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On R-quadratic Finsler spaces

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Abstract. In this paper, we introduce the notion of R-quadratic Finsler metric. It is known that every Berwald metric is R-quadratic and Landsbergian. We show that every compact R-quadratic Finsler space must be Landsbergian.

1. Introduction

In Finsler geometry, there are several notions of curvatures. Among them, the Riemann curvature is an important quantity. For a Finsler space (M, F), the Riemann curvature is a family of linear transformations $\mathbf{R}_y : T_xM \to T_xM$, where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ (the definition will be given in S2). If F is Riemannian, i.e., $F(y) = \sqrt{g(y,y)}$ for some Riemannian metric g, then $\mathbf{R}_y := \mathrm{R}(\cdot, y)y$, where $\mathrm{R}(u,v)z$ denotes the Riemannian curvature tensor of g. In this case, $\mathbf{R}_y$ is quadratic in $y \in T_xM$. A Finsler metric is said to be R-*quadratic* if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$. There are many non-Riemannian R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Thus R-quadratic Finsler spaces form a rich class of Finsler spaces. The main purpose of this paper is to prove the following

Theorem 1.1. *Let (M, F) be a positively complete Finsler space with bounded Cartan torsion. Suppose that F is* R*-quadratic, then F must be a*

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Landsberg metric. In particular, every compact R*-quadratic Finsler space must be Landsbergian.*

The second part of Theorem 1.1 is true because that Finsler metrics on a compact manifold must be positively complete with bounded Cartan torsion. Theorem 1.1 tells us that for Finsler metrics on a compact manifold, the following holds

$$\{\text{Berwald metrics}\} \subset \{\text{R-quadratic metrics}\} \subset \{\text{Landsberg metrics}\}.$$

It is an open problem in Finsler geometry whether or not there is a Landsberg metric which is not Berwaldian. Theorem 1.1 throws a light into this problem.

We will see that a Finsler metric is R-quadratic if and only if the h-curvature of the Berwald connection depends on position only in the sense of Bácsó–Matsumoto (see Remark 3.1 below). In [BM], Bácsó and Matsumoto classify Finsler metrics in the form $F(y) = \sqrt{g(y,y)} + \beta(y)$ (Randers metrics) whose h-curvature depend on position only. Their results indicates that there are possibly local R-quadratic Finsler metrics which are not Landsbergian.

For a submanifold M in a Minkowski space (V, F), the Cartan torsion must be bounded [Sh1]. We obtain the following

Corollary 1.2. *For any positively complete submanifold M in a Minkowski space (V, F), if the induced Finsler metric $\bar{F}$ is* R*-quadratic, then $\bar{F}$ must be a Landsberg metric.*

A Finsler space is said to be R-*flat*, if the Riemann curvature $\mathbf{R}_y = 0$. R-flat Finsler metrics are of course R-quadratic. According to Akbar-Zedah [AZ], for positively complete R-flat Finsler space (M, F), if the Cartan torsion and its vertical covariant derivative are bounded, then F is locally Minkowski. The conditions on the Cartan torsion are satisfied by submanifolds in a Minkowski space. We conclude that for a positively complete submanifold in a Minkowski space, if the induced Finsler metric is R-flat, then it must be locally Minkowski.

2. Preliminaries

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties

(a) F is C^∞ on $TM \setminus \{0\}$;

(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;

(c) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\big[F^2(y+su+tv)\big]\,|_{s,t=0}, \qquad u,v \in T_xM. \tag{1}$$

At each point $x \in M$, $F_x := F\,|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\big[\mathbf{g}_{y+tw}(u,v)\big]\,|_{t=0}, \qquad u,v,w \in T_xM. \tag{2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM\setminus\{0\}}$ is called the *Cartan torsion*. E. Cartan got this quantity when he introduced his metric-compatible connection. Since then, it is called the Cartan tensor or the Cartan torsion in literatures.

F is said to be *positively complete* if every geodesic on (a,b) can be extended to a geodesic on (a,∞). A curve $c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2}(t) + 2G^i(\dot{c}(t)) = 0, \tag{3}$$

where $G^i(y)$ are local functions on TM given by

$$G^i(y) := \frac{1}{4}g^{il}(y)\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l}(y)y^k - \frac{\partial[F^2]}{\partial x^l}(y)\right\}, \qquad y \in T_xM. \tag{4}$$

F is called a *Berwald metric* if $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$.

