Conformally Berwald and conformally flat Finsler spaces

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Abstract. A condition for a Finsler space to be conformal to a Berwald space or a locally Minkowski space is given in terms of a conformally invariant Finsler connection independently of the dimension number.

1. Introduction

We have had many publications on the theory of conformal changes of Finsler spaces, following M. HASHIGUCHI's theory [2] based on the modern treatment of Finsler spaces. H. IZUMI [7], [8] considered some special conformal changes, M. HASHIGUCHI and Y. ICHIJYO [3] dealt with the conformal flatness from the standpoint of the theory of Wagner spaces, and the present author [10] showed that the existence of special Finsler connections is closely related to conformal flatness. Next the problem of conformal flatness was solved for Finsler spaces with some remarkable metrics: Y. ICHIJYO and M. HASHIGUCHI [4] considered a Randers space, the present author [11] a Kropina space. Further the present author [12] treated of a Finsler space with 1-form metric. It is worthy of note that they considered the conformal flatness based on conformally invariant Finsler connections.

Recently we had S. KIKUCHI's epoch-making paper [9] on the conformal flatness. He found a conformally invariant Finsler connection for all Finsler spaces satisfying a certain condition by excellent idea and an conformal flatness was stated in terms of this connection. Though F. IKEDA's

papers [5], [6] seem to be written as explanatory notes to Kikuchi's paper, it rather brought a week point of Kikuchi's into the open.

The present paper deals with conformal flatness based on an idea similar to Kikuchi's. We introduce a conformally invariant Finsler connection for Finsler spaces having a tensor which satisfies conditions weaker than Kikuchi's, and the condition for a Finsler space to be conformal to a Berwald space or a locally Minkowski space is stated in terms of this connection. These statments are quite similar to those for a Finsler space to be a Bewald space or a locally Minkowski space stated in terms of the Berwald connection. It cannot be too much emphasized that our theory and theorems are independent of the dimension number.

2. B-contracting tensor

We consider a Finsler space $F^n = (M, L)$ of dimension n on an underlying manifold M and a conformal change of metric $L \to {}^*L = e^{c(x)}L$. Then we have the conformally changed space ${}^*F^n = (M, {}^*L)$ on the same manifold M. We have

$$ {}^*g_{ij} = e^{2c}g_{ij}, \qquad {}^*\ell_i = e^c\ell_i, \qquad {}^*\ell^i = e^{-c}\ell^i. $$

Hence we get conformally invariant tensors [2]:

$$ B_{ij} = (2/L^2)(g_{ij} - 2\ell_i\ell_j), \qquad B^{ij} = (L^2/2)(g^{ij} - 2\ell^i\ell^j). $$

The matrix (B^{ij}) is the inverse of (B_{ij}). Then we get a series of conformally invariant tensors as follows:

$$ B^{ij}{}_k = \dot{\partial}_k B^{ij}, \qquad B^{ij}{}_{k\ell} = \dot{\partial}_\ell B^{ij}{}_k, \ \ldots . $$

Now we treat of the well-known quantities

$$ 2G^i = \gamma_j{}^i{}_k(x,y)y^jy^k = g^{ij}\{y^r\partial_r\dot{\partial}_j(L^2/2) - \partial_j(L^2/2)\}, $$

where $\gamma_j{}^i{}_k(x,y)$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i. By the conformal change above we obtain

$$ {}^*\gamma_j{}^i{}_k = \gamma_j{}^i{}_k + \delta_j{}^i c_k + \delta_k{}^i c_j - g_{jk}c^i, $$

where $c_k = \partial_k c(x)$ and $c^i = g^{ik} c_k$. Hence we have $^*G^i$ of $^*F^n$ as

$$^*G^h = G^h - B^{hr} c_r.$$

We shall be concerned with the Berwald connection $B\Gamma = (G_i{}^h{}_j, G^h{}_i)$. Then the above gives rise to

