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Graded Lie algebra associated to a SODE

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Abstract. In this paper we introduce a graded Lie algebra associated to a second order differential equation which gives a powerful tool to the study of the inverse problem of the calculus of variations. We give effective generalizations of Douglas' criteria for the existence of a regular Lagrangian associated to a SODE.

1. Introduction

The inverse problem of the calculus of variations is an old problem of Differential Geometry consisting of the characterization of second order ordinary differential equations (SODE) derivable from a variational principle. In this problem one wants to determine whether a given SODE expresses that the unknown is the critical point of a functional.

One of the most important contribution to this problem is a paper of J. Douglas [4], where he classifies systems of variational differential equations of second order in the two-dimensional case. He showed that the Euler–Lagrange partial differential system

$$\frac{d}{dt}\frac{\partial E}{\partial y^i} - \frac{\partial E}{\partial x^i} = 0, \tag{1.1}$$

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(where the unknown function is E) associated to a SODE $\ddot{x}^i = f^i(x,\dot{x})$, $i = 1,\dots,n$, is equivalent to the first order partial differential system

$$\begin{aligned} \frac{d}{dt}g_{ij} + \frac{1}{2}\frac{\partial f^k}{\partial y^j}g_{ik} + \frac{1}{2}\frac{\partial f^k}{\partial y^i}g_{jk} &= 0,\\ A^k_j g_{ik} - A^k_i g_{jk} &= 0,\\ \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^j} &= 0,\\ g_{ij} - g_{ji} &= 0,\\ \det(g_{ij}) &\neq 0, \end{aligned} \tag{1.2}$$

where the unknown functions are g_{ij}, $i,j = 1,\dots,n$, and A^i_j are the components of the Douglas tensor (called also Jacobi endomorphism). A solution E of (1.1) gives a solution of (1.2) by taking

$$g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j} \tag{1.3}$$

and conversely, for every solution of (1.2) there exists a regular a solution E of (1.1) so that (1.3) holds. A solution of (1.2) is called *variational multiplier*.

Obstructions to the existence of a variational principle, i.e. integrability conditions of the Euler–Lagrange PDE, are in general very complex. However, in his article [4] Douglas gives a simple criteria on the existence of the variational multiplier and therefore on the existence of a variational principle for SODEs on 2-dimensional manifolds. This criteria can be easily carried over to the n-dimensional case (see [2], [8], [11]). In [12] the authors found a double hierarchy of algebraic conditions for the variational multiplier which is determined by the Douglas tensor, the curvature tensor and their derivatives.

In this paper we introduce a graded Lie-algebra associated in a natural way with the SODE using a differential algebraic characterization of connections and derivations. It contains algebraic conditions on the variational multiplier and in generic cases it gives a significant part of the obstruction to the existence of a variational principle (Theorem 2). This concept is of particular interest when the dimension of the base manifold is large, because we are able to obtain new information about the structure of the obstructions (Theorems 4 and 5).

2. Frölicher–Nijenhuis theory

In this paper we use extensively the Frölicher–Nijenhuis' theory of the derivation associated to vector valued forms. A complete description can be found in [5] or [10]. We recall here only the basic elements of this theory.

We denote by $\Lambda(M)$ (resp. $\Psi(M)$) the $C^\infty(M)$ modulus of the scalar (resp. vector valued) forms. The Frölicher–Nijenhuis theory gives a complete description of the derivation of $\Lambda(M)$ with the help of $\Psi(M)$.

Definition 1. A morphism $D : \Lambda(M) \longrightarrow \Lambda(M)$ is a derivation of $\Lambda(M)$ of degree r if it satisfies the following conditions:

a) $D(\Lambda^p(M)) \subset \Lambda^{p+r}(M)$,

b) $D(a\omega + b\omega') = aD\omega + bD\omega', \quad a, b \in \mathbb{R}$

c) $D(\omega \wedge \pi) = D\omega \wedge \pi + (-1)^{r \deg \omega} \omega \wedge D\pi$.

