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A generalization of Apostol's Möbius functions of order k

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Abstract. Apostol's Möbius functions $\mu_k(n)$ of order k are generalized to depend on a second integer parameter $m \geq k$. Asymptotic formulas are obtained for the partial sums of these generalized functions.

1. Introduction

Möbius functions of order k, introduced by T. M. Apostol [1], are defined by the formulas

$$\mu_k(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^{k+1} \mid n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{\alpha_i}, \text{ with } 0 \leq \alpha_i < k, \\ 1 & \text{otherwise.} \end{cases}$$

In [1] Apostol obtained the asymptotic formula

$$\sum_{n \leq x} \mu_k(n) = A_k x + O(x^{\frac{1}{k}} \log x), \tag{1}$$

where

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right).$$

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Later, Suryanarayana [3] showed that, on the assumption of the Riemann hypothesis, the error term in (1) can be improved to

$$O\left(x^{\frac{4k}{4k^2+1}}\omega(x)\right), \tag{2}$$

where

$$\omega(x)=\exp\{A\log x(\log\log x)^{-1}\}$$

for some positive constant k.

This paper generalizes Möbius functions of order k and establishes asymptotic formulas for their partial sums.

2. Preliminary lemmas

The generalization in question is denoted by $\mu_{k,m}(n)$, where $1<k\le m$.

If $m=k$, $\mu_{k,k}(n)$ is defined to be $\mu_k(n)$, and if $m>k$ the function is defined as follows:

$$\mu_{k,m}(n)=\begin{cases}1 & \text{if } n=1,\\ 1 & \text{if } p^k\nmid n \text{ for each prime } p,\\ (-1)^r & \text{if } n=p_1^m\cdots p_r^m\prod\limits_{i>r}p_i^{\alpha_i}, \quad \text{with } 0\le\alpha_i<k,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

This generalization, like Apostol's $\mu_k(n)$, is a multiplicative function of n, so it is determined by its values at the prime powers. We have

$$\mu_k(p^\alpha)=\begin{cases}1 & \text{if } 0\le\alpha<k,\\ -1 & \text{if } \alpha=k,\\ 0 & \text{if } \alpha>k,\end{cases}$$

whereas

$$\mu_{k,m}(p^\alpha)=\begin{cases}1 & \text{if } 0\le\alpha<k,\\ 0 & \text{if } k\le\alpha<m,\\ -1 & \text{if } \alpha=m,\\ 0 & \text{if } \alpha>m,\end{cases} \tag{4}$$

Lemma 2.1. *For $k \le m$ we have*

$$\mu_{k,m}(n) = \sum_{\substack{\delta d^m = n \\ (d,\delta)=1}} \mu(d)\; q_k(\delta) \tag{5}$$

where $\mu(n)$ is the Möbius function and $q_k(n)$ is the caracteristic function of the k-free integers:

$$q_k(n) = \begin{cases} 0 & \text{if } \; p^k \mid n \text{ for some prime } p \\ 1 & \text{if } \; p^\alpha \mid n \text{ implies } \alpha < k. \end{cases}$$

Proof. Because $\mu(n)$ and $q_k(n)$ are multiplicative functions of n, the sum in the lemma is also multiplicative, so to complete the proof we simply note that when $n = p^\alpha$ the sum has the values indicated in (4). □

The next two lemmas, proved in [4], involve the following functions:

$$\theta(n) = \text{ the number of square-free divisors of } n,$$

$$\psi_k(n) = n \prod_{p \mid n} \left(1 + \frac{1}{p} + \dots + \frac{1}{p^{k-1}}\right),$$

where k is an integer ≥ 2,

$$\delta_k(x) = \exp\{-Ak^{-\frac{8}{5}} \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\},$$

where $A > 0$ is an absolute constant,

$$\omega_k(x) = \exp\{B_k \log x (\log\log x)^{-1}\},$$

where B_k is a positive constant.

Lemma 2.2. *For $x \ge 3$ we have*

$$Q_k(x,n) = \sum_{\substack{r \le x \\ (r,n)=1}} q_k(r) = \frac{xn}{\zeta(k)\psi_k(n)} + 0\left(\theta(n) x^{\frac{1}{k}} \delta_k(x)\right) \tag{6}$$

uniformly in x, n and k.