$$(2.1) \qquad ^*G^h{}_i = G^h{}_i - B^{hr}{}_i c_r, \qquad ^*G_i{}^h{}_j = G_i{}^h{}_j - B^{hr}{}_{ij} c_r,$$

and the relation of the hv-curvature tensor $G = (G_i{}^h{}_{jk})$ as

$$(2.2) \qquad ^*G_i{}^h{}_{jk} = G_i{}^h{}_{jk} - B^{hr}{}_{ijk} c_r.$$

We have the well-known relations

$$^*C_{ijk} = e^{2c} C_{ijk}, \qquad ^*C_i{}^j{}_k = C_i{}^j{}_k,$$

$$^*C^{ijk} = e^{-4c} C^{ijk}, \qquad ^*C_i = C_i, \qquad ^*C^i = e^{-2c} C^i.$$

As a consequence we get conformally invariant tensors as follows:

$$(2.3) \qquad L^4 C^{ijk}, \quad L^4 g^{ij} C^k, \quad L^4 g^{ij} \ell^k.$$

It is noted that the third tensor of (2.3) satisfies $B^{hr}{}_{ijk}(L^4 g^{ij} \ell^k) = 0$ from $B^{hr}{}_{ijk} y^k = 0$, because $B^{hr}(x,y)$ are positively homogeneous in (y^i) of degree two.

Definition 2.1. A tensor field S of $(3,0)$-type of a Finsler space is called a *B-contracting tensor*, if

(a) S is conformally invariant,

(b) $B^{hr}{}_{ijk} S^{ijk} = \beta^{hr}$ is non-singular, that is, $\det(\beta^{hr}) \neq 0$. Let (Φ_{hr}) be the inverse of (β^{hr}).

S. KIKUCHI's idea [9] is to write $c_i(x)$ as the difference between a tensor of F^n and the tensor of $^*F^n$ which corresponds to it by conformal change. Here we shall solve $c_i(x)$ from (2.2) to realize his idea. Multiplying a B-contracting tensor S, (2.2) yields

$$(2.4) \qquad c_i = \phi_i - {}^*\phi_i, \qquad \phi_i = \Phi_{ir} G_j{}^r{}_{hk} S^{jhk}.$$

As M. HASHIGUCHI gave in the paper [2], we have

$$B^{ir}{}_j = y_j g^{ir} - \delta_j{}^i y^r - \delta_j{}^r y^i - L^2 C^{ir}{}_j,$$

$$B^{ir}{}_{jk} = g^{ir} g_{jk} - \mathfrak{S}_{(jk)}\{\delta_j{}^i \delta_k{}^r + 2C^{ir}{}_j y_k\} + L^2 C^{ir}{}_{j.k},$$

where $\mathfrak{S}_{(jk)}$ denotes the interchange of j, k and summation, and $(.k) = \dot{\partial}_k = \partial/\partial y^k$. Further we have

$$B^{ir}{}_{jkh} = 2g^{ir} C_{jkh} - 2\mathfrak{S}_{(jkh)}\{g_{jk} C^{ir}{}_h + y_j C^{ir}{}_{k.h}\} - L^2 C^{ir}{}_{j.k.h},$$

where $\mathfrak{S}_{(jkh)}$ denotes the cyclic interchange of j, k, h and summation.

Let us consider the two-dimensional case in detail. Then we can refer to the Berwald orthonormal frame field $(1, m)$ [1]. First we have

$$L\ell^i{}_{.j} = \varepsilon m^i m_j, \qquad\qquad L\ell_{i.j} = \varepsilon m_i m_j,$$

$$Lm^i{}_{.j} = -(\ell^i + \varepsilon I m^i)m_j, \qquad Lm_{i.j} = -(\ell_i - \varepsilon I m_i)m_j,$$

where I is the main scalar defined by $LC_{ijk} = Im_i m_j m_k$. By long but direct calculations we obtain

$$(2.5) \qquad (a) \qquad LB^{ir}{}_{jkh} = D^{ir} m_j m_k m_h,$$

$$(b) \qquad D^{ir} = 2J(\ell^i m^r + \ell^r m^i) - (J_{;2} - 2\varepsilon I J)m^i m^r,$$

where $K_{;2} = \varepsilon L K_{.i} m^i$ for a scalar K and $J = I_{;2}$. It is easy to show $\det(D^{ir}) = -4J^2$. Consequently, if we take a conformally invariant tensor S of $(3, 0)$-type having the $m^j m^k m^h$-component s, then $B^{ir}{}_{jhk} S^{jhk} = \beta^{ir} = \varepsilon s D^{ir}/L$. Therefore we have

