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Measurable solutions of a functional equation related to (2,2)–additive entropies of degree α

By LÁSZLÓ LOSONCZI (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. In this paper we find the general measurable solutions of the functional equation

$$F(xy) + F(x(1-y)) + F((1-x)y) + F((1-x)(1-y)) = G(x)H(y) \quad (x, y \in]0, 1[)$$

where $F, G, H :]0, 1[\to \mathbf{C}$. We specialize this result to obtain the measurable solutions of the equation

$$\begin{aligned} g(xy) + g(x(1-y)) + g((1-x)y) + g((1-x)(1-y)) = \\ = (g(x) + g(1-x))(g(y) + g(1-y)) \quad (x, y \in]0, 1[) \end{aligned}$$

and through this we determine all measurable normalized entropies having the sum property and satisfying (2,2)–additivity of degree $\alpha \neq 1$. These entropies are: the entropy of degree α and certain linear combinations of entropies of degree 5,4,3,2,0,∞.

1. Introduction

Let

$$\Gamma_n^0 := \{X = (x_1, \ldots, x_n) \mid x_i > 0 \, (i = 1, \ldots, n); \ \sum_{i=1}^{n} x_i = 1\} \tag{1}$$

denote the set of all discrete probability distributions of length n having positive probabilities and let Γ_n be its closure in $\mathbf{R}^n$ ($\mathbf{R}^n$ is the n–

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dimensional euclidean space), i.e.

$$\Gamma_n := \{X = (x_1, \dots, x_n) \mid x_i \geq 0\,(i = 1, \dots, n);\ \sum_{i=1}^{n} x_i = 1\,\}.$$

Let further

$$I_n : \Gamma_n^0 \to \mathbf{R} \qquad (n = 2, \dots) \tag{2}$$

be a sequence of real functions. Such a sequence is called *measure of information* or *entropy*. The sequence (2) is said to have the *sum property* if there exists a function $f :]0,1[\to \mathbf{R}$ such that

$$I_n(X) = \sum_{i=1}^{n} f(x_i) \quad (X \in \Gamma_n^0;\ n = 2, 3, \dots). \tag{3}$$

f is called the *generating function* of the sequence (see [1]). I_n is called (k,ℓ)–additive if

$$I_{k\cdot\ell}(X * Y) = I_k(X) + I_\ell(Y) \qquad \big(X \in \Gamma_k^0,\, Y \in \Gamma_\ell^0\big) \tag{4}$$

holds where $X * Y := (x_1 y_1, \dots, x_k y_\ell) \in \Gamma_{k\cdot\ell}^0$, and $k, \ell \geq 2$ are fixed integers. Properties (3), (4) are satisfied by the Shannon entropy. A generalization of (4) is the (k,ℓ)–additivity of degree α. We say that I_n is (k,ℓ)*–additive of degree* α if

$$I_{k\cdot\ell}(X * Y) = I_k(X) + I_\ell(Y) + (2^{1-\alpha} - 1) I_k(X) I_\ell(Y) \tag{5}$$

holds for all $X \in \Gamma_k^0$, $Y \in \Gamma_\ell^0$ and for some $\alpha \in \mathbf{R}$. If $\alpha = 1$ then (5) reduces to (4). Properties (3), (5) are satisfied by the entropy of degree α. The sequence I_n is called *normalized* if

$$I_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1. \tag{6}$$

From (3), (5) we can see that a sequence (2) having the sum property is (k,ℓ)–additive of degree α if and only if its generating function satisfies the functional equation

$$\begin{aligned} &\sum_{i=1}^{k} \sum_{j=1}^{\ell} f(x_i y_j) = \\ &= \sum_{i=1}^{k} f(x_i) + \sum_{j=1}^{\ell} f(y_j) + \big(2^{1-\alpha} - 1\big) \sum_{i=1}^{k} f(x_i) \sum_{j=1}^{\ell} f(y_j) \end{aligned} \tag{7}$$

for all $X \in \Gamma_k^0$, $Y \in \Gamma_\ell^0$. It is normalized if and only if $f\left(\frac{1}{2}\right) = \frac{1}{2}$. Introducing the function $g(x) := x + \left(2^{1-\alpha} - 1\right) f(x)$ $(x \in]0,1[)$ and supposing $\alpha \neq 1$ we can rewrite (7) in the form

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} g(x_i y_j) = \sum_{i=1}^{k} g(x_i) \sum_{j=1}^{\ell} g(y_j) \quad \left(X \in \Gamma_k^0,\, Y \in \Gamma_\ell^0\right).$$

The general solution of this equation is known if $k, \ell \geq 3$ are fixed integers (Losonczi [5], see also Losonczi–Maksa [9] where the same equation is treated with $g : [0,1] \to \mathbf{R}$, $X \in \Gamma_k$, $Y \in \Gamma_\ell$). In the case $k \geq 3$, $\ell = 2$ we refer to Losonczi [4], Maksa [10]. If $k = \ell = 2$ this equation has the form

$$\begin{aligned} (8) \quad & g(xy) + g(x(1-y)) + g((1-x)y) + g((1-x)(1-y)) = \\ & \qquad = (g(x) + g(1-x))(g(y) + g(1-y)) \quad (x, y \in]0,1[). \end{aligned}$$

The solutions of this equation have been found by the author [6] on the "closed domain" i.e. if $g : [0,1] \to \mathbf{R}$, (8) holds for $x, y \in [0,1]$ and if g is in $C_3[0,1]$, the space of all complex valued functions three times continuously differentiable on $[0,1]$.

The aim of this paper is to present the general measurable solution of equation (8). We shall obtain the solutions of (8) from the more general equation

$$(9) \quad F(xy) + F(x(1-y)) + F((1-x)y) + F((1-x)(1-y)) = G(x)H(y)$$

where $(x, y \in]0,1[)$ and $F, G, H :]0,1[\to \mathbf{C}$ are unknown functions. The $C_3[0,1]$ solutions of this equation on the closed domain $[0,1]$ have been found in [8]. We shall apply the results to determine all normalized entropies having the sum property with measurable generating function and satisfying (2,2)–additivity of degree $\alpha(\neq 1)$.

In Section 2 we collect the auxiliary results we need and reduce (9) to a simpler equation having symmetric right hand side. This is followed by the solution of this simpler equation in Section 3 and by the solution of (9), (8) and the applications in Section 4.

2. Auxiliary results and the reduction of (9)

The determination of measurable solutions has been made possible by the following two theorems ($\mathbf{R}$, $\mathbf{C}$ and $\mathbf{N}$ denote the set of real, complex and natural numbers respectively).

Theorem L. (see Losonczi [7], theorem 3) *Suppose that the functions* $f_{ij}, g_t, h_t :]0,1[\to \mathbf{C}$ $(i, j = 1, 2;\ t = 1, \ldots, N)$

(i) *satisfy the functional equation*

$$f_{11}(xy) + f_{12}\left(x(1-y)\right) + f_{21}((1-x)y) + f_{22}\left((1-x)(1-y)\right) =$$
$$= \sum_{t=1}^{N} g_t(x)h_t(y) \qquad (x, y \in \,]0,1[\,)$$

(ii) $f_{11}, f_{12}, f_{21}, f_{22}$ *are measurable on* $]0,1[$

(iii) *the functions* $\{g_1, \dots, g_N\}$ *and* $\{h_1, \dots, h_N\}$ *are linearly independent on* $]0,1[$.

Then there exists distinct complex numbers $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4, \dots, \lambda_M$ *and natural numbers* $m_1, \dots, m_M$ *with* $\sum_{j=1}^{M} m_j \le 30N + 7$ *such that*

$$f_{ij} \in \mathcal{E}(\lambda_1, \dots, \lambda_M; m_1, \dots, m_M)$$

where $\mathcal{E}(\lambda_1, \dots, \lambda_M; m_1, \dots, m_M)$ *denotes the linear space of all functions*

$$x \to \sum_{j=1}^{M} \sum_{k=0}^{m_j - 1} c_{jk} x^{\lambda_j} \log^k x \quad (x \in \,]0,1[\,)$$

with c_{jk} *being arbitrary complex constants.*

Theorem EL. *(*see Ebanks and Losonczi [3]*) Let* $\lambda_1, \dots, \lambda_M$; $m_1, \dots, m_M$ *be the same as in* theorem L *and let*

$$A_{jk}(x) := x^{\lambda_j} \log^k x + (1-x)^{\lambda_j} \log^k (1-x) \quad (x \in \,]0,1[\,)$$
$$B_{jk}(x) := x^{\lambda_j} \log^k x - (1-x)^{\lambda_j} \log^k (1-x) \quad (x \in \,]0,1[\,).$$

for $j = 1, \dots, M; k = 0, \dots, m_j - 1$. *If*

$$\sum_{j=1}^{M} \sum_{k=0}^{m_j - 1} a_{jk} A_{jk}(x) = 0 \quad (x \in \,]0,1[\,)$$

or

$$\sum_{j=1}^{M} \sum_{k=0}^{m_j - 1} a_{jk} B_{jk}(x) = 0 \quad (x \in \,]0,1[\,)$$

holds with some constants $a_{jk} \in \mathbf{C}$ *then for all* $j \in \{1, \dots, M\}$

$$a_{jk} = 0 \ (k = 0, \dots, m_j - 1) \quad \textit{if} \quad \lambda_j \notin \mathbf{N} \cup \{0\},$$

and

$$a_{jk} = 0 \ (k = 1, \dots, m_j - 1) \quad \textit{if} \quad \lambda_j \in \mathbf{N} \cup \{0\} \text{ and } m_j > 1.$$

We shall also use the following theorem due to Daróczy and Járai [2].