The bracket of two derivations D_1 and D_2 is defined by

$$[D_1, D_2] = D_1 D_2 - (-1)^{(\deg D_1)(\deg D_2)} D_2 D_1.$$

Definition 2. A derivation is called of i_* type or algebraic, if its action is trivial on $\Lambda^o(M)$, and of d_* type, if it commutes with the operator d.

Every derivation is determined by his action on $\Lambda^o(M)$ and $\Lambda^1(M)$. So the i_* type derivations are those which are completely determined by their action on $\Lambda^1(M)$, and the d_* type derivations are determined by their action on $C^\infty(M)$.

An i_* and a d_* type derivation can be associated to a vector valued l-form $L \in \Psi^l(M)$, denoted by i_L and d_L, in the following way:

1. if $\deg L = 0$, (i.e. $L \in \mathfrak{X}(M)$ is a vector field on M):
   $i_L \omega := \omega(L)$, and
   $d_L \omega := \mathcal{L}_L \omega$;
2. if $\deg L = l > 1$:
   $i_L \omega(X_1, \ldots, X_l) := \omega(L(X_1, \ldots, X_l)), \quad$ where $\omega \in \Lambda^1(M)$;
   $d_L f(X_1, \cdots, X_l) := df(L(X_1, \cdots, X_l)), \quad$ where $f \in \Lambda^o(M)$,

where $\omega \in \Lambda^1(TM)$, and $\mathcal{L}_L$ denotes the Lie derivation with respect to L. Conversely, it is easy to show that every i_* or d_* type derivation can be written in the above form with the help of some vector valued form. Therefore we arrive at the following:

Proposition 3. *Let L and M be vector valued differential l- and m-forms. Then there exists a unique vector valued $(l+m)$-form (denoted by $[L,M]$) which satisfies the equation*

$$[d_L, d_M] = d_{[L,M]}.$$

By this bracket, $\Psi(M)$ is a graded Lie algebra.

3. Inverse problem of the calculus of variations

We turn our attention to the inverse problem of the variational calculus. A coordinate free formulation of this problem can be given by the notion of sprays introduced by Klein in [9].

Let J be the canonical vertical endomorphism and C the canonical vertical vector field. In the local coordinate system (x^i) on M and (x^i, y^i) on TM,

$$J = dx^\alpha \otimes \frac{\partial}{\partial y^\alpha}, \quad C = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

A vector field S on TM is a spray if $JS = C$. A curve $\gamma : I \longrightarrow M$ is associated to the spray S if γ' is an integral curve of S i.e. $S_{\gamma'} = S\gamma''$. In a coordinate system the expression of a spray is

$$S = y^\alpha \frac{\partial}{\partial x^\alpha} + f^\alpha(x,y) \frac{\partial}{\partial y^\alpha}, \tag{3.1}$$

and the path $(x^\alpha(t))$ is associated to the spray (3.1) if and only if the second order differential equation

$$\frac{d^2 x^\alpha}{dt^2} = f^\alpha \left(x, \frac{dx}{dt} \right) \tag{3.2}$$

holds. Therefore a spray is a coordinate free presentation of a SODE on the manifold M.

Definition 4. The Lagrangian function $E : TM \to \mathbb{R}$ is called regular, if the 2-form $\Omega_E := dd_J E$ is symplectic.

Using a coordinate system, a Lagrangian E is regular if and only if the matrix $\left(\frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} \right)$ is not singular. It is well known that to any Lagrangian which defines a regular variational problem, a spray (i.e. a SODE) can be associated, in the following way:

Theorem 1 [6]**.** *Let $E \in C^\infty(TM)$ be a regular Lagrangian. The vector field S defined by the equation*

$$i_S\Omega_E = d(E - \mathcal{L}_C E) \tag{3.3}$$

is a spray, and the paths associated to the spray S are solutions of the corresponding variational problem.

So the spray S is variational if there exists a regular function $E \in C^\infty(TM)$, so that the equation (3.3) holds.

Remark. Let E be an arbitrary Lagrangian on M and S be a spray. The associated semi-basic 1-form

$$\omega_E := i_S\Omega_E + d\mathcal{L}_C E - dE \tag{3.4}$$

is called *Euler–Lagrange* form. The solution of the inverse problem for a given SODE is a regular Lagrangian such that the equation $\omega_E = 0$ holds.

