theorem 4.5]. $\square$

Remark. Using our quite algebraic approach presented above we can now give a short proof of Wigner's original theorem. Let H be a complex Hilbert space and let $\phi : P_1(H) \to P_1(H)$ be a bijective function which preserves the transition probabilities, that is, satisfies (1). One can extend ϕ to a linear transformation Φ on $F(H)$ in a similar way as in the proof of our previous theorem. Just as there, we find that Φ is bijective. On the other hand, for any $P, Q \in P_1(H)$ we have $PQ = QP = 0$ if and only if $\operatorname{tr} PQ = 0$. Therefore, Φ preserves the orthogonality between rank-one projections which, by the linearity of Φ, implies that Φ sends projections to projections. It is now an easy algebraic argument to verify that Φ is a Jordan *-automorphism of $F(H)$, that is, Φ satisfies $\Phi(A^2) = \Phi(A)^2$ and $\Phi(A^*) = \Phi(A)^*$ for every $A \in F(H)$ (see [1, Remark 2.2]). By a classical result of HERSTEIN [4], Φ is either a *-automorphism or a *-antiautomorphism of $F(H)$. But the forms of those morphisms are well-known (see, for example, [5, Proposition]). Namely, we have an either unitary or antiunitary operator U on H such that

$$\phi(P) = UPU^* \qquad (P \in P_1(H)).$$

This completes the proof of Wigner's theorem.

In the finite dimensional case we can get rid of the assumption on the bijectivity of the transformation ϕ. In fact, we have the following Wigner-type result for matrix algebras over general fields.

Theorem 2. *Let $\mathbb{F}$ be a field of characteristic different from 2 and let $n \in \mathbb{N}$. Suppose ϕ is a transformation on the set $I_1(\mathbb{F}^n)$ of all rank-one idempotents in $M_n(\mathbb{F})$ into itself with the property that*

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in I_1(\mathbb{F}^n)).$$

Then there exists a nonsingular matrix $A \in M_n(\mathbb{F})$ such that ϕ is either of the form
$$\phi(P) = APA^{-1} \qquad (P \in I_1(\mathbb{F}^n))$$

or of the form
$$\phi(P) = AP^t A^{-1} \qquad (P \in I_1(\mathbb{F}^n)).$$

PROOF. We first show that the range of ϕ linearly generates $M_n(\mathbb{F})$. As usual, denote by $E_{ij} \in M_n(\mathbb{F})$ the matrix whose ij entry is 1 and its all other entries are 0. Define

$$E'_{ij} = \begin{cases} \phi(E_{ii} + E_{ij}) - \phi(E_{ii}) & \text{if } i \neq j; \\ \phi(E_{ii}) & \text{if } i = j. \end{cases}$$

Suppose that

$$\sum_{i,j} \lambda_{ij} E'_{ij} = 0$$

for some scalars $\lambda_{ij} \in \mathbb{F}$. Fix indices $k, l \in \{1, \dots, n\}$. We have

$$\sum_{i,j} \lambda_{ij} E'_{ij} E'_{kl} = 0.$$

Taking trace, we obtain

$$\sum_{i,j} \lambda_{ij} \operatorname{tr} E'_{ij} E'_{kl} = 0.$$

By the preserving property of ϕ, it follows that

$$\sum_{i,j} \lambda_{ij} \operatorname{tr} E_{ij} E_{kl} = 0.$$

Since $E_{ij} E_{kl} = \delta_{jk} E_{il}$, from this equality we easily deduce that $\lambda_{lk} = 0$. As k, l were arbitrary, it follows that the matrices E'_{ij}, $i, j \in \{1, \dots, n\}$ form a linearly independent set in $M_n(\mathbb{F})$. This implies that the range of ϕ linearly generates $M_n(\mathbb{F})$.

Similarly to the proof of Theorem 1 we define a transformation $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by

$$\Phi\left(\sum_i \lambda_i P_i\right) = \sum_i \lambda_i \phi(P_i),$$

where the P_i's are rank-one idempotents and the λ_i's are scalars. First, Φ is well-defined. Indeed, let Q_j be rank-one idempotents and $\mu_j \in \mathbb{F}$ be such that $\sum_i \lambda_i P_i = \sum_j \mu_j Q_j$. Then we have

$$\sum_i \lambda_i \operatorname{tr} P_i E_{kl} = \operatorname{tr}\left(\sum_i \lambda_i P_i\right) E_{kl} = \operatorname{tr}\left(\sum_j \mu_j Q_j\right) E_{kl} = \sum_j \mu_j \operatorname{tr} Q_j E_{kl}.$$

By the preserving property of ϕ we infer that

$$\sum_i \lambda_i \operatorname{tr} \phi(P_i) E'_{kl} = \sum_j \mu_j \operatorname{tr} \phi(Q_j) E'_{kl}.$$