Theorem DJ. *Let* $f:]0,1[\to \mathbf{C}$ *be a measurable function satisfying the functional equation*

$$f(xy)+f(x(1-y))+f((1-x)y)+f((1-x)(1-y))=0 \quad (x,y\in]0,1[).$$

Then

$$f(x)=a_1\left(x-\frac{1}{4}\right) \quad (x\in]0,1[)$$

where $a_1\in\mathbf{C}$ *is an arbitrary constant.*

Let us *return to equation* (9). We may assume that G,H are not zero functions (otherwise the solutions could easily be obtained by theorem DJ). By the symmetry of the left hand side of (9) in the variables x,y we get $G(x)H(y)=G(y)H(x)$ hence $G(x)=\frac{G(y_0)}{H(y_0)}H(x)$ where $y_0\in]0,1[$ is such a value that $H(y_0)\neq 0$. The constant $G(y_0)/H(y_0)$ cannot be zero otherwise G would be the zero function. Thus, with the notation $p=H(y_0)/G(y_0)$ we have

$$G(x)=\frac{1}{p}H(x) \quad (x\in]0,1[). \tag{10}$$

Introducing the functions

$$f(x):=\frac{1}{p}F(x) \qquad h(x):=\frac{1}{p}H(x) \quad (x\in]0,1[) \tag{11}$$

we obtain from (9) that

$$f(xy)+f(x(1-y))+f((1-x)y)+f((1-x)(1-y))=h(x)h(y) \qquad (x,y\in]0,1[) \tag{12}$$

and $h\neq O$ (here and in what follows O denotes the zero function). Conversely, if $f,h\neq O$ satisfy (12) then the functions F,G,H obtained from (10), (11) satisfy (9) with any value $p\neq 0$ and $G\neq O, H\neq O$.

3. Measurable solutions of equation (12)

The measurable solutions of equation (12) are given by

Theorem 1. *Suppose that* $f,h:]0,1[\to\mathbf{C}$ *satisfy the functional equation*

$$f(xy)+f(x(1-y))+f((1-x)y)+f((1-x)(1-y))=h(x)h(y) \qquad (x,y\in\ ]0,1[\,) \tag{12}$$

the function f is measurable on $]0,1[$ and $h \neq O$. Then for all $x \in]0,1[$ we have

$$\begin{cases} f(x) = a_1\left(x - \frac{1}{4}\right) + cx^\lambda \\ h(x) = c^{1/2}[x^\lambda + (1-x)^\lambda] \end{cases} \tag{13}$$

where a_1, $c \neq 0$, λ are arbitrary complex constants, or

$$\begin{cases} f(x) = c_3x^3 + c_2x^2 + c_1\left(x - \frac{1}{4}\right) - \frac{1}{4}c_2c_3(4c_2+9c_3)^{-1} \\ h(x) = (4c_2+9c_3)^{-1/2}[2c_2\left(x^2+(1-x)^2\right) + 3c_3\left(x^3+(1-x)^3\right.] \end{cases} \tag{14}$$

where c_1, c_2, c_3 are arbitrary contants with $c_2c_3 \neq 0$, $4c_2 + 9c_3 \neq 0$, or

$$\begin{cases} f(x) = & d_5x^5 + d_4x^4 - \frac{1}{3}d_4d_5(4d_4+25d_5)^{-1}\times \\ & \times\left(40x^3 - 15x^2 + \frac{1}{2}\right) + d_1\left(x - \frac{1}{4}\right) \\ h(x) = & (4d_4+25d_5)^{-1/2}\left[2d_4\left(x^4+(1-x)^4\right) + \right. \\ & \left. +5d_5\left(x^5+(1-x)^5\right)\right] \end{cases} \tag{15}$$

where d_1, d_4, d_5 are arbitrary complex constants with $d_4d_5 \neq 0$, $4d_4 + 5d_5 \neq 0$.

Conversely the functions (13), (14), (15) satisfy equation (12).

Proof. Theorem L is applicable for equation (12) hence there are distinct complex numbers $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2, \dots, \lambda_M$ and natural numbers $m_1, \dots, m_M$ (for which we have $\sum_{k=1}^{M} m_k \leq 37$) such that

$$f(x) = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} c_{jk}x^{\lambda_j}\log^k x \quad (x \in]0,1[\,) \tag{16}$$

holds with suitable constants $c_{jk} \in \mathbf{C}$. Let $(\mathcal{L}f)(x,y)$ denote the left hand side of equation (12). Using the representation (16), the binomial theorem and the identity

$$\sum_{k=0}^{m_j-1}\sum_{\ell=0}^{k} u_{k\ell} = \sum_{\ell=0}^{m_j-1}\sum_{k=\ell}^{m_j-1} u_{k\ell}$$

we obtain that

$$(\mathcal{L}f)(x,y) = \sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} A_{j\ell}(x) \left[\sum_{k=\ell}^{m_j-1} c_{jk} \binom{k}{\ell} A_{j,k-\ell}(y) \right] \tag{17}$$
$$(x, y \in \,]0,1[\,)$$

where $A_{j\ell}$ are the functions defined in theorem EL. Our aim is to show that all but a few coefficients c_{jk} are zero while the nonzero coefficients have to be of special forms. This will lead to the stated formulae (13)–(15).

In the proof we shall need some lemmas.

Lemma 1. *Let*

$$d_{jl} := \sum_{k=\ell}^{m_j-1} c_{jk} \binom{k}{\ell} A_{j,k-\ell}(y_0) \tag{18}$$
$$\left(j = 1, \dots, M;\ \ell = 0, \dots, m_j - 1;\ y_0 = \frac{1}{2} \right).$$

If for any fixed value $j \in \{1, \dots, M\}$

$$d_{j\ell} = 0 \quad \text{for } \ell = \ell', \ell' + 1, \dots, m_j - 1 \tag{19}$$

where $\ell' = 0$ or $\ell' = 1$ (and $m_j \geq 2$) then we have

$$c_{j\ell} = 0 \quad \text{for } \ell = \ell', \ell' + 1, \dots, m_j - 1. \tag{20}$$

Proof of lemma 1. Assume first that $\ell' = 0$. Using (18) we can write (19) as

$$\begin{aligned}
d_{j0} = 0 = \\
= c_{j0}\binom{0}{0}A_{j0}(y_0) + c_{j1}\binom{1}{0}A_{j1}(y_0) + \dots + c_{j,m_j-1}\binom{m_j-1}{0}A_{j,m_j-1}(y_0)
\end{aligned}$$

$$\begin{aligned}
d_{j1} = 0 = \\
= \qquad c_{j1}\binom{1}{1}A_{j0}(y_0) + \dots + c_{j,m_j-1}\binom{m_j-1}{1}A_{j,m_j-2}(y_0) \\
\vdots \\
d_{j,m_j-1} = 0 = \\
= \qquad c_{j,m_j-1}\binom{m_j-1}{m_j-1}A_{j,0}(y_0).
\end{aligned}$$

This is a linear homogeneous algebraic system of equations for the unknowns $c_{j0}, c_{j1}, \dots, c_{j,m_j-1}$. The determinant of this system is

$$\left(A_{j0}(y_0) \right)^{m_j} = 2^{(1-\lambda_j)m_j} \neq 0.$$

Therefore our system has only trivial solutions proving (20). If $\ell' = 1$ the proof is similar. □

Lemma 2. *Suppose that $f, h =]0,1[\to \mathbf{C}$ satisfy (12) and f is measurable on $]0,1[$ and $h \neq O$. If $h\left(\frac{1}{2}\right) = 0$ then f is a polynomial.*

Proof of lemma 2. Using (16) we can write (12) in the form

$$(\mathcal{L}f)(x,y) = h(x)h(y) \qquad (x, y \in]0,1[) \tag{21}$$

where $\mathcal{L}f$ is given by (17). Substituting $y = y_0 = \dfrac{1}{2}$ here we obtain that

$$(\mathcal{L}f)(x,y_0) = \sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} A_{j\ell}(x) \left[\sum_{k=\ell}^{m_j-1} c_{jk} \binom{k}{\ell} A_{j,k-\ell}(y_0) \right] \quad (x \in]0,1[)$$

or by (18)

$$\sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} d_{j\ell} A_{j\ell}(x) = 0 \qquad (x \in]0,1[). \tag{22}$$

Theorem EL and lemma 1 implies that if $\lambda_j \notin \mathbf{N} \cup \{0\}$ then $d_{j\ell} = 0$ for $\ell = 0, \dots, m_j - 1$ hence $c_{j\ell} = 0$ for $\ell = 0, \dots, m_j - 1$. If $\lambda_j \in \mathbf{N} \cup \{0\}$ and $m_j \geq 2$ then $d_{j\ell} = 0$ for $\ell = 1, \dots, m_j - 1$ hence $c_{j\ell} = 0$ for $\ell = 1, \dots, m_j - 1$.

