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Some Cauchy-like functional equations on the natural numbers

By T. M. K. DAVISON (Hamilton)

Abstract. The equation $f(am+bn)+f(0)=f(am)+f(bn)$ is solved, where a,b are fixed relatively prime positive integers and m,n are arbitrary natural numbers.

1. Introduction

In this paper we give necessary and sufficient conditions that a function f from the natural numbers (denoted $\mathbb{N}_0$) to an additive abelian group (denoted Γ) satisfy

$$f(am+bn)+f(0)=f(am)+f(bn); \quad (m,n)\in\mathbb{N}_0^2. \tag{1}$$

Here a, b are fixed positive integers that are relatively prime. If $a=1$ and $b=1$ then equation (1) becomes the affine version of Cauchy's equation; namely

$$f(m+n)+f(0)=f(m)+f(n); \quad (m,n)\in\mathbb{N}_0^2. \tag{2}$$

It is clear that if f satisfies equation (2) then it also satisfies equation (1). For this reason we call solutions of equation (1) (a,b)-*Cauchy functions*.

We need some elementary number theory to enable us to complete the characterization of (a,b)-Cauchy functions. References for this are Dickson [1: Chapter III], Hua [2: Chapters 1, 2, 11], Rosen [3: Chapter 2] and Uspensky and Heaslet [4: Chapter III].

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The following sets of natural numbers are significant in the understanding of (a,b)-Cauchy functions:

$$S = S(a,b) := \{ax + by : (x,y) \in \mathbb{N}_0^2\}, \tag{3}$$

and

$$T = T(a,b) := \mathbb{N}_0 \setminus S(a,b). \tag{4}$$

Since a and b are relatively prime T is finite: more precisely, for all $n \in \mathbb{N}_0$

$$n \geq (a-1)(b-1) \Rightarrow n \in S(a,b). \tag{5}$$

(See [1: p. 65], [3: p. 109].) Indeed, the largest element of T is $ab - a - b$ [3: p. 109] and the number of elements in T is $\frac{(a-1)(b-1)}{2}$ [3: p. 109]. We see that T is empty if $a = 1$ or $b = 1$. Now letting $p \in \mathbb{N}$ we say a function $f : \mathbb{N}_0 \to \Gamma$ is *p-quasiperiodic* if

$$f(m+p) + f(0) = f(m) + f(p); \quad m \in \mathbb{N}_0. \tag{6}$$

It is easy to see that equation (6) implies

$$f(m+pn) + f(0) = f(m) + f(pn); \quad (m,n) \in \mathbb{N}_0^2. \tag{7}$$

Hence a p-quasi-periodic function is none other than a $(1,p)$-Cauchy function. We require two more bits of terminology prior to stating our first theorem. An (a,b)-Cauchy function g is *singular* if g has finite support: that is to say

$$\operatorname{supp}(g) := \{n \in \mathbb{N}_0 : g(n) \neq 0\} \tag{8}$$

is a finite set. An (a,b)-Cauchy function h is *regular* if h is also an $(1,ab)$-Cauchy function: in other words h is regular if it is an (a,b)-Cauchy function that is also ab-quasi-periodic. We observe that the sum/difference of singular/regular functions is singular/regular.

We now state our main results: the proofs are deferred to the second section of the paper.

Theorem 1. *Let $\mathbb{N}_0 \to \Gamma$ be an (a,b)-Cauchy function. Then f can be written uniquely as $g + h$ where g is a singular (a,b)-Cauchy function, and h is a regular (a,b)-Cauchy function.*

Thus, to understand (a,b)-Cauchy functions we need only characterize the special types: singular and regular. Singular functions are relatively easy:

Theorem 2. *The function* $g : \mathbb{N}_0 \to \Gamma$ *is a singular* (a,b)*-Cauchy function if, and only if,* $\mathrm{supp}(g) \subset T$*. (Here* $\mathrm{supp}(g)$ *is defined by* (8)*).*

To characterize regular (a,b)-Cauchy functions we require a supply of quasi-periodic functions: indeed those defined below are purely periodic. Let $p \in \mathbb{N}$. For $j \in \mathbb{N}_0$ we define the characteristic function $\chi_p^j : \mathbb{N}_0 \to \{0,1\}$ by $\chi_p^j(m) = 1$ if, and only if, $m \equiv j \mod p$. It is clear that

$$\chi_p^j(m+p) = \chi_p^j(m); \qquad m \in \mathbb{N}_0,$$

so χ_p^j is certainly p-quasi-periodic for all $j \in \mathbb{N}_0$. Finally we define $N_{a,b}(n)$ as the number of pairs $(x,y) \in \mathbb{N}_0^2$ satisfying the linear Diophantine equation

$$ax + by = n. \tag{9}$$

Our third main result is:

