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# Functional equations in the theory of conditionally specified distributions

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**Abstract.** The functional equation

$$G_1(xy+x)+F_1(y)=G_2(xy+y)+F_2(x)$$

related to the characterizations of bivariate distributions is investigated for functions $F_i, G_i : \mathbb{R} \to \mathbb{R}$ or $F_i, G_i : \mathbb{R}_+ \to \mathbb{R}$, respectively.

## 1. Introduction

Functional equations have many interesting applications in the characterization problems of probability theory.

In [1] Arnold, Castillo and Sarabia showed how solutions of functional equations can be used in characterizing joint distributions from conditional distributions and also an array of conditionally specified models was presented and analysed.

Let $(X, Y)$ be an absolutely continuous bivariate random variable. Let us denote the joint, mariginal and conditional densities by $f_{(X,Y)}$, $f_X$, $f_Y$, $f_{X|Y}$, $f_{Y|X}$, respectively. One can write $f_{(X,Y)}$ in two ways and obtain the functional equation

$$f_{X|Y}(x,y)f_Y(y)=f_{Y|X}(x,y)f_X(x) \tag{1}$$

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for all $x, y \in \mathbb{R}$ (or for all $x, y \in \mathbb{R}_+$ if we restrict our search to the random variable $(X, Y)$ with support in the positive quadrant).

For example, it is natural to inquire about the nature of all joint densities whose conditional densities satisfy

$$f_{X|Y}(x, y) = g_1((\alpha + y)x); \quad f_{Y|X}(x, y) = g_2((\beta + x)y) \tag{2}$$

for all $x, y \in \mathbb{R}$ or $x, y \in \mathbb{R}_+$, where $\alpha, \beta \in \mathbb{R}$ or $\alpha, \beta \in \mathbb{R}_+$ are arbitrary constants, respectively (see [1]). We ask for what functions $g_1$ and $g_2$ can we have (1) holding for $x, y \in \mathbb{R}$ or $x, y \in \mathbb{R}_+$, respectively.

From (1) and (2) we get that the functions $g_1, g_2, f_X, f_Y : \mathbb{R}$ (or $\mathbb{R}_+$) $\to$ $\mathbb{R}_+$ satisfy the functional equation

$$g_1((\alpha + y)x) f_Y(y) = g_2((\beta + x)y) f_X(x) \tag{3}$$

for all $x, y \in \mathbb{R}$ (or for all $x, y \in \mathbb{R}_+$).

The solution of (3) can be reduced to the solution of the functional equation

$$G_1(xy + x) + F_1(y) = G_2(xy + y) + F_2(x) \tag{4}$$

($x, y \in \mathbb{R}$ or $x, y \in \mathbb{R}_+$) for functions $G_i, F_i : \mathbb{R}$ (or $\mathbb{R}_+$) $\to \mathbb{R}$.

In this paper, we present the general solution of (4) when the functions are defined on $\mathbb{R}$, and the measurable solution of (4) when all the functions are defined on $\mathbb{R}_+$.

## 2. The general solution of (4) on $\mathbb{R}$

Here we shall use the following result of D. Blanuša and Z. Daróczy (see [2], [3]).

**Theorem B–D.** *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation*

$$f(x + y - xy) + f(xy) = f(x) + f(y), \qquad x, y \in \mathbb{R} \tag{H}$$

*if and only if*

$$f(x) = A(x) + b, \quad x \in \mathbb{R}, \tag{5}$$

*where $A$ is an additive function on $\mathbb{R}^2$ and $b \in \mathbb{R}$ is an arbitrary constant.*

**Theorem 1.** *The functions $F_i, G_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ satisfy the functional equation* (4) *for all $x, y \in \mathbb{R}$ if and only if*

$$F_i(x) = A(x) + b_i, \qquad x \in \mathbb{R} \quad (i = 1, 2), \tag{6}$$

$$G_i(x) = A(x) + c_i, \qquad x \in \mathbb{R} \quad (i = 1, 2), \tag{7}$$

*where $A : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$ and $b_i, c_i \in \mathbb{R}$ $(i = 1, 2)$ are arbitrary constants with $b_1 + c_1 = b_2 + c_2$.*