Thus the only possible nonzero coefficients c_{jk} in (16) are exactly those c_{j0} where j is such that $\lambda_j \in \mathbf{N} \cup \{0\}$. This means exactly that f is a polynomial. □

Lemma 3. *Suppose that $f, h :]0,1[\to \mathbf{C}$ satisfy (12) f is measurable on $]0,1[$ and $h \neq O$. If $h\left(\frac{1}{2}\right) \neq 0$ then*

$$f(x) = cx^{\lambda} + P(x) \tag{23}$$

where $c, \lambda \in \mathbf{C}$ are constants $\lambda \notin \mathbf{N} \cup \{0\}$ and P is a polynomial.

Proof of lemma 3. From (21) with $y = \frac{1}{2}$ we obtain that

$$h(x) = \alpha \sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} d_{j\ell} A_{j\ell}(x) \quad (x \in]0,1[), \tag{24}$$

where $\alpha = h(\frac{1}{2})^{-1}$ and $d_{j\ell}$ are the same as in (18). The substitution of (24), (17) into (21) leads to the equation

$$\sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} D_{j\ell}(y)A_{j\ell}(x) = 0 \quad (x, y \in \,]0,1[\,) \tag{25}$$

where for $y \in \,]0,1[$

$$D_{j\ell}(y) := \sum_{k=\ell}^{m_j-1} c_{j\ell}\binom{k}{\ell} A_{j,k-\ell}(y) - \alpha^2 d_{j\ell} \sum_{p=1}^{M} \sum_{q=0}^{m_j-1} d_{pq}A_{pq}(y). \tag{26}$$

By theorem EL (25) implies that

$$D_{j\ell}(y) = 0 \quad (\ell = 1, \dots, m_j - 1;\ y \in \,]0,1[\,) \tag{27}$$

for all $j = 1, \dots, M$ with $m_j \geq 2$.

First we show that (27) *holds if and only if*

$$c_{j\ell} = 0 \quad (\ell = 1, \dots, m_j - 1) \tag{28}$$

for all $j = 1, \dots, M$ *with* $m_j \geq 2$. By (26) and (18) it is clear that (28) implies (27). We show that (28) is the consequence of (27).

Let $\ell = m_j - 1 > 0$ in (27). Since by (18)

$$d_{j,m_j-1} = c_{j,m_j-1}A_{j0}(y_0),$$

we can rewrite (27) as

$$\begin{aligned} D_{j,m_j-1}(y) =& c_{j,m_j-1} \Big[\Big(1 - \alpha^2 A_{j0}(y_0) d_{j0} \Big) A_{j0}(y) - \\ & - \alpha^2 A_{j0}(y_0) \sum\nolimits^* d_{pq}A_{pq}(y) \Big] = 0, \end{aligned} \tag{29}$$

where $\sum^*$ denotes summation for all $p = 1, \dots, M$; $q = 0, \dots, m_p - 1$ except for $p = j$ and $q = 0$.

We claim that

$$c_{j,m_j-1} = 0. \tag{30}$$

Suppose this is not true. Then we may divide (29) by it and conclude that the function in the bracket in (29) is zero for $y \in]0,1[$. By theorem EL this implies that

$$d_{jq} = 0 \quad (q = 1, \dots, m_j - 1) \tag{31}$$

hence by lemma 1 we have

$$c_{jq} = 0 \quad (q = 1, \dots, m_j - 1), \tag{32}$$

in particular $c_{j,m_j-1} = 0$ which contradicts to our assumption, proving (30).

Supose now that we have proved

$$c_{j,m_j-1} = c_{j,m_j-2} = \ldots = c_{j,\ell'} = 0 \qquad (\ell' \geq 2), \tag{33}$$

then we show that

$$c_{j,\ell'-1} = 0 \tag{34}$$

holds too.

By (18), (33) we have

$$d_{j,\ell'-1} = c_{j,\ell'-1} A_{j0}(y_0),$$

thus by (27) for $y \in \,]0,1[$

$$D_{j,\ell'-1}(y) = c_{j,\ell'-1} \Big[\Big(1 - \alpha^2 A_{j0}(y_0) d_{j0} \Big) A_{j0}(y) - \\ - \alpha^2 A_{j0}(y_0) \sum\nolimits^* d_{pq} A_{pq}(y) \Big] = 0.$$

If (34) were not true then the expression in the bracket would vanish on $]0,1[$. Repeating the arguments above we get (31), (32) and in particular $c_{j,\ell'-1} = 0$ which is a contradiction proving (34).

Letting ℓ' run over $m_j - 1, m_j - 2, \ldots, 2$ we obtain (28).

Due to (27) we can write (25) in the form

$$\sum_{j=1}^{M} D_{j0}(y) A_{j0}(x) = 0$$

or by (26), (28), (18)

$$\sum_{j=1}^{M} c_{j0} \left[A_{j0}(y) - \alpha^2 A_{j0}(y_0) \sum_{p=1}^{M} c_{p0} A_{p0}(y_0) A_{p0}(y) \right] A_{j0}(x) = 0. \tag{35}$$

We claim that there is at most one nonzero coefficient c_{j0} *for which* $\lambda_j \notin \mathbf{N} \cup \{0\}$.

(i) *If we have* $c_{j0} = 0$ for all j with $\lambda_j \notin \mathbf{N} \cup \{0\}$ then f is a polynomial i.e. (23) holds with $c = 0$.

(ii) *If there is one such nonzero coefficient*, say $c_{10} \neq 0$, $\lambda_1 \notin \mathbf{N} \cup \{0\}$, then from (35) by theorem EL it follows that the bracket following c_{10} in (35) should vanish for all $y \in \,]0,1[$ i.e.

$$\Big(1 - \alpha^2 A_{10}^2(y_0) c_{10} \Big) A_{10}(y) - \alpha^2 A_{10}(y_0) \sum_{p=2}^{M} c_{p0} A_{p0}(y_0) A_{p0}(y) = 0 \\ (y \in \,]0,1[\,).$$

By theorem EL

$$1-\alpha^2 A_{10}^2(y_0)c_{10}=0,$$

and

$$\alpha^2 A_{10}(y_0)A_{p0}(y_0)c_{p0}=0, \quad \text{from which} \quad c_{p0}=0$$

for those subscripts $p\in\{2,\dots,M\}$ for which $\lambda_p\notin \mathbf{N}\cup\{0\}$. Thus

$$f(x)=c_{10}x^{\lambda_1}+P(x),$$

where P is a polynomial. With $\lambda=\lambda_1$, $c=c_{10}$ (23) holds again, completing the proof of lemma 3. $\square$

Lemma 4. *Let*

$$f(x):=cx^\lambda+P(x) \qquad (x\in\,]0,1[\,), \tag{36}$$

where $c\neq 0$, $\lambda\notin \mathbf{N}\cup\{0\}$ are complex constants and P is a polynomial. In order to exists a function $h:\,]0,1[\,\to\mathbf{C}$ $h\neq O$ such that f,h is a solution of (12) *it is necessary and sufficient that $P(x)=a\left(x-\frac14\right)$ i.e.*

$$f(x)=cx^\lambda+a\left(x-\frac14\right) \quad (x\in\,]0,1[\,) \tag{37}$$

where $a\in\mathbf{C}$ is a constant and in this case

$$h(x)=c^{\frac12}\left(x^\lambda+(1-x)^\lambda\right) \qquad (x\in\,]0,1[\,). \tag{38}$$

Proof of lemma 4. If the function f given by (36) and $h\neq O$ is a solution of (12) then $h\left(\frac12\right)\neq 0$ otherwise lemma 2 would lead to $c=0$ which contradicts to our assumption.