Theorem 3. *Let* $h : \mathbb{N}_0 \to \Gamma$*. Then* h *is a regular* (a,b)*-Cauchy function if, and only if, there are elements* $\alpha_1, \dots \alpha_{a-1}$*,* $\beta_1, \dots \beta_{b-1}$*,* γ_0*,* γ_{ab} *of* Γ *such that, for all* $m \in \mathbb{N}_0$

$$h(m) = \sum_{j=1}^{a-1} \chi_a^{jb}(m)\alpha_j + \sum_{k=1}^{b-1} \chi_b^{ka}(m)\beta_k + \gamma_0 + N_{a,b}(m)\gamma_{ab}.$$

In the final section of the paper we show how (a,b)-Cauchy functions over $\mathbb{Z}$ can easily be characterized using our results over $\mathbb{N}_0$.

2. Properties of (a,b)-Cauchy functions

We show first that (a,b)-Cauchy functions are ab-quasi-periodic on S.

Lemma 1. *Let* f *be an* (a,b)*-Cauchy function. Then for all* $s \in S$

$$f(s+ab) + f(0) = f(s) + f(ab). \tag{10}$$

Proof. Let $s \in S$; so $s = ax + by$ for some $x, y \in \mathbb{N}_0$. Then $f(s+ab) + f(0) = f(ax + b(y+a)) + f(0) = f(ax) + f(ab + by) = f(ax) + f(ab) + f(by) - f(0) = f(ax + by) + f(ab) = f(s) + f(ab)$, using equation (1) repeatedly. □

Let $f : \mathbb{N}_0 \to \Gamma$ be arbitrary. We define functions $\check{f}$, $\hat{f} : \mathbb{N}_0 \to \Gamma$ as follows:

$$\check{f}(m) := f(m) + f(ab) - f(m+ab) - f(0); \qquad m \in \mathbb{N}_0 \tag{11}$$

$$\hat{f}(m) := f(m+ab) + f(0) - f(ab); \qquad m \in \mathbb{N}_0. \tag{12}$$

We see that, for all $m \in \mathbb{N}_0$

$$f(m) = \check{f}(m) + \hat{f}(m). \tag{13}$$

If f is assumed to be an (a,b)-Cauchy function then equation (13) is, as we will show, the decomposition of f into singular and regular parts.

Lemma 2. *Let f be an (a,b)-Cauchy function*

(i) *$\check{f}$ is a singular (a,b)-Cauchy function and* $\operatorname{supp}(\check{f}) \subseteq T$

(ii) *$\hat{f}(s) = f(s)$ for all $s \in S$*

(iii) *$\hat{f}$ is a regular (a,b)-Cauchy function.*

Proof. (i) Let $s \in S$. Then $\check{f}(s) = f(s)+f(ab)-f(s+ab)-f(0) = 0$ by Lemma 1. Thus $\operatorname{supp}(\check{f}) \subseteq T$. But $|T| = \frac{(a-1)(b-1)}{2}$ so $\operatorname{supp}(\check{f})$ is finite. Now $\check{f}$ is clearly an (a,b)-Cauchy function as, in equation (1), $am+bn$, am, bn all belong to S so we require $0+0 = 0+0$ which is certainly true.

(ii) Since $\hat{f}(s) = f(s) - \check{f}(s)$ by equation (13) we deduce that $\hat{f}(s) = f(s)$ for all $s \in S$ from part (i).

Since $\hat{f} = f - \check{f}$ and both f, $\check{f}$ are (a,b)-Cauchy functions we see that $\hat{f}$ is also an (a,b)-Cauchy function. We have to show that $\hat{f}$ is ab-quasi-periodic. Let $m \in \mathbb{N}_0$ Then $m+ab \in S$ since $m+ab \geq (a-1)(b-1)$ using the criterion for S-membership in equation (5). Hence

$$\begin{aligned}
\hat{f}(m+ab) + \hat{f}(0) &= f(m+ab) + f(0) && \text{(by part (i) above)}\\
&= f(m) + f(ab) && \text{(by equation (12))}\\
&= \hat{f}(m) + \hat{f}(ab) && \text{(since } ab \in S\text{)}.
\end{aligned}$$

This proves that $\hat{f}$ is ab-quasi-periodic, and completes the proof that $\hat{f}$ is regular. □

One more result is useful in proving Theorem 1: only the zero function is both singular and regular.