4. Identities satisfied by variational sprays

From now on we shall work on TM, the tangent manifold of M. Where there is no possibility of confusion, TTM, T^*TM and T^vTM will be noted as T, T^* and T^v respectively.

The space of semi-basic scalar (resp. vectorial) l-forms is denoted by $\Lambda_v^l T_v^*$ (resp. $\Psi^l T_v^*$). We recall that a p-form $\omega \in \Lambda^p T^*$ is semi-basic if $\omega(X_1, \dots, X_p) = 0$ when one of the vectors X_i is vertical, and a vector valued l-form L is semi-basic if $L(X_1, \dots, X_p) = 0$ when one of the vectors X_i is vertical and its value is vertical.

A connection Γ can be associated to every spray S defined by the formula $\Gamma := [J, S]$ (see [7]): it is easy to check that $\Gamma^2 = I$ and the eigenspace corresponding to the eigenvalue -1 is the vertical space T^v. If we denote the eigenspace corresponding to the eigenvalue $+1$ by T^h, then TTM can be decomposed as

$$TTM = T^h \oplus T^v.$$

Let h and v be the corresponding projectors: $h := \frac{1}{2}(I + \Gamma)$, $v := \frac{1}{2}(I - \Gamma)$. The curvature of the connection Γ is the vector valued 2-form

$$R := -\frac{1}{2}[h, h].$$

The almost complex structure F associated to Γ which exchanges the horizontal and the vertical space is defined by

$$F = h[S,h] - J.$$

Property 5. *Let E be a Lagrangian, Γ a connection on M, h its associated horizontal projection, and F the associated almost-complex structure. The following properties are equivalent:*

a) $i_\Gamma \Omega_E = 0$

b) $i_F \Omega_E = 0$

c) $\Omega_E(hX, hY) = 0 \ \forall X, Y \in TTM$ *(i.e. the horizontal distribution is Lagrangian).*

A connection is called Lagrangian with respect to E, if it satisfies the above conditions.

Indeed, we have:

$$\begin{aligned}
i_\Gamma \Omega_E(hX, hY) &= 2\Omega_E(hX, hY)\\
i_\Gamma \Omega_E(hX, JY) &= \Omega_E(hX, JY) - \Omega_E(hX, JY) = 0\\
i_\Gamma \Omega_E(JX, JY) &= -2\Omega_E(JX, JY) = 0,
\end{aligned}$$

so a) $\iff$ c). On the other hand:

$$\begin{aligned}
i_F \Omega_E(hX, hY) &= -\Omega_E(JX, hY) - \Omega_E(hX, JY) = -i_J \Omega_E(hX, hY) = 0\\
i_F \Omega_E(hX, JY) &= -\Omega_E(JX, JY) + \Omega_E(hX, hY) = \Omega_E(hX, hY)\\
i_F \Omega_E(JX, JY) &= \Omega_E(hX, JY) + \Omega_E(JX, hY) = 0,
\end{aligned}$$

so b) $\iff$ c).

Property 6. *Let E be a Lagrangian on the manifold M. Then*

$$d_J \omega_E = i_\Gamma \Omega_E. \tag{4.1}$$

Consequently, if the spray S is variational and E is a Lagrangian associated to S, then the horizontal distribution associated to the spray S must be Lagrangian with respect to the symplectic 2-form Ω_E.