This implies that

$$\operatorname{tr}\Big(\sum_i \lambda_i \phi(P_i)\Big) E'_{kl} = \operatorname{tr}\Big(\sum_j \mu_j \phi(Q_j)\Big) E'_{kl}$$

and, as the matrices E'_{kl} linearly generate $M_n(\mathbb{F})$, we can conclude that $\sum_i \lambda_i \phi(P_i) = \sum_j \mu_j \phi(Q_j)$. Therefore, Φ is well-defined. Since the rank-one idempotents linearly generate $M_n(\mathbb{F})$, we obtain that Φ is a linear transformation from $M_n(\mathbb{F})$ into itself which preseves the rank-one idempotents. The form of such transformations is described in [3, Theorem 3] and this gives us the form of ϕ. $\qquad\square$

Wigner's classical result concerns transformations on the set of all rank-one (orthogonal) projections on a Hilbert space. In case the Hilbert space in question is finite dimensional and real, then the result reduces to the description of all bijective transformations on the set of all symmetric rank-one idempotents in $M_n(\mathbb{R})$ which preserve the trace of the product. In our last assertion using a result on linear preservers once again, we can generalize this statement for the case of general fields.

Theorem 3. *Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2 and let $n \in \mathbb{N}$. Suppose ϕ is a transformation on the set $P_1(\mathbb{F}^n)$ of all rank-one idempotents in $S_n(\mathbb{F})$ (the set of all symmetric elements of $M_n(\mathbb{F})$) into itself with the property that*

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in P_1(\mathbb{F}^n)).$$

Then there exists an orthogonal matrix $U \in M_n(\mathbb{F})$ such that ϕ is either of the form

$$\phi(P) = UPU^{-1} \qquad (P \in P_1(\mathbb{F}^n))$$

or of the form

$$\phi(P) = UP^tU^{-1} \qquad (P \in P_1(\mathbb{F}^n)).$$

PROOF. The proof is quite similar to the proof of our previous result. Define

$$
E'_{ij} = \begin{cases} 2\phi((E_{ii} + E_{jj} + E_{ij} + E_{ji})/2) - (\phi(E_{ii}) + \phi(E_{jj})) & \text{if } i < j; \\ \phi(E_{ii}) & \text{if } i = j. \end{cases}
$$

Just as in the proof of Theorem 2, one can show that the matrices E'_{ij}, $i \leq j$ are linearly independent in $S_n(\mathbb{F})$. This gives us that the range of ϕ linearly generates $S_n(\mathbb{F})$.

Next we define

$$
\Phi\left(\sum_i \lambda_i P_i\right) = \sum_i \lambda_i \phi(P_i)
$$

for every finite system λ_i of scalars and symmetric rank-one idempotents P_i. One can prove that Φ is well-defined in a way very similar to the corresponding part of the proof of Theorem 2. Since the symmetric rank-one idempotents linearly generate $S_n(\mathbb{F})$, Φ is a linear transformation from $S_n(\mathbb{F})$ into itself which preserves the rank-one idempotents in $S_n(\mathbb{F})$. The form of such transformations on $S_n(\mathbb{F})$ is described in [3, Theorem 4] and this gives us the form of ϕ. $\square$

References

[1] M. BREŠAR and P. ŠEMRL, Mappings which preserve idempotents, local automorpisms, and local derivations, *Canad. J. Math.* **45** (1993), 483–496.

[2] G. CASSINELLI, E. DE VITO, P. J. LAHTI and A. LEVRERO, Symmetry groups in quantum mechanics and the theorem of Wigner on the symmetry transformations, *Rev. Math. Phys.* **9** (1997), 921–941.

[3] G. H. CHAN and M. H. LIM, Linear preservers on powers of matrices, *Linear Algebra Appl.* **162–164** (1992), 615–626.

[4] I. N. HERSTEIN, Jordan homomorphisms, *Trans. Amer. Math. Soc.* **81** (1956), 331–341.

[5] L. MOLNÁR, Wigner's unitary-antiunitary theorem via Herstein's theorem on Jordan homomorphisms, *J. Nat. Geom.* **10** (1996), 137–148.

[6] L. MOLNÁR, A generalization of Wigner's unitary-antiunitary theorem to Hilbert modules, *J. Math. Phys.* **40** (1999), 5544–5554.

[7] L. MOLNÁR, Generalization of Wigner's unitary-antiunitary theorem for indefinite inner product spaces, *Commun. Math. Phys.* **201** (2000), 785–791.

[8] L. Molnár, A Wigner-type theorem on symmetry transformations in type II factors, *Int. J. Theor. Phys.* **39** (2000), 1463–1466.

[9] M. Omladič and P. Šemrl, Additive mappings preserving operators of rank one, *Linear Algebra Appl.* **182** (1993), 239–256.

[10] A. Pietsch, Operator ideals, *North-Holland Publishing Company*, 1980.

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