Lemma 2.3. *If the Riemann hypothesis is true, then for $x \geq 3$ we have*

$$Q_k(x,n) = \sum_{\substack{r \leq x \\ (r,n)=1}} q_k(r) = \frac{xn}{\zeta(k)\psi_k(n)} + 0\left(\theta(n) x^{\frac{2}{2k+1}} \omega_k(x)\right) \tag{7}$$

uniformly in x, n and k.

Our derivation of an asymptotic formula for the summatory function of $\mu_{k,m}(n)$ will also make use of the following lemma.

Lemma 2.4. *For $k \geq 2$ we have*

$$\sum_{d \mid n} \frac{\mu(d)\psi_{k-1}(d)}{d\psi_k(d)} = \frac{n}{\psi_k(n)}. \tag{8}$$

Proof. Both sides of (8) are multiplicative functions of n so it suffices to verify the equation when n is a prime power. If $n = p^\alpha$ we have

$$\sum_{d \mid p^\alpha} \frac{\mu(d)\psi_{k-1}(d)}{d\psi_k(d)} = 1 - \frac{1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-2}}}{p\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-1}}\right)}$$

$$= \frac{p^\alpha}{p^\alpha\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-1}}\right)} = \frac{n}{\psi_k(n)}. \qquad \square$$

The next lemma is proved in [5].

Lemma 2.5. *For $x \geq 3$, $n \geq 1$, and every $\epsilon > 0$ we have*

$$L_n(x) \equiv \sum_{\substack{r \leq x \\ (r,n)=1}} \frac{\mu(r)}{r} = O\left(\sigma^*_{-1+\epsilon}(n)\delta(x)\right) \tag{9}$$

uniformly in x and n, where $\sigma^*_\alpha(n)$ is the sum of the αth powers of the square-free divisors of n, and

$$\delta(x) = \exp\left\{-A \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right\},$$

for some absolute constant $A > 0$.

We note that if $\alpha < 0$ we have $\sigma^*_\alpha(n) \leq \sigma^*_0(n) = \theta(n)$. Also, $x^\epsilon \delta(x)$ is an increasing function of x for every $\epsilon > 0$ and $x > x_0(\epsilon)$. Using the method described in [5], it can be shown that if the Riemann hypothesis is true the factor $\delta(x)$ in the error term in (9) can be replaced by $\omega(x)x^{-\frac{1}{2}}$, where

$$\omega(x) = \exp\{A \log x (\log\log x)^{-1}\},$$

for some absolute constant $A > 0$.

Lemma 2.6. *For $x \geq 3$ and every $\epsilon > 0$ we have*

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \frac{\mu(r)}{\psi_k(r)} = 0\left(\sigma^*_{-1+\epsilon}(n)\delta(x)\right) \tag{10}$$

uniformly in x, n and k.

Proof. We write

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \frac{\mu(r)}{\psi_k(r)} = \sum_{\substack{r \leq x \\ (r,n)=1}} \frac{\mu(r)}{r} \frac{r}{\psi_k(r)},$$

then use (8) to obtain

$$\begin{aligned} \sum_{\substack{r \leq x \\ (r,n)=1}} \frac{\mu(r)}{\psi_k(r)} &= \sum_{\substack{d\delta \leq x \\ (d,\delta)=1 \\ (d\delta,n)=1}} \frac{\mu^2(d)\mu(\delta)\psi_{k-1}(d)}{d^2\delta\psi_k(d)} \\ &= \sum_{\substack{d \leq x \\ (d,n)=1}} \frac{\mu^2(d)\psi_{k-1}(d)}{d^2\psi_k(d)} \sum_{\substack{\delta \leq \frac{x}{d} \\ (\delta,nd)=1}} \frac{\mu(\delta)}{\delta} \\ &= \sum_{\substack{d \leq x \\ (d,n)=1}} \frac{\mu^2(d)\psi_{k-1}(d)}{d^2\psi_k(d)} L_{dn}\left(\frac{x}{d}\right). \end{aligned}$$

Using (9) and the inequality $\psi_{k-1}(d) \leq \psi_k(d)$ we find that the last sum is

$$O\left(\sigma^*_{-1+\epsilon}(n) \sum_{\substack{d \leq x \\ (d,n)=1}} \frac{\mu^2(d)\delta\left(\frac{x}{d}\right)\sigma^*_{-1+\epsilon}(d)}{d^2}\right).$$