Proposition 2.1. *In a two-dimensional Finsler space with non-zero $I_{;2}$, a conformally invariant tensor S of $(3,0)$-type is B-contracting if and only if S has a surviving $m^j m^k m^h$-component.*

We have $I_{;2} = 0$ if and only if I is a function of position alone and the fundamental function L of such a space is well-known ([1], Theorem 3.5.3.2). Both $L^4 C^{ijk}$ and $L^4 g^{ij} C^k$ of (2.3) are B-contracting, provided that the space is not a Riemannian space.

Substituting from (2.4) into (2.1) we obtain the conformally invariant connection ${}^cB\Gamma = ({}^cG_i{}^h{}_j, {}^cG^h{}_i)$, where

$$(2.6) \qquad {}^cG^h{}_i = G^h{}_i - B^{hr}{}_i \phi_r, \quad {}^cG_i{}^h{}_j = G_i{}^h{}_j - B^{hr}{}_{ij} \phi_r.$$

Definition 2.2. The conformally invariant connection $^cB\Gamma$ defined by (2.6) is called the *HMO-connection* with respect to the *B*-contracting tensor S.

Remark. The name HMO results from the initals of Prof. S. Hojo, Prof. K. Okubo and the present author who are members of the study group on Finsler geometry at Doshisha University.

From (2.2) we obtain a conformally invariant tensor

(2.7) $$^cG_i{}^h{}_{jk} = G_i{}^h{}_{jk} - B^{hr}{}_{ijk}\phi_r.$$

Since the *hv*-curvature tensor of the HMO-connection $^cB\Gamma$ is given by $\dot{\partial}_k{}^cG_i{}^h{}_j$, (2.6) shows that if ϕ_i depends on position alone, then $^cG_i{}^h{}_{jk}$ is nothing but the *hv*-curvature tensor of $^cB\Gamma$.

We shall return to the discussion of the two-dimensional case. From $\beta^{ir} = \varepsilon sD^{ir}/L$ and (2.5) it follows that the inverse Φ_{ir} is given by

$$\Phi_{ir} = (\varepsilon L/4sJ^2)(J_{;2} - 2\varepsilon IJ)\ell_i\ell_r + (L/2sJ)(\ell_i m_r + \ell_r m_i).$$

On the other hand we have the formula ([1], (3.5.2.7))

(2.8) $$LG_j{}^r{}_{hk} = \{-2I_{,1}\ell^r + (I_{,1;2} + I_{,2})m^r\}m_j m_h m_k,$$

where we put

$$K_{;i} = K_{,1}\ell_i + K_{,2}m_i,$$

for the *h*-covariant derivative $K_{;i}$ of a scalar K with respect to $B\Gamma$. We have the Ricci identities

(2.9)
$$K_{,1;2} - K_{;2,1} = K_{,2},$$
$$K_{,2;2} - K_{;2,2} = -\varepsilon(K_{,1} + IK_{,2} + I_{,1}K_{;2}).$$

Hence we have $I_{,1;2} + I_{,2} = J_{,1} + 2I_{,2}$ and consequently

$$LG_j{}^r{}_{hk} = \{-2I_{,1}\ell^r + (J_{,1} + 2I_{,2})m^r\}m_j m_h m_k.$$

Therefore we obtain $\phi_i = \Phi_{ir}G_j{}^r{}_{hk}S^{jhk}$ in the form

(2.10)
$$\phi_i = (1/2J^2)\{K_1 + 2J(I_{,2} + \varepsilon II_{,1})\}\ell_i - (\varepsilon I_{,1}/J)m_i,$$
$$K_1 = JJ_{,1} - J_{;2}I_{,1}.$$

We have to pay attention to (2.8). The scalar s does not appear in it.