Lemma 3. *Suppose f is an (a,b)-Cauchy function function that is both singular and regular. Then $f=0$.*

Proof. We note that $f=0$ if, and only if, $\mathrm{supp}(f)$ is the empty set. So suppose $\mathrm{supp}(f)\neq\emptyset$. Since $\mathrm{supp}(f)$ is finite (f is singular) there is a largest element in $\mathrm{supp}(f)$: call it m_0. Then $f(m_0+ab)+f(ab)=f(m_0+2ab)+f(0)$, since f is ab-quasi-periodic. Since $m_0+2ab>m_0+ab>m_0$ we have that $f(m_0+2ab)=0$ and $f(m_0+ab)=0$ (else m_0 is not largest in $\mathrm{supp}(f)$). We deduce that $f(ab)=f(0)$, and so for all $m\in\mathbb{N}_0$ $f(m+ab)=f(m)$. But this implies $0=f(m_0+ab)=f(m_0)\neq 0$. This contradiction shows that $\mathrm{supp}(f)=\emptyset$ and hence that $f=0$, as claimed. □

We can now prove

Theorem 1. *Let $f:\mathbb{N}_0\to\Gamma$ be an (a,b)-Cauchy function. Then f can be written uniquely as $g+h$ where g is a singular (a,b)-Cauchy function, and h is a regular (a,b)-Cauchy function.*

Proof. From equation (13) we know that $f=\check{f}+\hat{f}$, and from Lemma 2 we know that $\check{f}$ is a singular (a,b)-Cauchy function and $\hat{f}$ is a regular (a,b)-Cauchy function if f is an arbitrary (a,b)-Cauchy function. This proves the existence of the claimed decomposition.

For the uniqueness suppose $g+h=g'+h'$ where g, g' are singular (a,b)-Cauchy functions and h, h' are regular (a,b)-Cauchy functions. Then $g-g'=h'-h$. Moreover $g-g'$ is a singular (a,b)-Cauchy function and $h'-h$ is a regular (a,b)-Cauchy function. Thus the function $g-g'$ is both singular and regular. By Lemma 3 it follows that $g-g'=0$. Hence $g=g'$ and so, $h=h'$. This proves the uniqueness of the decomposition. □

A consequence of this theorem is that we need only characterize the special types: singular and regular. We characterize the singular functions in

Theorem 2. *A function $g:\mathbb{N}_0\to\Gamma$ is a singular (a,b)-Cauchy function if, and only if* $\mathrm{supp}(g)\subseteq T$.

Proof. Suppose g is a singular (a,b)-Cauchy function. Then $g=\check{g}+\hat{g}$ by Theorem 1. Since this decomposition is unique $\hat{g}=0$. Thus $\mathrm{supp}(g)=\mathrm{supp}(\check{g})\subseteq T$ by Lemma 2 (i).

Conversely suppose $g : \mathbb{N}_0 \to \Gamma$ satisfies $\mathrm{supp}(g) \subseteq T$. Then $g(am + bn) + g(0) - g(am) - g(bn) = 0 + 0 - 0 - 0 = 0$ since $am + bn \notin T$, $0 \notin T$, $am \notin T$; $bn \notin T$ and $x \notin T$ implies $g(x) = 0$. Thus g is an (a, b)-Cauchy function. It is a singular one since $\mathrm{supp}(g)$ is a finite set. □

It remains to characterize regular (a, b)-Cauchy functions. As a first step we show that there are many such.

Lemma 4. *The functions N_{ab}, χ_a^j, χ_b^k are regular (a, b)-Cauchy functions, for all j, $k \in \mathbb{N}_0$.*

Proof. We show first that $N_{a,b}$ is ab-quasi-periodic. Since a, b are relatively prime $ax + by = au + bv$ implies that $x \equiv u \bmod b$ and $y \equiv v \bmod a$. Hence all the non-negative solutions of $ax + by = n$ are in the list

$$(x_0, y_0),\ (x_0 + b,\ y_0 - a), \ldots, (x_0 + kb,\ y_0 - ka)$$

where $k = N_{a,b}(n) - 1$. So all the non-negative solutions of $ax + by = n + ab$ are in the list $(x_0,\ y_0 + a)$, $(x_0 + b,\ y_0), \ldots, (x_0 + kb,\ y_0 - ka)$. Thus $N_{a,b}(n + ab) = k + 2 = N_{a,b}(n) - 1 + 2$, and so

$$N_{a,b}(n + ab) + N_{a,b}(0) = N_{a,b}(n) + N_{a,b}(ab)$$

since $N_{a,b}(0) = 1$ and $N_{a,b}(ab) = 2$. This proves that $N_{a,b}$ is ab-quasi-periodic.