PROOF. The Euler–Lagrange form can be written in the following form:

$$\begin{aligned}\omega_E &= i_S d d_J E + d\mathcal{L}_S E - dE = \mathcal{L}_S d_J E - dE \\ &= d_J \mathcal{L}_S E - i_{[J,S]} dE = d_J \mathcal{L}_S E - 2 d_h E.\end{aligned}$$

Since the vertical distribution is integrable, we get $[J, J] = 0$, we have $d_J^2 = d_J \circ d_J = d_{[J,J]} = 0$. So

$$\begin{aligned}d_J \omega_E &= -2 d_J d_h E = 2 d_h d_J E = 2(i_h d d_J E - d i_h d_j E) \\ &= 2 i_h \Omega_E - 2\Omega_E = i_\Gamma \Omega_E.\end{aligned}$$

If the spray is variational and E is a Lagrangian associated with S, we have $\omega_E = 0$, then $i_\Gamma \Omega_E = 0$, so the connection associated to the spray is Lagrangian. □

Definition 7. Let S be a spray on M, $L \in \Psi_v(TM)$. The semi-basic derivation of L with respect to the spray S is

$$L' := h^* v[S, L] \tag{4.2}$$

where $h^* L(X_1, \ldots, X_l) := L(hX_1, \ldots, hX_l)$.

We have the following

Proposition 8. *Let S be a spray on M and $L \in \Psi_v(TM)$. We have the formula*

$$L' = [S, L] + FL - L \overline{\wedge} F. \tag{4.3}$$

In particular, suppose that S is variational, E being a Lagrangian associated to S. If the equation $i_L \Omega_E = 0$ holds, then the equations

$$i_{L'} \Omega_E = 0, \quad i_{L''} \Omega_E = 0, \quad i_{L'''} \Omega_E = 0, \quad \textit{etc.} \tag{4.4}$$

hold too.

Proof. To show the first expression, we note that

$$\begin{aligned}
L'(X_1,\dots,X_l) &= v[S,L](hX_1,\dots,hX_l)\\
&= v[S,L(X_1,\dots,X_l)] - \sum_{i=1}^{l} L(X_1,\dots,[S,hX_i],\dots,X_l)\\
&= [S,L(X_1,\dots,X_l)] - h[S,L(X_1,\dots,X_l)]\\
&\quad - \sum_{i=1}^{l} L(X_1,\dots,[S,h]X_i,\dots,X_l) - \sum_{i=1}^{l} L(X_1,\dots,[S,X_i],\dots,X_l)\\
&= [S,L](X_1,\dots,X_l) + FL(X_1,\dots,X_l)\\
&\quad - \sum_{i=1}^{l} L(X_1,\dots,h[S,h]X_i,\dots,X_l).
\end{aligned}$$

Using the identity $h[S,h]=F+J$ and the hypothesis that L is semi-basic, we obtain (4.3). Secondly, by the formula (4.3) we have

$$\begin{aligned}
i_{L'}\Omega_E &= i_{[S,L]}\Omega_E + i_{FL}\Omega_E - i_{F\overline{\wedge}}\Omega_E = i_{[S,L]}\Omega_E + i_F i_L\Omega_E - i_L i_F\Omega_E\\
&= \mathcal{L}_S i_L\Omega_E - d_L\omega_E + i_F i_L\Omega_E - i_L i_F\Omega_E.
\end{aligned}$$

When S is variational and the function E is a Lagrangian associated to S, then $\omega_E=0$ and the connection Γ is Lagrangian, so we have $i_F\Omega_E=0$ (Properties 5 and 6). If the equation $i_L\Omega_E=0$ holds, we have also $i_{L'}\Omega_E=0$ and recursively we obtain (4.4). □

Definition 9. Let h be the horizontal projection associated to the connection $\Gamma=[J,S]$, and $L\in\Psi_v^l(TM)$ a vector valued semi-basic l-form. The operator

$$d^hL := [h,L] \tag{4.5}$$

is the *semi-basic derivation of L with respect to h.*

Proposition 10. *Let L be a semi-basic vector valued l-form. Then d^hL is semi-basic. Moreover assume that S is variational, and E is a Lagrangian associated to S. If the equation $i_L\Omega_E=0$ holds, then the equation $i_{d^hL}\Omega_E=0$ holds too.*

Proof. It is not difficult to check that if L is a semi-basic vector valued l-form, then d^hL is also semi-basic. Let us show the second part of the proposition. Let us assume that S is variational, E is a Lagrangian associated to S, and L is a vector-valued semi-basic l-form. By the relation

$$(-1)^l i_{[h,L]} = i_h d_L - d_L i_h - d_{L\overline{\wedge}h}$$

and taking into account that $L\overline{\wedge}h = l\,L$, because L is semi-basic, we have

$$(-1)^l i_{d^h L}\Omega_E = (-1)^l i_{[h,L]} dd_J E = i_h d_L dd_J E - d_L i_h dd_J E - l\, d_L dd_J E.$$