The Riemann curvature can be defined using geodesic fields and the induced Riemannian metrics. A local vector field Y is called a *geodesic field* if the integral curves of Y are geodesics. Fix a vector $y \in T_xM \setminus \{0\}$ and extend it to a local geodesic field Y on a neighborhood U_x of x. Y defines a Riemannian metric on U_x by

$$\hat{g}_z(u,v) := g_{ij}(Y_z)u^i v^j, \qquad u, v \in T_zU.$$

Let $\hat{\mathrm{R}}(u,v)z$ denote the Riemannian curvature tensor of $\hat{g}$. Define $\mathbf{R}_y : T_xM \to T_xM$ by

$$\mathbf{R}_y(u) := \hat{\mathrm{R}}(u,y)y, \qquad u \in T_xM. \tag{5}$$

$\mathbf{R}_y$ is a well-defined linear transformation independent of the geodesic extension Y of y. The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM\setminus\{0\}}$ is called the *Riemann curvature*. The notion of Riemann curvature was first extended to Finsler metrics by L. Berwald from a different approach [Bw]. F is said to be R-*quadratic* if $\mathbf{R}_y$ is quadratic in $y \in T_xM$ at each point $x \in M$. Berwald metrics are always R-quadratic (see Remark 3.1).

Let $U(t)$ be a vector field along a curve $c(t)$. The canonical covariant derivative $\mathrm{D}_{\dot{c}}U(t)$ is defined by

$$\mathrm{D}_{\dot{c}}U(t) := \left\{ \frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}(\dot{c}(t)) \right\} \frac{\partial}{\partial x^i}\,|_{c(t)}\ . \tag{6}$$

$U(t)$ is said to be *parallel* along c if $\mathrm{D}_{\dot{c}(t)}U(t) = 0$.

To measure the changes of the Cartan torsion $\mathbf{C}$ along geodesics, we define $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\left[\mathbf{C}_{\dot{c}(t)}(U(t),V(t),W(t))\right]|_{t=0} \tag{7}$$

where $c(t)$ is a geodesic and $U(t)$, $V(t)$, $W(t)$ are parallel vector fields along $c(t)$ with $\dot{c}(0) = y$, $U(0) = u$, $V(0) = v$, $W(0) = w$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM\setminus\{0\}}$ is called the *Landsberg curvature*. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$. An important fact is that if F is Berwaldian, then it is Landsbergian. See Remark 3.3.

3. Structure equations

To find the relationship among various quantities, usually we go to the slit tangent bundle $\pi : \mathcal{T}M := TM \setminus \{0\} \to M$ and choose a linear connection (such as the Berwald connection, the Cartan connection and the Chern connection) on an appropriate vector bundle (such as the vertical tangent bundle $\mathcal{V}TM$ and the pull-back tangent bundle π^*TM). In this paper, we will study them using differential forms on $\mathcal{T}M$ instead.

Let F be a Finsler metric on an n-dimensional manifold M. Let (x^i, y^i) be a standard coordinate system in $\mathcal{T}M$ and $G^i(y)$ denote the geodesic coefficients of F in (4). Put

$$\omega_j{}^i := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y) dx^k. \tag{8}$$

In literatures, $\frac{\partial^2 G^i}{\partial y^j \partial y^k}$ are denoted by G^i_{jk}. $\{\omega_j{}^i\}$ are called the *Berwald connection forms*. Put

$$g_{ij}(y) := \mathbf{g}_y(e_i, e_j), \tag{9}$$

$$C_{ijk}(y) := \mathbf{C}_y(e_i, e_j, e_k), \quad L_{ijk}(y) := \mathbf{L}_y(e_i, e_j, e_k). \tag{10}$$

where $\{e_i = \frac{\partial}{\partial x^i}|_{\pi(y)}\}$ is a natural local frame on M. They are local functions on $\mathcal{T}M$. With the Berwald connection forms, we define $C_{ijk;l}$ and $C_{ijk\cdot l}$ by

$$dC_{ijk} - C_{pjk}\omega_i{}^p - C_{ipk}\omega_j{}^p - C_{ijp}\omega_k{}^p = C_{ijk;l}\omega^l + C_{ijk\cdot l}\omega^{n+l}. \tag{11}$$

The definition of $\mathbf{L}$ in (7) is equivalent to the following

$$L_{ijk}(y) := C_{ijk;l}(y) y^l. \tag{12}$$

In literatures, $C_{ijk;l}(y)y^l$ are also denoted by $C_{ijk|0}(y)$. Thus $L_{ijk} = C_{ijk|0}$.