Now to prove that $N_{a,b}$ is (a, b)-Cauchy let $m, n \in \mathbb{N}_0$. By the division theorem we can write $m = bm' + u$ with $0 \le u \le b - 1$, and $n = an' + v$ with $0 \le v \le a - 1$. Then an easy computation using the ab-periodicity of $N_{a,b}$ (in particular equation (7))

$$\begin{aligned} &N_{a,b}(am + bn) + N_{a,b}(0) - N_{a,b}(am) - N_{a,b}(bn) \\ &\quad = N_{a,b}(au + bv + (m' + n')ab) + N_{a,b}(0) \\ &\qquad - N_{a,b}(au + m'ab) - N_{a,b}(bv + n'ab) \\ &\quad = N_{a,b}(au + bv) + N_{a,b}((m' + n')ab) - N_{a,b}(au) - N_{a,b}(m'ab) \\ &\qquad + N_{a,b}(0) - N_{a,b}(bv) - N_{a,b}(n'ab) + N_{a,b}(0) \\ &\quad = N_{a,b}(au + bv) + N_{a,b}(0) - N_{a,b}(au) - N_{a,b}(bv) \\ &\qquad + N_{a,b}(m'ab + n'ab) + N_{a,b}(0) - N_{a,b}(m'ab) - N_{a,b}(n'ab) \\ &\quad = N_{a,b}(au + bv) + N_{a,b}(0) - N_{a,b}(au) - N_{a,b}(bv). \end{aligned}$$

Thus $N_{a,b}$ is an (a,b)-Cauchy function if, and only if,

$$N_{a,b}(au+bv)+N_{a,b}(0)=N_{a,b}(au)+N_{a,b}(bv) \tag{14}$$

for all u, v in $\mathbb{N}_0$ satisfying $0\le u\le b-1$, $0\le v\le a-1$. Now $N_{a,b}(0)=N_{a,b}(au)=N_{a,b}(bv)=1$. It remains to prove that $N_{a,b}(au+bv)=1$ also for (14) to be satisfied. If $ax+by=au+bv$ with $(x,y)\in\mathbb{N}_0^2$ and $x>u$ then $v>y$; but then $y<0$ – which is a contradiction [$v\equiv y \bmod a$ and $y<v<a$ implies $y<0$]. Similarly if $x<u$ then $y>v$ and $x<0$; also a contradiction. Hence $N_{a,b}(au+bv)=1$, and equation (14) has been shown to be satisfied. Thus $N_{a,b}$ is an (a,b)-Cauchy function.

Next $\chi_a^j(am+bn)=\chi_a^j(bn)$ since χ_a^j is purely a-periodic, as noted in the introduction. Thus $\chi_a^j(am+bn)+\chi_a^j(0)-\chi_a^j(am)-\chi_a^j(bn)=\chi_a^j(bn)+\chi_a^j(0)-\chi_a^j(0)-\chi_a^j(bn)=0$. Hence χ_a^j is an (a,b)-Cauchy function. Now χ_a^j is also trivially ab-quasi-periodic since $\chi_a^j(m+ab)+\chi_a^j(0)-\chi_a^j(m)-\chi_a^j(ab)=\chi_a^j(m)+\chi_a^j(0)-\chi_a^j(m)-\chi_a^j(0)=0$. Thus χ_a^j is a regular (a,b)-Cauchy function. Similarly, χ_b^k is a regular (a,b)-Cauchy function. □

We can now prove

Theorem 3. *The function $h:\mathbb{N}_0\to\Gamma$ is a regular (a,b)-Cauchy function if, and only if, there are elements $\alpha_1,\dots,\alpha_{a-1}$, $\beta_1,\dots,\beta_{b-1}$, γ_0, γ_{ab} in Γ such that for all $m\in\mathbb{N}_0$*

$$h(m)=\sum_{j=1}^{a-1}\chi_a^{jb}(m)\alpha_j+\sum_{k=1}^{b-1}\chi_b^{ka}(m)\beta_k+\gamma_0+N_{a,b}(m)\gamma_{ab}. \tag{15}$$

Proof. Let the elements $\alpha_1,\dots,\gamma_{ab,}$ be given. Then the functions $\chi_a^{jb}\,\alpha_j$, $\chi_b^{ka}\beta_k$, γ_0 and $N_{a,b}\gamma_{ab}$ are regular (a,b)-Cauchy functions from $\mathbb{N}_0$ to Γ using Lemma 4. Hence so is $h(m)$ as defined by equation (15).