If the equation $i_L\Omega_E = 0$ holds, then

$$\begin{aligned}(-1)^l i_{d^h L}\Omega_E &= i_h d i_L dd_J E - d_L i_{\frac{1}{2}(I+\Gamma)} dd_J E - l\, d i_L dd_J E \\ &= -l\, d i_L dd_J E - \frac{1}{2} d_L i_\Gamma dd_J E = 0.\end{aligned}$$

□

5. Graded Lie algebra associated to a SODE

Definition 11. The graded Lie algebra $\mathcal{A}_S$ associated to the spray S is the graded Lie sub-algebra of the vector-valued forms spanned by the vertical endomorphism J, the Douglas tensor $A := v[h,S]$, and generated by the action of the semi-basic derivation defined in (4.2), the derivation d^h, and the Frölicher–Nijenhuis bracket $[\ ,\ ]$. The graduation of $\mathcal{A}_S$ is given by

$$\mathcal{A}_S = \oplus_{k=1}^{n} \mathcal{A}_S^k \tag{5.1}$$

where $\mathcal{A}_S^k := \mathcal{A}_S \cap \Psi^k(TM)$.

Remark. Note that J and A are semi-basic and that, as we showed in the preceding paragraph, the space of semi-basic forms is stable by semi-basic derivation defined in (4.2), by the derivation d^h, and by the Frölicher–Nijenhuis bracket $[\ ,\ ]$. It follows that $\mathcal{A}_S$ is a graded Lie sub-algebra of the vector-valued *semi-basic* forms.

The importance of the graded Lie algebra associated to a spray is given by the following:

Theorem 2. *Let S be a variational spray and E a Lagrangian associated to S. Then for every element L of $\mathcal{A}_S$ the equation*

$$i_L \Omega_E = 0 \tag{5.2}$$

holds. Therefore elements of $\mathcal{A}_S$ give algebraic conditions on the variational multiplier.

Proof. To prove Theorem 2 we will first show that J and A satisfy the equation (5.2). Then we will prove that all the vector-valued forms obtained from J and A by a finite number of successive operations which define $\mathcal{A}_S$, also satisfy the equation (5.2).

1. From $[J, J] = 0$ we can easily obtain :

$$i_J \Omega_E = i_J dd_J E = d_J^2 E = d_{[J,J]} E = 0, \tag{5.3}$$

   so the equation (5.2) holds for J.

2. For the Douglas tensor we find

$$\begin{aligned} i_A \Omega_E &= i_{[h,S]} \Omega_E + i_F \Omega_E = i_h \mathcal{L}_S \Omega_E - \mathcal{L}_S i_h \Omega_E + i_F \Omega_E \\ &= i_h d\omega_E - \mathcal{L}_S \left(\Omega_E + \frac{1}{2} d_J \omega_E \right) + i_F \Omega_E \\ &= i_h d\omega_E - d\omega_E - \frac{1}{2} \mathcal{L}_S d_J \omega_E + i_F \Omega_E \\ &= d_h \omega_E - \frac{1}{2} \mathcal{L}_S d_J \omega_E + i_F \Omega_E \end{aligned}$$

   If S is variational and E is a Lagrangian associated to S, then $\omega_E = 0$ and the connection Γ is Lagrangian. Therefore every term vanishes, and the equation (5.2) also holds for $A = L$.

3. From Propositions 8 and 10, respectively we know that if $i_L \Omega_E = 0$ holds for $L \in \mathcal{A}_S$ then $i_{L'} \Omega_E = 0$, $i_{d^h L} \Omega_E = 0$, hold too.