Therefore we have

Proposition 2.2. *In a two-dimensional Finsler space with non-zero* $I_{;2}(= J)$, ϕ_i *defined by* (2.4) *is uniquely determined by* (2.7), *independent of the choice of a B-contracting tensor S.*

3. Conformal flatness

We have had a conformally invariant Finsler connection $^cB\Gamma$, called the HMO-connection. Now we are concerned with the conformal flatness based on this connection.

Definition 3.1. A Finsler space $F^n = (M, L)$ is called *conformally Berwald*, if for any point p of M there exist a local coordinate neighbourhood (U, x) containing p and a function $c(x)$ on U such that $^*L = e^c L$ is a metric of a Berwald space.

Assume that the conformally changed space $^*F^n = (M, {}^*L)$ of the last section is a Berwald space, namely, $^*G_i{}^h{}_{jk} = 0$. Then (2.4) gives $^*\phi_i = 0$ and $c_i = \phi_i$. Thus $\phi_i = \phi_i(x)$ must be a gradient vector. Since (2.7) leads to $^cG_i{}^h{}_{jk} = 0$ in $^*F^n$, we have $^cG_i{}^h{}_{jk} = 0$ in F^n. In this case $^cG_i{}^h{}_{jk}$ is the hv-curvature tensor of $^cB\Gamma$.

Conversely, we consider F^n such that $\phi_i = \phi_i(x)$ is a gradient vector and the hv-curvature tensor $^cG_i{}^h{}_{jk}$ of $B\Gamma$ vanishes. Since we have a function $c(x)$ satisfying $\partial_i c = \phi_i$, we apply to F^n the conformal change $L \to {}^*L = e^c L$ and get the conformally changed space $^*F^n = (M, {}^*L)$. Then we have (2.4) which gives $^*\phi_i = 0$ and (2.7) leads to $^*G_i{}^h{}_{jk} = 0$, that is, $^*F^n$ is a Berwald space.

Consequently we have

Theorem 3.1. *Suppose that a Finsler space* $F^n = (M, L)$ *has a B-contracting tensor S and let* $^cB\Gamma$ *be the HMO-connection with respect to S. Then* F^n *is conformally Berwald, if and only if*

(1) ϕ_i, *defined by* (2.4), *is a gradient vector* $\phi_i(x)$,

(2) *the* hv-*curvature tensor* ^cG *of* $^cB\Gamma$ *vanishes, that is, the* hv-*curvature tensor G of the Berwald connection BΓ is of the form*
$$G_i{}^h{}_{jk} = B^{hr}{}_{ijk}\phi_r.$$

Next we are concerned with the conformal flatness. Similarly to the "conformally Berwald" case, we can state as follows:

Definition 3.2. A Finsler space $F^n = (M, L)$ is called *conformally flat*, if for any point p of M there exist a local coordinate neighbourhood (U, x) containing p and a function $c(x)$ on U such that $^*L = e^c L$ is a locally Minkowski metric.

Let the conformally changed space $^*F^n = (M, {}^*L)$ be locally Minkowski. Since $^*F^n$ is a Berwald space with the vanishing h-curvature tensor H of $B\Gamma$ [1], all the facts mentioned in Theorem 3.1 and its proof hold. Further we can refer to an adapted coordinate system (x^i) of $^*F^n$ in which $^*G_i{}^h{}_j = 0$ [1]. Hence (2.6) and $^*\phi_i = 0$ give $^cG_i{}^h{}_j = 0$. Therefore the h-curvature tensor ^cH of $^cB\Gamma$ vanishes.