Suppose that

$$P(x)=\sum_{k=0}^{n}a_kx^k \quad (x\in\,]0,1[\,;\ a_k\in\mathbf{C}). \tag{39}$$

(36) shows that

$$(\mathcal{L}f)(x,y)=cA_\lambda(x)A_\lambda(y)+\sum_{k=0}^{n}a_kA_k(x)A_k(y) \quad (x,y\in\,]0,1[\,) \tag{40}$$

where

$$A_t(x):=x^t+(1-x)^t \quad (x\in\,]0,1[\,;\ t\in\mathbf{C}).$$

From (12) with $y=y_0=\frac12$

$$h(x)=\alpha(\mathcal{L}f)(x,y_0)=\alpha\left(c\,2^{1-\lambda}A_\lambda(x)+\sum_{k=0}^{n}\frac{a_k}{2^{k-1}}A_k(x)\right) \tag{41}$$

where $\alpha = h\left(\frac{1}{2}\right)^{-1}$. Using (40), (41) the functional equation (12) can be written as

$$\begin{aligned} & cA_\lambda(x)\left[\left(1-\alpha^2 c\, 2^{2-2\lambda}\right)A_\lambda(y) - \alpha^2 2^{1-\lambda}\sum_{\ell=0}^{n}\frac{a_\ell}{2^{\ell-1}}A_\ell(y)\right] + \\ & + \sum_{k=0}^{n} A_k(x)\left[a_k A_k(y) - \alpha^2 c\, 2^{1-\lambda}\frac{a_k}{2^{k-1}}A_\lambda(y) - \right. \\ & \left. -\alpha^2 \frac{a_k}{2^{k-1}}\sum_{\ell=0}^{n}\frac{a_\ell}{2^{\ell-1}}A_\ell(y)\right] = 0. \end{aligned} \tag{42}$$

Since $\lambda \notin \mathbf{N} \cup \{0\}$ theorem EL gives that the coefficient of $A_\lambda(x)$ is the zero function i.e.

$$\left(1-\alpha^2 c\, 2^{2-2\lambda}\right)A_\lambda(y) - \alpha^2 2^{1-\lambda}\sum_{\ell=0}^{n}\frac{a_\ell}{2^{\ell-1}}A_\ell(y) = 0 \quad (y \in\,]0,1[\,). \tag{43}$$

Applying theorem EL again we obtain from (43) that

$$1-\alpha^2 c\, 2^{2-2\lambda} = 0, \tag{44}$$

thus $\alpha = 2^{\lambda-1}c^{-\frac{1}{2}}$ and

$$\sum_{\ell=0}^{n}\frac{a_\ell}{2^{\ell-1}}A_\ell(y) = 0 \qquad (y \in\,]0,1[\,). \tag{45}$$

By (44), (45) equation (42) is simplified to

$$\sum_{k=0}^{n} a_k A_k(x) A_k(y) = 0 \qquad (x,y \in\,]0,1[\,)$$

or to

$$(\mathcal{L}P)(x,y) = 0 \qquad (x,y \in\,]0,1[\,).$$

This means that P is a (measurable) solution of the "homogeneous" functional equation

$$(\mathcal{L}f)(x,y) = f(xy) + f(x(1-y)) + f((1-x)y) + f((1-x)(1-y)) = 0.$$

Hence by theorem DJ

$$P(x) = a_1\left(x - \frac{1}{4}\right)$$

that is $a_k = 0$ for $k \ge 2$; $a_0 = -\dfrac{a_1}{4}$, $a_1 \in \mathbf{C}$ arbitrary. This proves (37). With these coefficients a_k and α calculated from (44) the formula (41) goes over into $h(x) = c^{\frac{1}{2}} A_\lambda(x)$ which is exactly (38).

The functions (37), (38) satisfy equation (12) since (with the above values of the coefficients a_k and α) equation $(\mathcal{L}P)(x,y) = 0$ can be rewritten in the forms (42) or (21). $\square$

Let $f, h \neq O$ be solutions of equation (12) and f be measurable. By lemmas 2–4 either f, h are of the form (37), (38) with $\lambda \notin \mathbf{N} \cup 0$ or f is a polynomial. In the latter case h must be a polynomial too. Namely substituting a value $y_1 \in]0,1[$ with $h(y_1) \neq 0$ in (12) we obtain $h(x) = h(y_1)_{-1}\mathcal{L}f)(x, y_1)$.

If f is a polynomial then $(\mathcal{L}f)(x, y_1)$ and thus h too is a polynomial whose degree is not greater than that of f.

One possibility to find the polynomial solutions of (12) is the application of a result of [8]. There those functions $f, h : [0,1] \in \mathbf{C}$ were determined which satisfy (12) for all $x, y \in [0,1]$ and $f \in C_3[0,1]$. If f, h are polynomials satisfying (12) for $x, y \in]0,1[$ then they satisfy (12) for all $x, y \in \mathbf{R}$ and also the condition $f \in C_3[0,1]$ is satisfied. Thus the polynomial solutions could be obtained from theorem 1 of [8]. However, that theorem rests on a series of lemmas which are extraneus to our situation. In our opinion it is desirable to find the polynomial solutions directly, giving some insight to the structure of such solutions.

The next two lemmas give the polynomial solutions of (12).

Lemma 5. *Suppose that* $f(x) = \sum\limits_{k=0}^{n} a_k x^k$ $(x \in]0,1[;\ a_n \neq 0)$, $h :]0,1[\to \mathbf{C}$, $h \neq O$. *If* f, h *is a solution of* (12) *then either* n *is even and*

$$
\begin{cases} f(x) = a_n x^n + a_1 \left(x - \dfrac{1}{4} \right) \\ h(x) = a_n^{1/2} \Big(x^n + (1-x)^n \Big) \qquad (x \in]0,1[;\ n > 1), \end{cases} \tag{46}
$$

or n *is odd* $a_{n-1} \neq 0$ *and necessarily* $n = 5$, $n = 3$ *or* $n = 1$.

Remark. It is clear that solution (46) is of the form (13) with $c = a_n$, $\lambda = n \in \mathbf{N}$.

Proof of lemma 5. As we have seen h is also a polynomial of the form

$$h(x) = \sum_{k=0}^{n} b_k x^k \qquad (x \in]0,1[;\ b_n \in \mathbf{C}).$$

By the Taylor formula

$$f\Big((1-x)y\Big) = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!} \left[(1-x)y - y\right]^k = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!} (-1)^k x^k y^k,$$

$$f\Big((1-x)(1-y)\Big) = \sum_{k=0}^{n} \frac{f^{(k)}(1-y)}{k!} \Big[\, (1-x)(1-y) - (1-y) \, \Big]^k =$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(1-y)}{k!} (-1)^k x^k (1-y)^k,$$

hence

$$(\mathcal{L}f)(x,y) =$$

$$= \sum_{k=0}^{n} x^k \left[a_k \big(y^k + (1-y)^k \big) + (-1)^k \left(\frac{f^{(k)}(y)}{k!} y^k + \frac{f^{(k)}(1-y)}{k!} (1-y)^k \right) \right].$$

We also have

$$h(x)h(y) = \sum_{k=0}^{n} x^k b_k h(y) \qquad (x, y \in \,]0,1[\,).$$

Comparing the coefficients of x^n in the equation

$$(\mathcal{L}f)(x,y) = h(x)h(y) \tag{47}$$

or in

$$\sum_{k=0}^{n} x^k \left[a_k \left(\, y^k + (1-y)^k \, \right) + (-1)^k \left(\frac{f^{(k)}(y)}{k!} y^k + \frac{f^{(k)}(1-y)}{k!} (1-y)^k \right) \right] =$$

$$= \sum_{k=0}^{n} x^k b_k h(y) \qquad (x, y \in \,]0,1[\,)$$

we obtain that

$$a_n \left(\, 1 + (-1)^n \, \right) \left(\, y^n + (1-y)^n \, \right) = b_n h(y) \quad (y \in \,]0,1[\,). \tag{48}$$

Case 1: n is even. Then from (48)

$$2a_n \left(\, y^n + (1-y)^n \, \right) = b_n h(y) \qquad (y \in \,]0,1[\,). \tag{49}$$

Comparing the coefficients of y^n here we get

$$4a_n = b_n^2, \tag{50}$$

thus from (49), (50)

$$h(y) = a_n^{1/2} \left(\, y^n + (1-y)^n \, \right).$$

With the notation $f_1(x) := f(x) - a_n x^n$ (47) can be written as

$$(\mathcal{L} f_1)(x,y) = 0$$

hence by theorem DJ $f_1(x) = a_1\left(x - \frac{1}{4}\right)$ and

$$f(x) = a_n x^n + a_1\left(x - \frac{1}{4}\right), \tag{51}$$

i.e (46) holds.