Let $\omega^i := dx^i$ and $\omega^{n+i} := dy^i + y^j \omega_j{}^i$. $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a natural coframe for $T^*(\mathcal{T}M)$. They satisfy the following structure equations

$$d\omega^i = \omega^j \wedge \omega_j{}^i, \tag{13}$$

$$dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{14}$$

See [Sh2] for a proof.

The Riemann curvature $\mathbf{R}_y$ defined in (5) gives rise to a set of local function R^i_k on $\mathcal{T}M$

$$\mathbf{R}_y(e_k) =: R^i_k(y) \frac{\partial}{\partial x^i}\,|_x, \qquad u \in T_xM. \tag{15}$$

In local coordinates, $R^i_k(y)$ can be expressed in terms of $G^i(y)$

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{16}$$

See [Sh2]. Thus the Riemann curvature defined in (5) coincides with the usual one defined in a different way [Bw].

Remark 3.1. If F is Berwaldian, i.e., $G^i(y)$ are quadratic in $y \in T_xM$, then $R^i_k(y)$ are quadratic in $y \in T_xM$. Put

$$R_j{}^i{}_{kl}(y) := \frac{1}{3}\frac{\partial}{\partial y^j}\left\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\right\}.$$

$R_j{}^i{}_{kl}$ are the coefficients of the h-curvature of the Berwald connection, which are also denoted by $H_j{}^i{}_{kl}$ in literatures. We have

$$R^i_k(y) = y^j R_j{}^i{}_{kl}(y) y^l.$$

Thus $R^i_k(y)$ is quadratic in $y \in T_xM$ if and only if $R_j{}^i{}_{kl}(y)$ are functions of x only.

There is another set of local functions B^i_{jkl} on $\mathcal{T}M$ defined by

$$B^i_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y).$$

B^i_{jkl} are also denoted by G^i_{jkl} in literatures. Because this quantity was introduced by L. Berwald first, I call it the Berwald curvature in my papers. Note that F is Berwaldian if and only if $B^i_{jkl} = 0$.

We have

$$\Omega_j{}^i := d\omega^i_j - \omega_j{}^k \wedge \omega_k{}^i \tag{18}$$

$$= \frac{1}{2} R_j{}^i{}_{kl}\omega^k \wedge \omega^l - B^i_{jkl}\omega^k \wedge \omega^{n+l}. \tag{19}$$

The following lemma is crucial in our proof of the main result.

Lemma 3.2.

$$L_{ijk;l}y^l = \frac{1}{2} y^s y^l g_{ps} R_i{}^p{}_{kl\cdot j}\,. \tag{20}$$

Proof. Define $g_{ij;k}$ and $g_{ij\cdot k}$ by

$$dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k = g_{ij;k}\omega^k + g_{ij\cdot k}\omega^{n+k}.$$

(14) means

$$g_{ij;k} = -2L_{ijk}, \quad g_{ij\cdot k} = 2C_{ijk}. \tag{21}$$

Define $y^i_{;k}$ and $y^i_{\cdot k}$ by

$$\omega^{n+i} = dy^i + y^j\omega_j{}^i = y^i_{;k}\omega^k + y^i_{\cdot k}\omega^{n+k}.$$

This means

$$y^i_{;k} = 0, \qquad y^i_{\cdot k} = \delta^i_k. \tag{22}$$

Differentiating (14) yields the following Ricci identities.