Proof. Putting $x = 0$ or $y = 0$ or $x = 0$, $y = 0$ in (4) we get

$$G_1(0) + F_1(y) = G_2(y) + F_2(0), \quad y \in \mathbb{R}, \tag{8}$$

and

$$G_1(x) + F_1(0) = G_2(0) + F_2(x), \quad x \in \mathbb{R}, \tag{9}$$

and

$$G_1(0) + F_1(0) = G_2(0) + F_2(0) \tag{10}$$

respectively. Using these identities and (4) we have

$$G_1(xy + x) + F_1(y) = F_1(xy + y) + G_1(x), \quad x, y \in \mathbb{R}. \tag{11}$$

Putting $x = -1$ here we obtain

$$G_1(-y - 1) + F_1(y) = F_1(0) + G_1(-1), \quad y \in \mathbb{R}. \tag{12}$$

Substituting this into (11) we get the functional equation

$$G_1(xy + x) - G_1(-y - 1) = -G_1(-(xy + y) - 1) + G_1(x), \quad x, y \in \mathbb{R}.$$

Replacing here $x$, $y$ by $-x$, $y - 1$, we get

$$G_1(-xy) + G_1(-(x + y - xy)) = G_1(-x) + G_1(-y), \quad x, y \in \mathbb{R},$$

which implies that the function $f$ defined by

$$f(x) = G_1(-x), \quad x \in \mathbb{R} \tag{13}$$

satisfies the functional equation (H).

So, by Theorem B–D, $f$ is of the form

$$f(x) = A_1(x) + c_1, \quad x \in \mathbb{R}, \tag{14}$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$ and $c_1 \in \mathbb{R}$ is an arbitrary constant.

Taking (13) and (14) into consideration, we have (7) for $G_1$ with the additive function $A = -A_1$.

Then from (12), (9) and (8) we obtain (6) and (7) for the functions $F_1$, $F_2$ and $G_2$, respectively, with real constants $b_1$, $b_2$, $c_2$.

An easy calculation shows that the functions (6) and (7) indeed satisfy (4) if $b_1 + c_1 = b_2 + c_2$. $\square$

### 3. The general mesurable solution of (4) on $\mathbb{R}_+$

We need the following result of A. Járai ([5] Theorem 2.7.2).

**Theorem J.** *Let $\mathcal{T}$ be a locally compact metric space, let $Z_0$ be a metric space, and let $Z_i$ $(i = 1, 2, \dots, n)$ be separable metric spaces. Suppose, that $D$ is an open subset of $\mathcal{T} \times \mathbb{R}^k$ and $X_i \subset \mathbb{R}^k$ for $i = 1, 2, \dots, n$. Let $f_0 : \mathcal{T} \to Z_0$, $f_i : X_i \to Z_i$, $g_i : D \to X_i$, $H : D \times Z_1 \times Z_2 \times \dots \times Z_n \to Z_0$ be functions. Suppose, that the following conditions hold:*

(1) *For every $(t, y) \in D$*

$$f_0(t) = H(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))).$$

(2) *$f_i$ is Lebesgue measurable over $X_i$ for $i = 1, 2, \dots, n$.*

(3) *$H$ is continuous on compact sets.*

(4) *For $i = 1, 2, \dots, n$, $g_i$ is continuous, and for every fixed $t \in \mathcal{T}$ the mapping $y \to g_i(t, y)$ is differentiable with the derivative $D_2 g_i(t, y)$ and with the Jacobian $J_2 g_i(t, y)$, moreover, the mapping $(t, y) \to D_2 g_i(t, y)$ is continuous on $D$ and for every $t \in \mathcal{T}$ there exists a $(t, y) \in D$ so that*

$$J_2 g_i(t, y) \neq 0 \quad \text{for} \quad i = 1, 2, \dots, n.$$

*Then $f_0$ is continuous on $\mathcal{T}$.*

**Lemma 1.** *If the measurable functions $G_i, F_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ satisfy the functional equation* (4) *for all $x, y \in \mathbb{R}_+$ then the functions $G_i, F_i$ are continuous.*