Case 2: n is odd. Then from (48) $b_n = 0$. Comparing the coefficients of x^{n-1} in (47) we get

$$\begin{aligned} na_n \Big(y^n + (1-y)^n \Big) + 2a_{n-1} \Big(y^{n-1} + (1-y)^{n-1} \Big) = \\ = b_{n-1} h(y) \qquad (y \in \,]0,1[\,). \end{aligned} \tag{52}$$

We claim that $b_{n-1} \neq 0$. If $n = 1$ then $b_1 = 0$ and $b_0 \neq 0$ otherwise h were the zero function O. Let $n > 1$ and suppose that $b_{n-1} = 0$. Then the left hand side of (52) vanishes for all $y \in \,]0,1[$. With $y = \frac{1}{2}$ and $y = \frac{1}{4}$ this gives a linear homogeneous system of equations for na_n and $2a_{n-1}$ whose determinant $\left(1 - 3^{n-1}\right) 2^{2-3n} \neq 0$ hence $na_n = 2a_{n-1} = 0$; $a_n = 0$ which is a contradiction.

Case 2.1: $a_{n-1} = 0$. Then the comparison of the coefficients of y^{n-1} in (52) gives that

$$n^2 a_n = b_{n-1}^2, \tag{53}$$

and thus from (52), (53)

$$h(y) = a_n^{1/2} \Big(y^n + (1-y)^n \Big).$$

Similarly to the previous case we obtain (51) and thus (46) holds again.

Case 2.2: $a_{n-1} \neq 0$. *We show that* $n = 5$, $n = 3$ *or* $n = 1$.

We prove that the assumption $n \geq 7$ (n odd!) leads to $a_n = 0$ which is a contradiction. Hence $n < 7$ and n is an odd natural number, giving the statement of Case 2.2.

Supposing $n \geq 7$ the comparison of the coefficients of x^{n-3} in (47) gives the equation

$$\begin{aligned} & a_{n-3}\left(y^{n-3} + (1-y)^{n-3}\right) + \\ & + (-1)^{n-3} \left(\frac{f^{(n-3)}(y)}{(n-3)!} y^{n-3} + \frac{f^{(n-3)}(1-y)}{(n-3)!} (1-y)^{n-3} \right) = \\ & = b_{n-3} h(y) \end{aligned} \tag{54}$$

Since

$$\frac{f^{(n-3)}(y)}{(n-3)!}y^{n-3} = a_{n-3}\binom{n-3}{0}y^{n-3} + a_{n-2}\binom{n-2}{1}y^{n-2} + \\ + a_{n-1}\binom{n-1}{2}y^{n-1} + a_n\binom{n}{3}y^n,$$

using (52) and the notation (39) we can write (54) as

$$\begin{aligned} &2a_{n-3}A_{n-3}(y) + a_{n-2}(n-2)A_{n-2}(y) + \\ (55) \qquad &+ a_{n-1}\binom{n-1}{2}A_{n-1}(y) + a_n\binom{n}{3}A_n(y) = \\ &= \frac{b_{n-3}}{b_{n-1}}\Big[\, n a_n A_n(y) + 2a_{n-1}A_{n-1}(y) \,\Big]. \end{aligned}$$

To express the ratio $\frac{b_{n-3}}{b_{n-1}}$ by the help of the coefficients a_k compare the coefficients of y^{n-1} and y^{n-3} in (52). We get

$$(56) \qquad n^2 a_n + 4a_{n-1} = b_{n-1}^2,$$

$$(57) \qquad n\binom{n}{3}a_n + \binom{n-1}{2}a_{n-1} = b_{n-1}b_{n-3},$$

therefore

$$\frac{b_{n-3}}{b_{n-1}} = \frac{n\binom{n}{3}a_n + \binom{n-1}{2}a_{n-1}}{n^2 a_n + 4a_{n-1}}.$$

Substituting this in (55) and rearranging the terms we can write (55) in the form

$$(58) \quad \alpha_n A_n(y) + \alpha_{n-1}A_{n-1}(y) + \alpha_{n-2}A_{n-2}(y) + \alpha_{n-3}A_{n-3}(y) = 0 \qquad (y \in\,]0,1[\,)$$

where

$$(59) \qquad \begin{cases} \alpha_n &= -2\binom{n}{3}a_n a_{n-1}, \\ \alpha_{n-1} &= \ n\binom{n}{3}a_n a_{n-1}, \\ \alpha_{n-2} &= \ (n-2)a_{n-2}\left(n^2 a_n + 4a_{n-1}\right), \\ \alpha_{n-3} &= \ 2a_{n-3}\left(n^2 a_n + 4a_{n-1}\right). \end{cases}$$

To complete the proof of lemma 5 we need

Proposition 1. *If $n \geq 7$ is an odd integer and*

$$\alpha_n A_n(y) + \alpha_{n-1}A_{n-1}(y) + \alpha_{n-2}A_{n-2}(y) + \alpha_{n-3}A_{n-3}(y) = 0 \qquad (y \in]0,1[\,) \tag{60}$$

holds with some constants α_n, α_{n-1}, α_{n-2}, $\alpha_{n-3} \in \mathbf{C}$ then

$$\alpha_n = \alpha_{n-1} = \alpha_{n-2} = \alpha_{n-3} = 0. \tag{61}$$

Proof of proposition 1. With $y = \frac{1}{2} - t$ we have by the binomial theorem

$$\begin{aligned} A_k(y) &= \left(\tfrac{1}{2} - t\right)^k + \left(\tfrac{1}{2} + t\right)^k = \\ &= \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{1+(-1)^\ell}{2^{k-\ell}} t^\ell = \sum_{\ell_1=0}^{[k/2]} \binom{k}{2\ell_1} 2^{2\ell_1+1-k}\, t^{2\ell_1}. \end{aligned} \tag{62}$$

This shows that the left hand side of (60) has only even powers of t where the coefficients of t^{n-1}, $2^{-2}t^{n-3}$, $2^{-4}t^{n-5}$, $2^{-6}t^{n-7}$ must vanish. This gives the following linear homogeneous system for the coefficients α_i ($i = n, n-1, n-2, n-3$):

$$C_n \left(\alpha_n, 2\alpha_{n-1}, 2^2\alpha_{n-2}, 2^3\alpha_{n-3}\right)^T = 0 \tag{63}$$

where

$$C_n = \begin{pmatrix} \binom{n}{1} & 2 & 0 & 0 \\ \binom{n}{3} & \binom{n-1}{2} & \binom{n-2}{1} & \binom{n-3}{0} \\ \binom{n}{5} & \binom{n-1}{4} & \binom{n-2}{3} & \binom{n-3}{2} \\ \binom{n}{7} & \binom{n-1}{6} & \binom{n-2}{5} & \binom{n-3}{4} \end{pmatrix}$$

and T denotes the transposition. A simple calculation shows that

$$\det C_n = 4725^{-1} n(n-1)(n-2)^2(n-3)^2(n-4)^2(n-5)(n-6) \neq 0, \tag{64}$$

hence (63) has only trivial solutions proving proposition 1. □

Returning to the proof of lemma 5 the application of proposition 1 for (58) with the coefficients (59) implies that

$$\alpha_n = -2\binom{n}{3} a_n a_{n-1} = 0$$

hence $a_{n-1} = 0$ which is a contradiction. □

Lemma 6. *Suppose that* $f(x) = \sum_{k=0}^{n} a_n x^k \quad (x \in]0,1[\,)$ *where* $n = 5, 3$ *or* 1 *and and* $a_n a_{n-1} \neq 0$. *If there is a* $h :]0,1[\to \mathbf{C}, h \neq O$ *such that* f, h *is a solution of* (12) *then*

in case $n = 5$ *the functions* f, h *are of the form* (15) *(where* $d_k = a_k$ *for* $k = 1, 4, 5$*),*

in case $n = 3$ *the functions* f, h *are of the form* (14) *(where* $c_k = a_n$ *for* $k = 1, 2, 3$*),*

in case $n = 1$ *the functions* f, h *are of the form* (13) *(where* $\lambda = 0,\ c = a_1 + 4a_0 \neq 0$*).*

Proof of lemma 6. We have

$$f(x) = \sum_{k=0}^{n} a_k x^k \quad (a_n a_{n-1} \neq 0) \tag{65}$$

and from (52), (56) we get that $n^2 a_n + 4a_{n-1} \neq 0$ and

$$h(x) = \big(n^2 a_n + 4a_{n-1}\big)^{-1/2} \big(n a_n A_n(x) + 2a_{n-1} A_{n-1}(x)\big). \tag{66}$$