Because $x^{\epsilon'}\delta(x)$ increases for every $\epsilon'>0$, we have $\left(\frac{x}{d}\right)^{\epsilon'}\delta\left(\frac{x}{d}\right)\le x^{\epsilon'}\delta(x)$ so $\delta\left(\frac{x}{d}\right)\le d^{\epsilon'}\delta(x)$ and the foregoing 0-term is

$$O\Bigg(\sigma^*_{-1+\epsilon}(n)\delta(x)\sum_{\substack{d\le x\\(d,n)=1}}\frac{\mu^2(d)\sigma^*_{-1+\epsilon}(d)}{d^{2-\epsilon'}}\Bigg).$$

But $\sigma^*_{-1+\epsilon}(d)\le\tau(d)=O(d^{\epsilon'})$ for every $\epsilon'>0$. If we choose $\epsilon'<\frac{1}{2}$ the last sum is $O(1)$ and we obtein (10). □

Applying (10) together with the remark following Lemma 2.5 we obtain

Lemma 2.7. *If the Riemann hypothesis is true, then for $x\ge 3$, $n\ge 1$ and every $\epsilon>0$ we have*

$$\sum_{\substack{r\le x\\(r,n)=1}}\frac{\mu(r)}{\psi_k(r)}=O\left(\sigma^*_{-1+\epsilon}(n)\omega(x)x^{-\frac{1}{2}}\right)\tag{11}$$

uniformly in x, n and k.

Applying partial summation in (10) we obtain

Lemma 2.8. *For $x\ge 3$, $k\ge 2$, and every $\epsilon>0$ we have*

$$\sum_{\substack{r\le x\\(r,n)=1}}\frac{\mu(r)}{\psi_k(r)r^{k-1}}=O\left(\sigma^*_{-1+\epsilon}(n)\delta(x)x^{1-k}\right)\tag{12}$$

uniformly in x, n and k.

Note: If the Riemann hypothesis is true, the error term in Lemma 2.8 holds with $\delta(x)x^{1-k}$ replaced by $\omega(x)x^{\frac{1}{2}-k}$.

3. Main results

Theorem 3.1. *For $x\ge 3$ and $m>k\ge 2$ we have*

$$\sum_{\substack{r\le x\\(r,n)=1}}\mu_{k,m}(r)=\frac{xn^2\alpha_{k,m}}{\zeta(k)\psi_k(n)\alpha_{k,m}(n)}+0\left(\theta(n)x^{\frac{1}{k}}\delta(x)\right)\tag{13}$$

uniformly in x, n and k, where

$$\alpha_{k,m} = \prod_p \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right)$$

and

$$\alpha_{k,m}(n) = n \prod_{p|n} \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right).$$

Proof. By (5) and (6) we have

$$\sum_{\substack{r \le x \\ (r,n)=1}} \mu_{k,m}(n) = \sum_{\substack{\delta d^m \le x \\ (d,\delta)=1 \\ (d\delta,n)=1}} \mu(d) q_k(\delta) = \sum_{\substack{d \le x^{\frac{1}{m}} \\ (d,n)=1}} \mu(d) \sum_{\substack{\delta \le \frac{x}{d^m} \\ (\delta,dn)=1}} q_k(\delta)$$

$$= \sum_{\substack{d \le x^{\frac{1}{m}} \\ (d,n)=1}} \mu(d) Q_k\left(\frac{x}{d^m}, dn\right)$$

$$= \sum_{\substack{d \le x^{\frac{1}{m}} \\ (d,n)=1}} \mu(d) \left\{ \frac{\left(\frac{x}{d^m}\right) dn}{\zeta(k)\psi_k(dn)} + 0\left(\theta(dn) \frac{x^{\frac{1}{k}}}{d^{\frac{m}{k}}} \delta\left(\frac{x}{d^m}\right)\right)\right\}$$

$$= \frac{xn}{\zeta(k)\psi_k(n)} \sum_{\substack{d=1 \\ (d,n)=1}}^{\infty} \frac{\mu(d)}{d^{m-1}\psi_k(d)} - \frac{xn}{\zeta(k)} \psi_k(n) \sum_{\substack{d > x^{\frac{1}{m}} \\ (d,n)=1}} \frac{\mu(d)}{d^{m-1}\psi_k(d)}$$

$$+ O\Bigg(\theta(n) x^{\frac{1}{k}-\epsilon} \sum_{\substack{d \le x^{\frac{1}{m}} \\ (d,n)=1}} \frac{\delta\left(\frac{x}{d^k}\right)\left(\frac{x}{d^k}\right)\mu^2(d)\theta(d)}{d^{\frac{m}{k}-\epsilon' k}}\Bigg).$$