$$\begin{aligned} g_{pj}\Omega_i{}^p + g_{ip}\Omega_j{}^p = &-2L_{ijk;l}\omega^k \wedge \omega^l - 2L_{ijk\cdot l}\omega^k \wedge \omega^{n+l} \\ &- 2C_{ijl;k}\omega^k \wedge \omega^{n+l} - 2C_{ijl\cdot k}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega_l{}^p y^l. \end{aligned} \tag{23}$$

It follows from (23) that

$$C_{ijl;k} + L_{ijk\cdot l} = \frac{1}{2}g_{pj}B^p_{ikl} + \frac{1}{2}g_{ip}B^p_{jkl}. \tag{24}$$

Contracting (24) with y^j and using (22) yield

$$L_{jkl} = -\frac{1}{2}y^m g_{im}B^i_{jkl}\,. \tag{25}$$

Remark 3.3. F is Berwaldian if and only if $B^i_{jkl} = 0$. Thus Berwald metrics are always Landsbergian.

Contracting (24) with y^k yields (12). Differentiating (18) yields

$$d\Omega_j{}^i = -\Omega_j{}^k \wedge \omega_k{}^i + \omega_j{}^k \wedge \Omega_k{}^i. \tag{26}$$

Define $R_j{}^i{}_{kl;m}$ and $R_j{}^i{}_{kl\cdot m}$ by

$$\begin{aligned} dR_j{}^i{}_{kl} - R_m{}^i{}_{kl}\omega_i{}^m - R_j{}^i{}_{ml}\omega_k{}^m - R_j{}^i{}_{km}\omega_l{}^m + R_j{}^m{}_{kl}\omega_m{}^i \\ =: R_j{}^i{}_{kl;m}\omega^m + R_j{}^i{}_{kl\cdot m}\omega^{n+m}. \end{aligned} \tag{27}$$

Similarly, we define $B^i_{jkl;m}$ and $B^i_{jkl\cdot m}$. From (26), one obtains the following Bianchi identity

$$R_j{}^i{}_{kl\cdot m} = B^i_{jml;k} - B^i_{jkm;l}. \tag{28}$$

Contracting (28) with $y^s g_{is}$ and using (25) yield

$$L_{jkm;l} - L_{jml;k} = \frac{1}{2} y^s g_{ps} R_j{}^p{}_{kl\cdot m}\,. \tag{29}$$

Contracting (29) with y^l yields (20). □

4. Proof of Theorem 1.1

Let (M, F) be a Finsler space and $c : [a, b] \to M$ a geodesic. For a parallel vector field $V(t)$ along c,

$$\mathbf{g}_{\dot c(t)}(V(t), V(t)) = \text{constant}. \tag{30}$$

Lemma 4.1. *Let (M, F) be a Finsler space. Suppose that F is* R-*quadratic. Then for any geodesic $c(t)$ and any parallel vector field $V(t)$ along c, the following functions*

$$\mathbf{C}(t) := \mathbf{C}_{\dot c}(V(t), V(t), V(t)) \tag{31}$$

must be in the following forms

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0). \tag{32}$$

Proof. By assumption, $R_j{}^i{}_{kl}(y)$ are functions of x only. Thus

$$R_j{}^i{}_{kl\cdot m} = \frac{\partial R_j{}^i{}_{kl}}{\partial y^m} = 0.$$

It follows from (20) that

$$L_{ijk;l} y^l = 0. \tag{33}$$

Let

$$\mathbf{L}(t) := \mathbf{L}_{\dot c}(V(t), V(t), V(t)). \tag{34}$$

From our definition of $\mathbf{L}_y$, we have

$$\mathbf{L}(t) = \mathbf{C}'(t).$$

By (33), we obtain

$$\mathbf{L}'(t) = L_{ijk;l}(\dot{c}(t))\dot{c}^l(t)V^i(t)V^j(t)V^k(t) = 0. \tag{35}$$

Then (32) follows. □

To prove Theorem 1.1, take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $V(t)$ the parallel vector field along c with $V(0) = v$. Define $\mathbf{C}(t)$ and $\mathbf{L}(t)$ as in (31) and (34), respectively. Then

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0).$$

Suppose that $\mathbf{C}_y$ is bounded, i.e., there is a constant $K < \infty$ such that

$$\|\mathbf{C}\|_x := \sup_{y\in T_xM\setminus\{0\}} \sup_{v\in T_xM} \frac{|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}} \le K.$$

By (30), we know that

$$Q := \mathbf{g}_{\dot{c}(t)}(V(t), V(t))$$

is a positive constant. Thus

$$|\mathbf{C}(t)| \le KQ^{\frac{3}{2}} < \infty.$$

and $\mathbf{C}(t)$ is a bounded function on $[0, \infty)$. This implies

$$\mathbf{L}_y(v,v,v) = \mathbf{L}(0) = 0.$$

Therefore $\mathbf{L} = 0$ and F is a Landsberg metric. This completes the proof of Theorem 1.1.