Case 2.2.1: $n = 5$. Substituting the functions f, h given by (65), (66) with $n = 5$ in equation (12) and rearranging the terms we obtain that

$$\sum_{k=0}^{5} \beta_n A_n(x) = 0 \quad (x \in]0,1[\,) \tag{67}$$

where

$$\begin{cases} \beta_k := a_k A_k(y) \quad (k = 0, 1, 2, 3) \\ \beta_4 := \dfrac{-5a_4 a_5}{25a_5 + 4a_4} \big(2A_5(y) - 5A_4(y)\big) \\ \beta_5 := \dfrac{2a_4 a_5}{25a_5 + 4a_4} \big(2A_5(y) - 5A_4(y)\big). \end{cases} \tag{68}$$

To continue the proof we need

Proposition 2. *The equation*

$$\sum_{k=0}^{5} \beta_n A_n(x) = 0 \quad (x \in]0,1[\,) \tag{69}$$

holds with some constants $\beta_k \in \mathbf{C}$ $(k = 0, \dots, 5)$ *if and only if*

$$\begin{cases} \beta_3 = -4\beta_2 - 10(\beta_1 + 2\beta_0) \\ \beta_4 = \quad 5\beta_2 + 15(\beta_1 + 2\beta_0) \\ \beta_5 = -2\beta_2 - \quad 6(\beta_1 + 2\beta_0) \end{cases} \tag{70}$$

is satisfied. (For example $\beta_1+2\beta_0$, β_2 are arbitrary and $\beta_3, \beta_4, \beta_5$ are given by (70).*)*

Proof of proposition 2. We can see from (72) that the left hand side of (69) is an even polynomial of degree ≤ 4 of the variable $t = \frac{1}{2} - y$. Thus (69) holds if and only if the coefficients of t^4, $2^{-2}t^2$, $2^{-4}t^0$ are zero, i.e. if

$$\begin{aligned} 5\beta_5 + 2\beta_4 &= \quad 0 \\ 10\beta_5 + 12\beta_4 + 12\beta_3 &= -8\beta_2 \\ \beta_5 + 2\beta_4 + 4\beta_3 &= -8\beta_2 - 16(\beta_1 + 2\beta_0). \end{aligned}$$

It is easy to check that this system is equivalent to (70). $\square$

The next proposition is a special case of the previous one.

Proposition 3. *The equation*

$$\sum_{k=0}^{3} \gamma_k A_k(x) = 0 \quad (x \in]0,1[\,) \tag{71}$$

holds with some constants $\gamma_k \in \mathbf{C}$ $(k = 0, \dots, 3)$ if and only if

$$\begin{aligned} \gamma_2 &= -3(\gamma_1 + 2\gamma_0) \\ \gamma_3 &= \quad 2(\gamma_1 + 2\gamma_0). \end{aligned} \tag{72}$$

is satisfied.

Indeed with $\gamma_k = \beta_k \quad k = 0, \dots, 3, \quad \beta_4 = \beta_5 = 0$ (70) can be rewritten in the form (72). $\square$

Returning to the proof of lemma 6 we apply proposition 2 to the system (67) with coefficients (68) ($y \in]0,1[$ is thought to be fixed). Since

$$2A_5(y) - 5A_4(y) = -4A_3(y) + A_2(y) \quad (y \in \mathbf{R}),$$

we can write (70) in the form

$$\begin{aligned} &a_3A_3(y) = \\ &\qquad = -4a_2A_2(y) - 10\big(a_1A_1(y) + 2a_0A_0(y)\big) \end{aligned} \tag{73}$$

$$\begin{aligned} &\frac{-5a_4a_5}{25a_5 + 4a_4}\big(-4A_3(y) + A_2(y)\big) = \\ &\qquad = 5a_2A_2(y) + 15\big(a_1A_1(y) + 2a_0A_0(y)\big) \end{aligned} \tag{74}$$

$$\frac{2a_4a_5}{25a_5+4a_4}\big(-4A_3(y)+A_2(y)\big)= \tag{75}$$
$$=-2a_2A_2(y)-6\big(a_1A_1(y)+2a_0A_0(y)\big).$$

Equation (75) is a constant multiple of (74) thus it can be omitted. The remaining equations (73), (74) both have the form (71). Thus, by proposition 3, we have

$$4a_2=-30(a_1+4a_0) \tag{76}$$

$$a_3=\quad 20(a_1+4a_0) \tag{77}$$

as the necessary and sufficient condition for (73) and

$$\frac{-5a_4a_5}{25a_5+4a_4}-5a_2=\quad 45(a_1+4a_0) \tag{78}$$

$$\frac{20a_4a_5}{25a_5+4a_4}=-30(a_1+4a_0) \tag{79}$$

as the necessary and sufficient condition for (74). The comparison of (76) and (79) gives

$$a_2=\frac{5a_4a_5}{25a_5+4a_4}, \tag{80}$$

while from the comparison of (77) and (79)

$$a_3=-\frac{40}{3}\frac{a_4a_5}{25a_5+4a_4}. \tag{81}$$

From (78), (80) we get

$$a_1+4a_0=-\frac{2}{3}\frac{a_4a_5}{25a_5+4a_4}\ . \tag{82}$$

It is easy to check that with arbitrary a_1,a_4,a_5 satisfying the conditions ($a_4a_5\neq 0, 25a_5+4a_4\neq 0$) and a_2,a_3,a_1+4a_0 given by (80), (81), (82) we obtained all solutions of (76)–(79), hence

$$f(x)=a_5x^5+a_4x^4+\frac{a_4a_5}{25a_5+4a_4}\left(-\frac{40}{3}x^3+5x^2\right)+a_1x-$$
$$-\frac{1}{4}\left(a_1+\frac{2}{3}\frac{a_4a_5}{25a_5+4a_4}\right),$$

and from (66)

$$h(x) = (25a_5 + 4a_4)^{-1/2}\bigl(5a_5A_5(x) + 2a_4A_4(x)\bigr),$$

which, apart from the notation of the constants, is identical with (15).

Case 2.2.2: n=3. Substituting (65), (66) with n=3 into (12) and rearranging the terms we obtain the equation

$$\sum_{k=0}^{3} \gamma_k A_k(x) = 0$$

where

$$\begin{aligned}
\gamma_3 &= \frac{2a_2a_3}{9a_3 + 4a_2}\bigl(2A_3(y) - 3A_2(y)\bigr),\\
\gamma_2 &= \frac{-3a_2a_3}{9a_3 + 4a_2}\bigl(2A_3(y) - 3A_2(y)\bigr),\\
\gamma_k &= a_kA_k(y) \quad (k = 0, 1).
\end{aligned}$$

Since

$$2A_3(y) - 3A_2(y) = A_1(y) = -1 \text{ and } A_0(y) = 2 \quad (y \in \,]0,1[\,)$$

proposition 3 shows that f, h given by (65), (66) (with $n = 3$) satisfy (12) if and only if

$$\begin{aligned}
\frac{3a_2a_3}{9a_3 + 4a_2} &= -3(a_1 + 4a_0),\\
\frac{-2a_2a_3}{9a_3 + 4a_2} &= \quad 2(a_1 + 4a_0),
\end{aligned}$$

hold. The second equation can be omitted since it is a constant multiple of the first. The solution of this system is: a_1, a_2, a_3 are arbitrary constants in $\mathbf{C}$ (with $a_2a_3 \neq 0, \quad 9a_3 + 4a_2 \neq 0$) and

$$a_0 = -\frac{a_1}{4} - \frac{1}{4}\frac{a_2a_3}{9a_3 + 4a_2}\ .$$

Hence

$$f(x) = a_3x^3 + a_2x^2 + a_1x - \frac{a_1}{4} - \frac{1}{4}\frac{a_2a_3}{9a_3 + 4a_2}\ ,$$

and from (66)

$$h(x) = (9a_3 + 4a_2)^{-1/2}\bigl(9a_3A_3(x) + 4a_2A_2(x)\bigr)$$

which, apart from the notation of the constants, is identical with (14).

Case 2.2.3: n=1. In this case (65), (66) give

$$f(x) = a_1 x + a_0 \quad (a_1, a_0 \neq 0)$$
$$h(y) = (a_1 + 4a_0)^{-1/2}\big(a_1 A_1(y) + 2a_0 A_0(y)\big) = (a_1 + 4a_0)^{1/2}.$$

Here a_1, a_0 are arbitrary constants satisfying the conditions $a_1 a_0 \neq 0$, $a_1 + 4a_0 \neq 0$. This solution is a special case of (13). □

With this the proof of theorem 1 is complete, namely by lemmas 2,3 f is of the form (23). If in (23) $c \neq 0$ the by lemma 4 the solution of (12) is given by (37), (38) (=(13)). If in (23) $c = 0$ then f and h are polynomials. The polynomial solutions of (12) have been determined in lemmas 5,6.
□

4. Applications

We can easily obtain the measurable solutions of (9) from (10), (11) and theorem 1.