Conversely, we consider F^n such that there exists a gradient vector $\phi_i(x) = \partial_i c(x)$ and $^cG = {}^cH = 0$. Then $^cG_i{}^h{}_j$ are functions of position alone and hence ^cH is of the form

$$^cH_i{}^h{}_{jk} = \widetilde{A}_{[jk]}\{\partial_k({}^cG_i{}^h{}_j) + ({}^cG_i{}^r{}_j)({}^cG_r{}^h{}_k)\} = 0,$$

where $\widetilde{A}_{[jk]}$ denotes the interchange of j, k and subtraction. Therefore we have a coordinate system (x^i) in which $^cG_i{}^h{}_j = 0$ identically. Then (2.6) with $^*\phi_i = 0$ yields $^*G_i{}^h{}_j = 0$ and hence $^*H = 0$. Thus $^*F^n$ becomes a locally Minkowski space.

Consequently we can state

Theorem 3.2. *Let a Finsler space $F^n = (M, L)$ have a B-contracting tensor S and let $^cB\Gamma$ be the HMO-connection with respect to S. Then F^n is conformally flat, if and only if*

(1) *ϕ_i defined by (2.4) is a gradient vector $\phi_i(x)$,*

(2) *the h and hv-curvature tensors ^cH, ^cG of $^cB\Gamma$ vanish.*

4. The two-dimensional case

In the two-dimensional case we have had a detailed discussion of conformal flatness by the present author [13]. At that time he expected some special situations of conformal flatness of Finsler spaces in the two-dimensional case, because any Riemannian space of dimension two is locally conformal to a flat space.

But we had two theorems independent of the dimension number. Now we apply these theorems to the two-dimensional case and compare them with the results given in the paper [13].

First we deal with the condition (1) of these theorems: ϕ_i is a gradient vector $\phi_i(x)$. Putting

$$\phi_i = \phi_1 \ell_i + \phi_2 m_i,$$

in the Berwald frame (ℓ, m), with respect to the Cartan connection $C\Gamma$ we have

$$\phi_{i|j} = \ell_i(\phi_{1,1}\ell_j + \phi_{1,2}m_j) + m_i(\phi_{2,1}\ell_j + \phi_{2,2}m_j).$$

Here we use the symbols

$$\phi_{1|j} = \partial_j \phi_1 - (\dot{\partial}_r \phi_1)G^r{}_j = \phi_{1,1}\ell_j + \phi_{1,2}m_j.$$

Thus $\phi_{1,1}, \ldots, \phi_{2,2}$ depend on the nonlinear connection $G^i{}_j$ only.

Next we have

$$L\phi_{i.j} = \phi_{1;2}m_j\ell_i + \phi_1 \varepsilon m_i m_j + \phi_{2;2}m_j m_i - \phi_2(\ell_i - \varepsilon I m_i)m_j$$
$$= (\phi_{1;2} - \phi_2)\ell_i m_j + \{\phi_{2;2} + \varepsilon(\phi_1 + I\phi_2)\}m_i m_j.$$

Consequently the condition (1) can be written as

(4.1)
$$\text{(a)} \quad \phi_{1,2} - \phi_{2,1} = 0,$$
$$\text{(b)} \quad \phi_{1;2} = \phi_2, \qquad \text{(c)} \quad \phi_{2;2} = -\varepsilon(\phi_1 + I\phi_2).$$

We have (2.10):

$$\phi_1 = (1/2J^2)\{K_1 + 2J(I_{,2} + \varepsilon II_{,1})\}, \quad \phi_2 = -\varepsilon I_{,1}/J.$$

First we deal with (c) of (4.1):

$$\phi_{2;2} = -\varepsilon(I_{,1;2}/J - I_{,1}J_{;2}/J^2), \quad \phi_1 + I\phi_2 = K_1/2J^2 + I_{,2}/J.$$

We use $I_{,1;2} = J_{,1} + I_{,2}$ from the first part of (2.9). Then (c) is equivalent to $K_1 = 0$.