4. Let $K \in \mathcal{A}_S^k$, $L \in \mathcal{A}_S^l$ be semi-basic vector-valued forms, such that $i_K \Omega_E = 0$ and $i_L \Omega_E = 0$. Since K and L are semi-basic, we have $L \overline{\wedge} K \equiv 0$ and hence

$$\begin{aligned} (-1)^l i_{[K,L]} \Omega_E &= \big(i_K d_L - (-1)^{l(m-1)} d_L i_K - d_{L \overline{\wedge} K} \big) \Omega_E \\ &= i_K (i_L d - d i_L) dd_J E - (-1)^{l(m-1)} d_L i_K dd_J E - d_{L \overline{\wedge} K} dd_J E \\ &= i_K d i_L \Omega_E - (-1)^{l(m-1)} d_L i_K \Omega_E = 0. \end{aligned}$$

On the other side, it is easy to see that (5.2) gives algebraic condition on the variational multiplier. Indeed, if the spray is variational and E is a regular Lagrangian associated with S, then locally one has

$$\Omega_E = \frac{1}{2}\left(\frac{\partial^2 E}{\partial x^\alpha \partial y^\beta} - \frac{\partial^2 E}{\partial y^\alpha \partial x^\beta}\right) dx^\alpha \wedge dx^\beta - \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} dx^\alpha \wedge dy^\beta. \tag{5.4}$$

If $L \in \Psi^l(TM)$ is semi-basic, then

$$i_L \Omega_E = \frac{1}{l!} \sum_{\alpha \in \mathfrak{S}_{l+1}} \varepsilon(\alpha) L^\beta_{\alpha_1 \ldots \alpha_l} \frac{\partial^2 E}{\partial y^\beta \partial y^{\alpha_{l+1}}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{l+1}},$$

where $\mathfrak{S}_{p+l-1}$ denotes the $(p+l-1)!$-order symmetric group and $\varepsilon(\alpha)$ the sign of α. Then the equation $i_L \Omega_E = 0$ is an algebraic equation

$$\sum_{\alpha \in \mathfrak{S}_{l+1}} \varepsilon(\alpha) L^\beta_{\alpha_1 \ldots \alpha_l} g_{\beta \alpha_{l+1}} = 0 \tag{5.5}$$

in the variational multiplier $g_{\alpha\beta} = \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}$. □

Remark. From the construction of $\mathcal{A}_S$ it is clear that $\mathcal{A}_S$ contains the Douglas tensor and its semi-basic derivatives with respect to S. On the other hand, the curvature tensor R is related to A by the equation $R = \frac{1}{3}[J, A]$, so $\mathcal{A}_S$ also contains R and its semi-basic derivatives. Therefore $\mathcal{A}_S$ contains the double hierarchy of algebraic conditions for the variational multiplier founded in [12].

Theorem 3. *If at $x \in TM$ one has*

$$\operatorname{rank}\big\{J, A, A', \ldots, A^{(k)}, \ldots\big\}_{k \in \mathbb{R}} > \frac{n(n+1)}{2},$$

then S is not variational in the neighborhood of x.

Proof. Let us suppose that S is variational, E is an associated regular Lagrangian, and $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ is a variational multiplier. For every $L \in \mathcal{A}^1_S$ the condition $i_L \Omega_E = 0$ gives

$$g_{ik} L^k_j = g_{jk} L^k_i.$$

i.e. L is symmetric with respect to g. Since the tensors $J, A, A', A'', \ldots$ $\ldots, A^{(\frac{n(n+1)}{2} - 1)}$ are elements of $\mathcal{A}_S$, we have $i_{A^{(k)}} \Omega_E = 0$. Therefore, if the spray is variational, then the tensors $J, A, A', A"$, … ,$A^{(\frac{n(n+1)}{2}-1)}$ are self-adjoint with respect to g. But the space of the $(1-1)$ tensors which are self-adjoint with respect to a regular matrix is $\frac{n(n+1)}{2}$-dimensional. Consequently if the spray is variational, then $J, A, A', A'', \dots, A^{(\frac{n(n+1)}{2}-1)}$ are linearly dependent. □

If $\dim M = 2$, then $\mathcal{A}_S$ only contains J, A and the hierarchy given by its semi-basic derivatives A', A'', etc. However, if $\dim M > 2$, then we find other hierarchies in $\mathcal{A}_S$ which give, in the generic case, new necessary conditions for the variational multipliers. We arrive at the following generalization of the Theorem 3:

Theorem 4. *Let S be a spray and $x \in TM$. If there exists an integer $k < n$ for which*

$$\dim \mathcal{A}_S^k(x) > k\binom{n+1}{k+1}, \tag{5.6}$$

then the spray is not variational in a neighborhood of x.