Corollary 4.2. *For any positively complete Randers metric $F = \alpha+\beta$ on a manifold M, if F is R-quadratic, then it must be a Berwald space.*

Proof. First we know that the Cartan torsion of F must be bounded. In fact, $\|\mathbf{C}\|_x \le 3/\sqrt{2}$ for any $x \in M$ (see Appendix below). By Theorem 1.1, F is a Landsberg metric. In a 1974 paper [M], Matsumoto showed that $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel. In a 1977 paper [HI], M. Hashiguchi and Y. Ichijyō showed that for a Randers metric $F = \alpha + \beta$, if β is parallel, then F is a Berwald metric. This completes the proof. □

5. Appendix

In this section, we will prove the following

Proposition 5.1. *For any Randers norm $F=\alpha+\beta$ in an n-dimensional vector space* V *with* $\|\beta\| := \sup_{\alpha(y)=1}\beta(y) < 1$, *the Cartan torsion satisfies*

$$\|\mathbf{C}\| \le \frac{3}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|^2}}. \tag{36}$$

Proposition 5.1 in dimension two is proved in Exercise 11.2.6 in [BCS]. We will first give a different argument in dimension two, then extend it to higher dimensions.

Assume that $\dim V = 2$. The unit circle $\mathrm{S} = F^{-1}(1)$ is a simple closed curve around the origin. For a unit vector $y \in \mathrm{S}$, there is a vector $y^\perp \in \mathrm{V}$ satisfying

$$g_y(y, y^\perp) = 0, \quad g_y(y^\perp, y^\perp) = 1. \tag{37}$$

The set $\{y, y^\perp\}$ is called the *Berwald basis* at y. Define

$$\mathbf{I}(y) := \mathbf{C}_y(y^\perp, y^\perp, y^\perp), \qquad y \in \mathrm{S}.$$

We call $\mathbf{I}$ the *main scalar*. Note that $\mathbf{I} = 0$ if and only if $\mathbf{C} = 0$. Moreover,

$$\|\mathbf{C}\| = \sup_{y\in \mathrm{S}} |\mathbf{I}(y)|.$$

Fix a basis $\{e_1, e_2\}$ for V. Parameterize S by a counter-clockwise map $c(t) = u(t)e_1 + v(t)e_2$. Then

$$\sigma(t) := g_{c(t)}\Big(\dot{c}(t), \dot{c}(t)\Big) = \frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)} > 0. \tag{38}$$

For a unit vector $y = c(t) \in \mathrm{S}$, we can take

$$y^\perp := \frac{1}{\sqrt{\sigma(t)}}\dot{c}(t).$$

The main scalar $\mathbf{I}(t) := \mathbf{I}(c(t))$ is given by

$$\mathbf{I}(t) = \frac{1}{\sqrt{\sigma(t)}}\frac{d}{dt}\left[\ln\frac{\sqrt{\sigma(t)}}{u(t)v'(t) - u'(t)v(t)}\right]. \tag{39}$$

The above formulas also hold for singular Minkowski norms in dimension two.