Using the Euler product representation for absolutely convergent series of multiplicative terms [2] we have

$$\sum_{\substack{d=1 \\ (d,n)=1}}^{\infty} \frac{\mu(d)}{d^{m-1}\psi_k(d)} = \prod_{p|n} \left(1 - \frac{1}{p^{m-k+1} + \cdots + p^m}\right) = \frac{\alpha_{k,m}}{\alpha_{k,m}(n)}.$$

Now use (12) and the fact that $\delta(x)x^{\epsilon'}$ is increasing for all $\epsilon' > 0$, then choose $\epsilon > 0$ so that $\frac{m}{k} - \epsilon' k > 1 + \epsilon$ and we obtain (13).

When $n = 1$, Theorem 3.1 gives the following corollary for $x \geq 3$ and $m > k \geq 2$:

$$\sum_{r \leq x} \mu_{k,m}(r) = \frac{x}{\zeta(k)} \alpha_{k,m} + 0\left(x^{\frac{1}{k}} \delta(x)\right) \tag{14}$$

uniformly in x and k. □

Applying the method used to prove Theorem 1, and making use of (7) and Lemma 2.9 we get

Theorem 3.2. *If the Riemann hypothesis is true, then for $x \geq 3$ and $m > k \geq 2$ we have*

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \mu_{k,m}(r) = \frac{x n^2 \alpha_{k,m}}{\zeta(k) \psi_k(n) \alpha_{k,m}(n)} + 0\left(\theta(n) x^{\frac{2}{2k+1}} \omega(x)\right) \tag{15}$$

uniformly in x, n and k.

In particular, if $n = 1$ we have

$$\sum_{r \leq x} \mu_{k,m}(r) = \frac{x}{\zeta(k)} \alpha_{k,m} + 0\left(x^{\frac{2}{2k+1}} \omega(x)\right) \tag{16}$$

uniformly in x and k.

4. Conjectures

Suryanarayana raised the question of improving the error term in Apostol's asymptotic formula (1), and notes that no improvement seems possible by this method. Our method gives no improvement in the error term but it does suggest the following conjectures:

For $x \geq 3$, $n \geq 1$ and $k \geq 2$ we have

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \mu_k(r) = \frac{x n^2 \alpha_{k,k}}{\zeta(k) \psi_k(n) \alpha_{k,k}(n)} + 0\left(\theta(n) x^{\frac{1}{k}} \delta(x)\right) \tag{17}$$

uniformly in x, n and k.

In particular, when $n=1$ the conjecture is

$$\sum_{r\le x}\mu_k(r)=\frac{x}{\zeta(k)}\alpha_{k,k}+0\left(x^{\frac{1}{k}}\delta(x)\right) \tag{18}$$

uniformly in x and k.

If the Riemann hypothesis is true, the conjectured formulas are

$$\sum_{\substack{r\le x\\(r,n)=1}}\mu_k(r)=\frac{xn^2\alpha_{k,k}}{\zeta(k)\psi_k(n)\alpha_{k,k}(n)}+0\left(\theta(n)x^{\frac{2}{2k+1}}\omega(x)\right) \tag{19}$$

uniformly in x, n and k, for $x\ge 3$, $n\ge 1$ and $k\ge 2$.

In particular, when $n=1$ the conjecture is

$$\sum_{r\le x}\mu_k(r)=\frac{x}{\zeta(k)}\alpha_{k,k}+0\left(x^{\frac{2}{2k+1}}\omega(x)\right) \tag{20}$$

uniformly in x and k.

It should be noted that $\alpha_{k,k}=\zeta(k)A_k$, where A_k is Apostol's constant in (1), so the leading term in (18) and (20) is the same as that in (1).

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