Secondly we consider (b). We use $I_{,2;2} = J_{,2} - \varepsilon(I_{,1} + II_{,2} + I_{,1}J)$ from the second part of (2.9). Then (b) is written as

$$JJ_{,2} + \varepsilon IJJ_{,1} - (I_{,2} + \varepsilon II_{,1})J_{;2} = 0.$$

Substituting $J_{,1} = I_{,1}J_{;2}/J$ from $K_1 = 0$, this is reduced to $K_2 = JJ_{,2} - J_{;2}I_{,2} = 0$.

Finally we consider (a): This written as

$$J^2(\varepsilon I_{,1,1} + I_{,2,2} + \varepsilon II_{,1,2} + \varepsilon I_{,1}I_{,2}) = \varepsilon JJ_{,1}I_{,1} + JJ_{,2}(I_{,2} + \varepsilon II_{,1}).$$

Substituting $JJ_{,a} = J_{;2}I_{,a}$, $a = 1, 2$, from $K_a = 0$, this is rewritten in the form

$$(4.2) \quad J_{;2}\{(I_{,1})^2 + \varepsilon(I_{,2})^2 + II_{,1}I_{,2}\} = J^2(I_{,1,1} + \varepsilon I_{,2,2} + II_{,1,2} + I_{,1}I_{,2}).$$

Next we deal with ${}^cG_i{}^h{}_{jk} = G_i{}^h{}_{jk} - B^{hr}{}_{ijk}\phi_r = 0$. Now we have ϕ_i of the form

$$(4.3) \qquad\qquad \phi_1 = (I_{,2} + \varepsilon II_{,1})/J, \quad \phi_2 = -\varepsilon I_{,1}/J.$$

Since we have (2.8) and (2.5), the above can be written as

$$-2I_{,1}\ell^h + (J_{,1} + 2I_{,2})m^h = 2\varepsilon J\phi_2\ell^h + \{2J\phi_1 - \varepsilon\phi_2(J_{;2} - 2\varepsilon IJ)\}m^h.$$

It is easy to show that this is equivalent to $K_1 = 0$.

Consequently we have

Theorem 4.1. *A two-dimensional Finsler space F^2 with non-zero J $(= I_{;2})$ is conformally Berwald, if and only if*

$$K_a = JJ_{,a} - J_{;2}I_{,a} = 0, \qquad a = 1, 2,$$

and (4.2). Then the conformally changed space ${}^*F^2$ with ${}^*L = e^{c(x)}L$ is a Berwald space, where $c(x)$ is given as $\partial_i c(x) = \phi_i$ from ϕ_i of (4.3).

Further we consider conformal flatness. It is well-known that all Berwald spaces of dimension two are classified as

$$B_1 = \{I = \text{const., and } R \neq 0\}, \qquad B_2 = \{I = \text{const., and } R = 0\},$$

$$B_3 = \{I \neq \text{const., and } R = 0\}.$$

A space $F^2 \in B_1 + B_2$ has $I_{;2} = 0$ and I is a conformal invariant. Therefore we are not concerned with $B_1 + B_2$ in the present paper. Thus the ${}^*F^2$ occurring in Theorem 4.1 belongs to B_3, which is locally Minkowski.

Therefore we obtain

Theorem 4.2. *A Finsler space of dimension two with non-zero J ($= I_{;2}$) is conformally flat, if and only if the conditions stated in Theorem 4.1 are satisfied.*

Thus it can be concluded that Theorem 3 of the paper [13] is again established, because it just coincides with Theorem 4.2.

It is remarked that the conformal flatness of two-dimensional Finsler spaces is uniquely treated independent of the choice of a B-contracting tensor. But we may have some complicated situations in higher dimensions. It may be possible that a Finsler space F^n, $n \geq 3$, has plural B-contracting tensors and, as a consequence, we have plural Berwald spaces or locally Minkowski spaces which are conformal to F^n. Therefore we have

Problem I. Are there two Berwald spaces which are conformal to each other?

and

Problem II. Are there two locally Minkowski spaces which are conformal to each other?

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