We now consider a Randers norm $F = \alpha + \beta$ in V. Take an orthonormal basis $\{e_1, e_2\}$ for (V, α) such that $\beta(ue_1 + ve_2) = bu$, where $b = \|\beta\| := \sup_{\alpha(y)=1} \beta(y) < 1$. Then

$$F(ue_1 + ve_2) = \sqrt{u^2 + v^2} + bu. \tag{40}$$

The indicatrix $\mathrm{S} = F^{-1}(1)$ is an ellipse determined by the following equation

$$(1-b^2)^2 \left(u + \frac{b}{1-b^2}\right)^2 + (1-b^2)v^2 = 1.$$

Parameterize S by $c(t) = u(t)e_1 + v(t)e_2$

$$u(t) = -\frac{b}{1-b^2} + \frac{1}{1-b^2}\cos(t), \quad v(t) = \frac{1}{\sqrt{1-b^2}}\sin(t).$$

Plugging $u(t)$ and $v(t)$ into (39), we obtain

$$\mathbf{I}(t) = -\frac{3}{2}\frac{b\sin(t)}{\sqrt{1 - b\cos(t)}}. \tag{41}$$

It is easy to see that

$$\|\mathbf{C}\| = \max_{F(y)=1} |\mathbf{I}(y)| = \max_{0 \le t \le 2\pi} |\mathbf{I}(t)| \le \frac{3}{\sqrt{2}}\sqrt{1 - \sqrt{1-b^2}}. \tag{42}$$

We obtain the same bound for $\mathbf{C}_y$ as in Exercise 11.2.6 in [BCS].

Now we consider a Randers norm $F = \alpha + \beta$ in an n-dimensional vector space V. We claim that the Cartan torsion still satisfies (36). To prove (36), we just need to simplify the problem to the two-dimensional case. Let y_0, v_0 with $F(y_0) = 1$ and $g_{y_0}(v_0, v_0) = 1$ such that

$$\|\mathbf{C}\| = \mathbf{C}_{y_0}(v_0, v_0, v_0).$$

Let $\bar{\mathrm{V}} = \mathrm{span}\left\{y_0, v_0\right\}$ and $\bar{F} := F|_{\bar{\mathrm{V}}}$. The Cartan torsion $\bar{\mathbf{C}}$ of $\bar{F}$ satisfies

$$\mathbf{C}_{y_0}(v_0, v_0, v_0) = \frac{1}{4}\frac{\partial^3}{\partial s^3}\left[F^2(y_0 + sv_0)\right]|_{s=0} = \bar{\mathbf{C}}_{y_0}(v_0, v_0, v_0).$$

Let $\bar{\alpha} := \alpha|_{\bar{V}}$ and $\bar{\beta} = \beta_{|\bar{V}}$. We have

$$\|\bar{\beta}\| = \sup_{\bar{\alpha}(y)=1} \bar{\beta}(y) \le \sup_{\alpha(y)=1} \beta(y) = \|\beta\|.$$

Let $\bar{\mathbf{I}}(y_0)$ denote the main scalar of $\bar{F}$ at y_0. By the above argument, we have

$$\|\bar{\mathbf{C}}\| = \max_{\bar{F}(y)=1} |\bar{\mathbf{I}}(y)| \le \frac{3}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\bar{\beta}\|^2}}\,. \tag{43}$$

Thus

$$\|\mathbf{C}\| \le \|\bar{\mathbf{C}}\| \le \frac{3}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\bar{\beta}\|^2}} \le \frac{3}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|^2}}\,.$$

References

[AZ] H. Akbar-Zadeh, Sur les espaces de Finsler á courbures sectionnelles constantes, *Bull. Acad. Roy. Bel. Cl, Sci, 5e Série - Tome* **LXXXIV** (1988), 281–322.

[BCS] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann-Finsler Geometry, *Springer-Verlag*, 2000.

[BM] S. Bácsó and M. Matsumoto, Randers spaces with the h-curvature tensor H dependent on position alone, *Publ. Math. Debrecen* **57** (2000), 185–192.

[Bw] L. Berwald, Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus, *Math. Z.* **25** (1926), 40–73.

[HI] M. Hashiguchi and Y. Ichijyō, On some special (α,β)-metrics, *Rep. Fac. Sci. Kagoshima Univ.* **8** (1975), 39–46.

[M] M. Matsumoto, On Finsler spaces with Randers metric and special forms of important tensors, *J. Math. Kyoto Univ.* **14** (1974), 477–498.

[Sh1] Z. Shen, On Finsler geometry of submanifolds, *Math. Ann.* **311** (1998), 549–576.

[Sh2] Z. Shen, Differential Geometry of Spray and Finsler Spaces, *Kluwer Academic Publishers*, 2001.

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