Proof. Let S be a spray and E a regular Lagrangian. We consider for every $k = 1, \dots, n$ the morphism

$$\begin{aligned} \Lambda^k T_v^* \otimes T^v &\xrightarrow{\psi_k} \Lambda^{k+1} T_v^* \\ L &\longrightarrow i_L \Omega_E. \end{aligned}$$

By the regularity of E the 2-form Ω_E is symplectic, and the morphism ψ_k is onto. Indeed, it is easy to see that if $\{X_1, \dots, X_n\}$ is a basis of T^v, then $\alpha_1, \dots, \alpha_n \in T^*$ defined by $\alpha_i = i_{X_i}\Omega_E$, gives a basis of T_v^*. Consequently

$$\left\{\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} \wedge \alpha_{i_{k+1}}\right\}_{1 \le i_1 < \dots < i_{k+1} \le n} \tag{5.7}$$

is a basis of $\Lambda^{k+1} T_v^*$ and

$$\left\{\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} \otimes X_{i_{k+1}}\right\}_{1 \le i_1 < \dots < i_k \le n,\ 1 \le i_{k+1} \le n} \tag{5.8}$$

gives a basis of $\Lambda^k T_v^* \otimes T^v$. Moreover, if the components of $\Lambda \in \Lambda^{k+1} T_v^*$ with respect to the basis (5.7) are $\Lambda^{i_1 \dots i_{k+1}}$, then $\Lambda = \psi_k(L)$ where $L = \Lambda_{i_1 \dots i_{k+1}} \alpha_{i_1} \wedge \dots \wedge \alpha_{i_k} \otimes X_{i_{k+1}}$. This proves that ψ_k is onto. Therefore

$$\operatorname{rank} \psi_k = \dim \Lambda^{k+1} T_v^* = \binom{n}{k+1} = \frac{n!}{(k+1)!(n+1-k)!},$$

and

$$\dim \operatorname{Ker} \psi_k = n \cdot \binom{n}{k} - \binom{n}{k+1} \tag{5.9}$$
$$= \frac{k(n+1)n!}{(k+1)!(n-k)!} = k \cdot \binom{n+1}{k+1}$$

On the other hand using Theorem 2 we have

$$\mathcal{A}_S^k \subset \operatorname{Ker} \psi_k.$$

But if the inequality (5.6) holds, then $\dim \mathcal{A}_S^k > \dim \operatorname{Ker} \psi_k$, and consequently the spray is not variational. □

Definition 12. Let S be a spray $x \in TM$, and let us consider the system of linear equations

$$\Big\{ \sum_{i \in \mathfrak{S}_{l+1}} \varepsilon(i) L^j_{i_1 \dots i_l} x_{j i_{l+1}} = 0 \;\Big|\; L \in \mathcal{A}_S(x) \Big\} \tag{5.10}$$

in the symmetric variables x_{ij} ($x_{ij} = x_{ji}$) where $L^j_{i_1 \dots i_l}$ are the components of $L \in \mathcal{A}_S(x)$. The rank of the linear equations (5.10) is called the *rank of the spray* at x.

Remark. As equation (5.5) shows, the rank of a spray gives the number of independent equations satisfied by the variational multipliers. Consequently, if the system (5.10) does not have a solution with $\det(x_{ij}) \neq 0$, then there is no variational multiplier for S, and therefore the spray is non-variational. Thus we arrive at

Theorem 5. *If at $x \in TM$ we have*

$$\operatorname{rank} S(x) \geq \frac{n(n+1)}{2}$$

then S is non-variational in a neighborhood of x.

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