Theorem 2. *Suppose that the functions* $F, G, H :]0,1[\to \mathbf{C}$ *are solutions of the functional equation*

$$F(xy)+F(x(1-y))+F((1-x)y)+F((1-x)(1-y)) = G(x)H(y) \qquad (x, y \in]0,1[) \tag{9}$$

and F *is measurable on* $]0,1[$. *Then for any* $x \in]0,1[$

$$F(x) = F^*(x), \quad G(x) = O(x), \quad H(x) = \text{arbitrary}, \tag{83}$$

or

$$F(x) = F^*(x), \quad G(x) = \text{arbitrary},\ G \neq O,\ H(x) = O(x), \tag{84}$$

or

$$\begin{cases} F(x) = F^*(x) + p_1 a_1 x^\alpha \\ G(x) = a_1(x^\alpha + (1-x)^\alpha) \\ H(x) = p_1(x^\alpha + (1-x)^\alpha), \end{cases} \tag{85}$$

where $a_1 \neq 0$, $p_1 \neq 0$, $\alpha \neq 0$ *are arbitrary complex constants, or*

$$\begin{cases} F(x) = F^*(x) + p_2\left[\dfrac{1}{6}(3a_3 + 2a_2)\big(2a_3 x^3 + 3a_2 x^2\big) - \dfrac{a_2 a_3}{24}\right] \\ G(x) = a_3\big(x^3 + (1-x)^3\big) + a_2\big(x^2 + (1-x)^2\big) \\ H(x) = p_2\left[a_3\big(x^3 + (1-x)^3\big) + a_2\big(x^2 + (1-x)^2\big)\right], \end{cases} \tag{86}$$

where $a_2 \neq 0,\ a_3 \neq 0,\ p_2 \neq 0$ *are arbitrary complex constants satisfying the condition* $3a_3 + 2a_2 \neq 0$, *or*

$$
(87)\qquad
\begin{cases}
F(x) = F^*(x) + p_3\left[\dfrac{1}{10}(5a_5+2a_4)\big(2a_5x^5+5a_4x^4\big)-\right.\\
\qquad\qquad \left. -\dfrac{a_4a_5}{30}\left(40x^3-15x^2+\dfrac{1}{2}\right)\right]\\
G(x) = a_5\big(x^5+(1-x)^5\big)+a_4\big(x^4+(1-x)^4\big)\\
H(x) = p_3\left[a_5\big(x^5+(1-x)^5\big)+a_4\big(x^4+(1-x)^4\big)\right],
\end{cases}
$$

where $a_4 \neq 0,\ a_5 \neq 0,\ p_3 \neq 0$ *are arbitrary complex constants satisfying the condition* $5a_5 + 2a_4 \neq 0$.

In (83)–(87)

$$
(88)\qquad F^*(x) = a\left(x-\frac{1}{4}\right) \quad (x \in\,]0,1[\,),
$$

is the measurable solution of the homogeneous equation corresponding to (9), $a \in \mathbf{C}$ *is an arbitrary constant.*

Conversely the functions given by (83)–(87) *are measurable solutions of* (9) *such that each solution* F, G, H *belong to only one of the solution classes* (83)–(87).

We remark that (83)–(87) are solutions of (9) *for any values of the constants* $a, a_1, \dots, a_5,\ p_1, \dots, p_3, \alpha$ but if we do not require any conditions for the constants then the solution classes (83)–(87) may not be disjoint anymore.

Proof. If G or H is the zero function O then $F = F^*$ is the solution of the corresponding homogeneous equation (whose solution is given by theorem DJ) and we obtain solutions (83), (84).

If $G \neq O,\ H \neq O$ then by (10), (11)

$$
(89)\qquad F(x) = pf(x), \quad G(x) = h(x), \quad H(x) = p\,h(x)
$$

where $f, h \neq O$ are the solutions of (12) and $p \neq 0$ is an arbitrary constant.

By the measurability of F the function f is measurable too, hence theorem 1 is applicable.

If (13) *holds* then replacing in (13) a_1 by aa_1/p_1 $(p_1 \neq 0)$ c by a_1^2 $(a_1 \neq 0)$ and λ by α we obtain from (89) with $p = p_1/a_1$ the solution (85). Here we may assume that $\alpha \neq 0$ since the solution (85) with $\alpha = 0$:

$$
F(x) = a\left(x-\frac{1}{4}\right) + p_1a_1, \quad G(x) = 2a_1, \quad H(x) = 2p_1
$$

can obviously obtained from the solution (85) with $\alpha = 1$:

$$F(x) = a\left(x - \frac{1}{4}\right) + p_1 a_1 x = (a + p_1 a_1)\left(x - \frac{1}{4}\right) + \frac{p_1 a_1}{4},$$
$$G(x) = a_1,\ H(x) = p_1$$

by introducing suitable new constants.

If (14) *holds* then with the notations $p = p_2,\ a = p_2 c_1,$

$$a_2 = 2c_2(4c_2 + 9c_3)^{-1/2}, \quad a_3 = 3c_3(4c_2 + 9c_3)^{-1/2}$$

we get from (14) and (89) the solution (86).

If (15) *holds* then with the notations $p = p_3,\ a = p_3 d_1,$

$$a_4 = 2d_4\left(4d_4 + 25d_5\right)^{-1/2}, \quad a_4 = 5d_5\left(4d_4 + 25d_4\right)^{-1/2}$$

we obtain from (15) and (89) the solution (87). $\square$

The measurable solutions of equation (8) are supplied by

Theorem 3. *The function* $g :]0,1[\to \mathbf{C}$ *measurable on* $]0,1[$ *satisfies the functional equation*

$$(8) \quad g(xy) + g(x(1-y)) + g((1-x)y) + g((1-x)(1-y)) = \\ = (g(x) + g(1-x))(g(y) + g(1-y)) \quad (x, y \in]0,1[\,).$$

if and only if

$$(90) \qquad g(x) = x^{\lambda}$$

or

$$(91) \qquad g(x) = (1 - p^2)x + \frac{p^2 - p}{2}$$

or

$$(92) \qquad g(x) = \frac{q^2 - 1}{3}x^3 + \frac{-q^2 + q + 2}{2}x^2 + \frac{q^2 - 3q + 2}{6}x$$

or

$$(93) \quad g(x) = \frac{r^2 - 1}{15}x^5 + \frac{-r^2 + 3r + 4}{6}x^4 + \frac{r^2 - 5r + 4}{270}(40x^3 - 15x^2 + 2x)$$

holds where $\lambda, p, q, r \in \mathbf{C}$ *are arbitrary constants. The solutions* (90)–(93) *are realvalued if and only if* $\lambda, p, q, r \in \mathbf{R}$.

Proof. First we find those solutions for which

$$(94) \qquad g(x) + g(1-x) = \text{ constant} = 1 - p \quad (x \in]0,1[\,).$$

From (8) we can see that in this case the function defined by

$$g_1(x) := g(x) - \frac{(1-p)^2}{4} \quad (x \in \,]0,1[\,)$$

satisfies the homogeneous equation

$$g_1(xy) + g_1(x(1-y)) + g_1((1-x)y) + g_1((1-x)(1-y)) = 0 \quad (x, y \in \,]0,1[\,)$$

hence, by theorem DJ $g_1(x) = a_1\left(x - \frac{1}{4}\right)$ and

$$g(x) = a_1\left(x - \frac{1}{4}\right) + \frac{(1-p)^2}{4} \quad (x \in \,]0,1[\,). \tag{95}$$

Determining a_1 from the equation (94) we get $a_1 = 1 - p^2$ and with this value (95) goes over into (91).

We may suppose now that

$$h(x) = g(x) + g(1-x) \quad x \in \,]0,1[\tag{96}$$

is not a constant function in particular $h \neq O$. $f = g$, $h \neq O$ are solutions of equation (12) therefore by the measurability of g theorem 2 can be applied. We have to find those solutions $g = f, h \neq O$ of (12) for which (96) holds.

Case 1: $g = f$, h are the functions given by (13). Then from (96)

$$2\left(c - c^{1/2}\right) A_\lambda(x) + a_1 = 0 \quad (x \in \,]0,1[\,) \tag{97}$$

where A_λ is the function defined by (39).

If $\lambda = 1$ then $A_1(x) = 1 \quad (x \in \,]0,1[\,)$ and from (97) with the notation $p := 1 - c^{1/2}$ we get $a_1 = 2\left(p - p^2\right)$ and thus (13) gives again solution (91).

If $\lambda = 0$ then $A_0(x) = 2 \quad (x \in \,]0,1[\,)$ and from (97) with $p := 1 - 2c^{1/2}$ we get $a_1 = 1 - p^2$ and (13) repeatedly gives solution (91).