Proof. First we prove the continuity of $G_1$. From (4), with $t = xy + x$, we obtain

$$G_1(t) = G_2\left(\frac{ty}{y+1} + y\right) + F_2\left(\frac{t}{y+1}\right) - F_1(y), \quad (t, y) \in \mathbb{R}_+^2. \tag{15}$$

Let $\mathcal{T} = \mathbb{R}_+$, $n = 3$, $Z_0 = Z_1 = Z_2 = Z_3 = \mathbb{R}$, $X_1, X_2, X_3 = \mathbb{R}_+$, $D = \mathbb{R}_+^2$. Define the functions $g_i$ on $\mathbb{R}_+^2$ by

$$g_1(t, y) = \frac{ty}{y+1} + y, \quad g_2(t, y) = \frac{t}{y+1}, \quad g_3(t, y) = y$$

and let $H(t, y, z_1, z_2, z_3) = z_1 + z_2 - z_3$.

It follows from (15) that the functions $f_i$ $(i = 1, 2, 3)$ given by

$$f_0 = G_1, \quad f_1 = G_2, \quad f_2 = F_2, \quad f_3 = F_1$$

satisfy the functional equation in (1) of Theorem J for all $t, y \in D = \mathbb{R}_+^2$ and $f_i$ $(i = 0, 1, 2, 3)$ is measurable by the conditions of our lemma. $H$ is continuous and condition (4) of Theorem J holds, too, since

$$D_2 g_1(t, y) = \frac{t}{(y+1)^2} + 1 \neq 0, \quad D_2 g_2(t, y) = -\frac{t}{(y+1)^2} \neq 0,$$
$$D_2 g_3(t, y) = 1 \neq 0$$

for all $(t, y) \in D = \mathbb{R}_+^2$.

Thus, by Theorem J, $f_0 = G_1$ is continuous on $\mathbb{R}_+$. The continuity of $G_2$ can be proved by making the substitutions $x \to y$, $y \to x$ in (4) and repeating the above argument.

Putting $x = 1$ or $y = 1$ in (4) and solving the equation obtained for $F_1$ and $F_2$, respectively, we get

$$F_1(y) = G_2(2y) - G_1(y+1) + F_2(1), \quad y \in \mathbb{R}_+, \tag{16}$$

and

$$F_2(x) = G_1(2x) - G_2(x+1) + F_1(1), \quad x \in \mathbb{R}_+, \tag{17}$$

respectively. Whence by the continuity of $G_1$, $G_2$ it follows that $F_1$ and $F_2$ are continuous as well. □

**Lemma 2.** *If the measurable functions $G_i, F_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ satisfy the functional equation* (4) *for all $x, y \in \mathbb{R}_+$ then they are differentiable infinitely many times on $\mathbb{R}_+$.*

Proof. Write (4) in the form (15) and let $[\alpha, \beta] \subset \mathbb{R}_+$ be arbitrary and choose the interval $[\lambda, \mu] \subset \mathbb{R}_+$ arbitrarily, too, then $[\alpha, \beta] \times [\lambda, \mu] \subset D = \mathbb{R}_+^2$ holds.

Integrating (15) with respect to $y$ on $[\lambda, \mu]$ we obtain

$$(\mu - \lambda) G_1(t) = \int_\lambda^\mu G_2\left(\frac{ty}{y+1} + y\right) dy + \int_\lambda^\mu F_2\left(\frac{t}{y+1}\right) dy - \int_\lambda^\mu F_1(y) dy.$$

We use the substitutions

$$g_1(t, y) = \frac{ty}{y+1} + y = u, \quad g_2(t, y) = \frac{t}{y+1} = u$$

in the first and second integral, respectively. It is easy to check that these equations can uniquely be solved for $y$ if $t \in [\alpha, \beta]$.

In the case $\frac{t}{y+1} = u$ this is clear. In the case $\frac{ty}{1+y} + y = u$ this uniqueness is ensured, namely the derivative of the function $y \to g_1(t, y)$:

$$D_2 g_1(t, y) = \frac{t}{(y+1)^2} + 1$$

is positive on $[\alpha, \beta] \times [\lambda, \mu]$, hence our function is strictly increasing.

The solutions

$$y = \frac{-(t - u + 1) + \sqrt{(t - u + 1)^2 + 4u}}{2} \doteq \gamma_1(t, u), \; y = \frac{t}{u} - 1 \doteq \gamma_2(t, u)$$

are infinitely many times differentiable functions of $t$ and $u$. Performing the substitutions we have

$$G_1(t) = \frac{1}{\mu - \lambda} \left[ \int_{\frac{\lambda t}{\lambda+1} + \lambda}^{\frac{\mu t}{\mu+1} + \mu} G_2(u) D_2 \gamma_1(t, u) du + \int_{\frac{t}{\lambda+1}}^{\frac{t}{\mu+1}} F_2(u) D_2 \gamma_2(t, u) du - C \right],$$

where $C = \int_\lambda^\mu F_1(y)\, dy$. The functions $G_2$, $F_2$ are at least continuous. Hence, by repeated application of the theorem concerning the differentiable of parametric integrals (see e.g. [4]), the right hand side is differentiable infinitely many times on $[\alpha, \beta]$. Since $[\alpha, \beta]$ is an arbitrary subinterval of $\mathbb{R}_+$, we have that $G_1$ is differentiable infinitely many times on $\mathbb{R}_+$. The differentiability of $G_2$ can be obtained similarly.

Finally from (16) and (17) we can deduce that $F_1$ and $F_2$ are also differentiable infinitely many times on $\mathbb{R}_+$. □

**Lemma 3.** *If the functions $G_i, F_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ satisfy the functional equation* (4) *for all $x, y \in \mathbb{R}_+$ and they are twice differentiable in $\mathbb{R}_+$, then there exist constants $C, \gamma, \delta_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$, with $\delta_1 + \delta_3 = \delta_2 + \delta_4$ such that*

$$G_1(x) = C \ln x + \gamma x + \delta_1, \qquad x \in \mathbb{R}_+, \tag{18}$$

$$F_1(x) = C \ln \frac{x}{x+1} + \gamma x + \delta_3, \quad x \in \mathbb{R}_+, \tag{19}$$

$$G_2(x) = C \ln x + \gamma x + \delta_2, \qquad x \in \mathbb{R}_+, \tag{20}$$

$$F_2(x) = C \ln \frac{x}{x+1} + \gamma x + \delta_4, \quad x \in \mathbb{R}_+. \tag{21}$$

Proof. Differentiating (4) with respect to $x$, then the resulting equation with respect to $y$, we have

$$G_1'(xy + x) + (xy + x)G_1''(xy + x) = G_2'(xy + y) + (xy + y)G_2''(xy + y), \qquad x, y \in \mathbb{R}_+.$$

This can hold if and only if

$$tG_1''(t) + G_1'(t) = \gamma = sG_2''(s) + G_2'(s), \quad t, s \in \mathbb{R}_+$$

for some constant $\gamma$.

The general solutions of the differential equations

$$tG_1''(t) + G_1'(t) = \gamma, \qquad t \in \mathbb{R}_+,$$

and

$$sG_2''(s) + G_2'(s) = \gamma, \qquad s \in \mathbb{R}_+,$$

have the following forms

$$G_1(t) = C \ln t + \gamma t + \delta_1, \qquad t \in \mathbb{R}_+,$$

$$G_2(s) = C \ln s + \gamma s + \delta_2, \qquad s \in \mathbb{R}_+,$$

where $C, \gamma, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary constants, thus $G_1$ and $G_2$ are of the forms (18) and (20), respectively. Then, from (16), (17), (18) and (20), we get (19) and (21) for $F_1$ and $F_2$, respectively.

It is easy to see that (18), (19), (20) and (21) satisfy (4) if $\delta_1 + \delta_3 = \delta_2 + \delta_4$. $\square$

We may sum up the results of Lemmas 1, 2, 3 in the following theorem.

**Theorem 2.** *If the measurable functions $G_i, F_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ satisfy the functional equation* (4) *for all $x, y \in \mathbb{R}_+$, then there exist constants $C, \gamma, \delta_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ such that $G_1$, $F_1$, $G_2$ and $F_2$ have the forms* (18), (19), (20) *and* (21), *respectively and $\delta_1 + \delta_3 = \delta_2 + \delta_4$.*

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