If $\lambda \neq 0$, $\lambda \neq 1$, then the functions $x \to 1$ and A_λ are linearly independent (see [6], lemma 1) thus (97) implies $a_1 = 0$, $c = c^{1/2}$. Hence $c = 1$ (namely in (13) $c \neq 0$) and $g(x) = f(x) = x^\lambda$ i.e. (90) holds.

Case 2: $g = f$, h are given by (14). Since g is a polynomial satisfying (8) for $x, y \in]0,1[$ its continuous extension $\tilde{g}$ will satisfy (8) for $x, y \in [0,1]$ too. From (8) with $y = 0$

$$[\tilde{g}(x) + \tilde{g}(1-x)][\tilde{g}(1) + \tilde{g}(0) - 1] = 2\tilde{g}(0).$$

Since $x \to \tilde{g}(x) + \tilde{g}(1-x) \quad (x \in [0,1])$ is not a constant function either we have $\tilde{g}(0) = 0$, $\tilde{g}(1) = 1$. By the continuity of $\tilde{g}$ and (14) these conditions go over into

$$c_1 = -c_2 c_3 (4c_2 + qc_3)^{-1} \tag{98}$$

$$c_3 + c_2 + c_1 = 1. \tag{99}$$

Eliminating c_1 from the system (98), (99) and solving the equation so obtained for c_2 we get

$$c_2 = \frac{1}{2}\left(1 - 3c_3 + (3c_3+1)^{1/2}\right). \tag{100}$$

Let $q := (3c_3+1)^{1/2}$ then

$$c_3 = \frac{q^2-1}{3} \tag{101}$$

and from (99), (100)

$$c_2 = \frac{1}{2}\left(-q^2+q+2\right), \quad c_1 = \frac{1}{6}\left(q^2-3q+2\right). \tag{102}$$

The restrictions $c_2c_3 \neq 0,\ 4c_2+9c_3 \neq 0$ in (14) give that $q \neq 1,\ -1,\ 2$. Using the constants c_1, c_2, c_3 given by (101), (102) we get solution (92) with the restrictions $q \neq 1,\ -1,\ 2$. These restrictions however can be omitted. If $q = 1, -1, 2$ then solution (92) goes over into solution (90) with $\lambda = 2, 1, 3$ respectively hence (92) *is a solution of* (8) *for every* $q \in \mathbf{C}$.

Case 3: $g = f, h$ are given by (15). Similarly to case 2 we have $\tilde{g}(0) = 0,\ \tilde{g}(1) = 1$, hence from (15) by the continuity of $\tilde{g}$

$$d_1 = -\frac{2}{3}d_4d_5\left(4d_4+25d_5\right)^{-1} \tag{103}$$

$$d_5 + d_4 - \frac{25}{3}d_4d_5\left(4d_4+25d_5\right)^{-1} + d_1 = 0. \tag{104}$$

Substituting d_1 from (103) into (104) and solving the equation so obtained for d_4 we get

$$d_4 = \frac{1}{2}\left(1 - 5d_5 + (15d_5+1)^{1/2}\right). \tag{105}$$

Let $r := (15d_5+1)^{1/2}$ then

$$d_5 = \frac{r^2-1}{15} \tag{106}$$

and from (103), (105)

$$d_4 = \frac{1}{6}\left(-r^2+3r+4\right), \quad d_1 = \frac{2}{270}\left(r^2-5r+4\right). \tag{107}$$

The restrictions $d_4d_5 \neq 0,\ \ 4d_4+5d_5 \neq 0$ in (15) give that $r \neq 1, -1, 4$. The one–parameter family of constants d_5, d_4, d_1 given by (106), (107) supply the the solution (93) with the above restrictions concerning r. We show that these restrictions can be omitted. If $r = 1, 4$ then (93) goes over into (90) with $\lambda = 4, 5$ while for $r = -1$ (93) gives

$$g(x) = \frac{1}{27}(40x^3 - 15x^2 + 2x).$$

This is a solution of (8) obtained from the family (92) with the parameter value $q = 7/3$. Therefore (93) *is a solution of* (8) *for every* $r \in \mathbf{C}$. We obtained all solutions of (8).

The solution (90) is realvalued if and only if $\lambda \in \mathbf{R}$. On the other hand a polynomial of a real variable with complex coefficients is realvalued if and only if its coefficients are real. This easily gives that the solutions (91)–(93) are realvalued if and only if $p, q, r \in \mathbf{R}$. $\square$

The next theorem shows that the class of normalized information measures having the sum property (with measurable generating function) and posessing (2,2)–additivity of degree α is wider than the entropies of degree α.

Theorem 4. *Suppose that* $I_n : \Gamma_n^0 \to \mathbf{R} \quad (n = 2, 3, \dots)$ *is a measure of information with the following properties:*

(i) $\{I_n\}$ *has the sum property and its generating function is measurable on* $]0, 1[$,

(ii) $\{I_n\}$ *is* (2,2)*–additive of degree* α, *i.e.*

$$I_4(X * Y) = I_2(X) + I_2(Y) + \left(2^{1-\alpha} - 1\right) I_2(X) I_2(Y) \quad \left(X, Y \in \Gamma_2^0\right)$$

holds where $1 \neq \alpha \in \mathbf{R}$,

(iii) $\{I_n\}$ *normalized, i.e.* $\quad I_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1.$

Then I_n *is one of the following functions:*

$$I_n(X) = H_n^{(\alpha)}(X) \tag{108}$$

$$I_n(X) = -\frac{1}{2}(\delta + 1) H_n^{(0)}(X) - \frac{1}{2}(3\delta + 1) H_n^{(\infty)}(X) \tag{109}$$

$$I_n(X) = -(4\delta + 2) H_n^{(3)}(X) + (4\delta + 3) H_n^{(2)}(X) \tag{110}$$

$$\begin{aligned} I_n(X) = &- 3\left(48\delta + 82 + 35.0\delta^{-1}\right) H_n^{(5)}(X) + \\ &+ 7\left(48\delta + 85 + 37.5\delta^{-1}\right) H_n^{(4)}(X) + \\ &- 4\left(64\delta + 116 + 52.5\delta^{-1}\right) H_n^{(3)}(X) + \\ &+ \left(64\delta + 116 + 52.5\delta^{-1}\right) H_n^{(2)}(X) + H_n^{(\infty)}(X), \end{aligned} \tag{111}$$

where $X \in \Gamma_n^0$, $\delta = 2^{1-\alpha} - 1$, *and*

$$H_n^{(\alpha)}(X) := \delta^{-1} \sum_{k=1}^{n} (x_k^\alpha - x_k) \quad (\alpha \in \mathbf{R},\, \alpha \neq 1)$$

is the entropy of degree α, $H_n^{(\infty)}(X) := \lim_{\alpha\to\infty} H_n^{(\alpha)}(X) = 1$.

Proof. If f is the generating function of $\{I_n\}$ then by (ii) the function

$$g(x) := \delta f(x) + x \quad (x \in \,]0,1[\,) \tag{112}$$

satisfies the functional equation (8) and (iii) implies that

$$g\left(\frac{1}{2}\right) = 2^{-\alpha}. \tag{113}$$

By theorem 3 g is of the form (90)–(93).

If (90) is valid then from (113) $\lambda = \alpha$ and by (90), (112) $f(x) = \delta^{-1}(x^\alpha - x)$ $(x \in \,]0,1[)$ which, by (3), gives (108).

If (88), (89) or (90) holds then from (113) we get $p = -\delta$, $q = -4\delta - 1$ or $r = -48\delta - 41$ respectively. Writing the function $f(x) = \delta^{-1}(g(x) - x)$ $(x \in \,]0,1[\,)$ in each case as a linear combination of the functions $f_5(x) = -\frac{16}{15}\left(x^5 - x\right)$, $f_4(x) = -\frac{8}{7}\left(x^4 - x\right)$, $f_3(x) = -\frac{4}{3}(x^3 - x)$, $f_2(x) = -2(x^2 - x)$, $f_0(x) = 1 - x$, $f_\infty(x) = x$, the generating functions of the entropies $H_n^{(5)}, \dots, H_n^{(0)}, H_n^{(\infty)}$, and using (3) we obtain that (109)–(111) hold. We omit the details.

We remark that because of the condition $\alpha \neq 1$ $(\delta \neq 0)$ (90), (91), (92) and (93) do not determine information measures satisfying (i)–(iii) if $\lambda = 1$, $p = 0$, $q = -1$ and $r = -41$ respectively. □

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LÁSZLÓ LOSONCZI
L. KOSSUTH UNIVERSITY
DEPARTMENT OF MATHEMATICS
4010 DEBRECEN, PF.12
HUNGARY

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