Assume conversely that h is a regular (a,b)-Cauchy function. Define elements $\alpha_j:=h(jb)-h(0)$, $\beta_k:=h(ka)-h(0)$, $\gamma_0:=2h(0)-h(ab)$ and $\gamma_{ab}:=h(ab)-h(0)$. Define $h':\mathbb{N}_0\to\Gamma$ by $h'(m):=\sum_{j=1}^{a-1}\chi_a^{jb}(m)\alpha_j+\sum_{k=1}^{b-1}\chi_b^{ka}(m)\beta_k+\gamma_0+N_{a,b}(m)\gamma_{ab}$. Then by the direct part of the theorem $h':\mathbb{N}_0\to\Gamma$ is a regular (a,b)-Cauchy function. Now define $\overline{h}:=h-h'$. Then $\overline{h}$ is also a regular (a,b)-Cauchy function.

It suffices to show that $\overline{h}$ vanishes on S. For then $\operatorname{supp}(\overline{h})\subseteq T$ and $\overline{h}$ would be a singular (a,b)-Cauchy function by Theorem 2. So $\overline{h}=0$, and thus $h = h'$, as described. First $\overline{h}(0) = h(0) - h'(0) = h(0) - \gamma_0 - \gamma_{ab} = 0$. (For $\chi_a^{jb}(0) = 0$ for $j = 1, 2, \ldots, a-1$ since a and b are relatively prime; similarly $\chi_b^{ka}(0) = 0$.) Second

$$\overline{h}(ab) = h(ab) - h'(ab) = h(ab) - \gamma_0 - 2\gamma_{ab} = 0.$$

Now for arbitrary $n \in \mathbb{N}$ we have

$$\overline{h}(nab) = \overline{h}((n-1)ab + ab) + \overline{h}(0) = \overline{h}((n-1)ab) + \overline{h}(ab) = \overline{h}((n-1)ab),$$

and so $\overline{h}(nab) = 0$ for all $n \in \mathbb{N}$ by induction.

Third, let $\ell \in \mathbb{N}_0$, $1 \leq \ell \leq a-1$. Then, for $1 \leq j \leq a-1$, $\chi_a^{jb}(\ell b) = 1$ iff $jb \equiv \ell b \bmod a$, iff $j \equiv \ell \bmod a$, iff $j = \ell$ since j, ℓ are both small. Hence $\sum_{j=1}^{a-1} \chi_a^{jb}(\ell b)\alpha_j = \alpha_\ell$. Next $\chi_b^{ka}(\ell b) = 0$. So $\overline{h}(\ell b) = h(\ell b) - h'(\ell b) = h(\ell b) - \alpha_\ell - \gamma_0 - \gamma_{ab} = 0$. Similarly, $\overline{h}(ma) = 0$ for $1 \leq m \leq b-1$. Finally, let $s = ax + by \in S$. Write $x = bx' + u$, $y = ay' + v$ where $0 \leq u \leq b-1$, $0 \leq v \leq a-1$. Then $\overline{h}(ax+by) = \overline{h}(au + bv + (x'+y')ab) = \overline{h}(au+bv)$ (since $\overline{h}(au) + \overline{h}(bv) = 0 + 0 = 0$). Thus $\overline{h}$ is zero on S, and the proof is complete. □

Corollary. *Let $p \in \mathbb{N}$. Then $f : \mathbb{N}_0 \to \Gamma$ is p-quasi-periodic if, and only if, there are elements $\beta_1, \ldots, \beta_{p-1}, \gamma_0, \gamma_p$ in Γ such that*

$$f(n) = \sum_{k=1}^{p-1} \chi_p^k(m)\beta_k + \gamma_0 + N_{1,p}(n)\gamma_p. \tag{16}$$

Proof. This is merely the case $a = 1$, $b = p$ of the theorem. □

It is easy to evaluate $N_{1,p}(n)$ with the help of a well known p-quasi-periodic function. Let $p \in \mathbb{N}$. There are p-quasi-periodic functions $q_p : \mathbb{N}_0 \to \mathbb{N}_0$ and $r_p : \mathbb{N}_0 \to \{0, 1, 2, \ldots, p-1\}$ for all $n \in \mathbb{N}_0$,

$$n = pq_p(n) + r_p(n). \tag{17}$$

Of course the notation is self-explanatory: q_p is the quotient after division by p, and r_p is the remainder. We can now state

Lemma 5. *Let $p \in \mathbb{N}$. Then*

$$N_{1,p}(n) = q_p(n) + 1; \qquad n \in \mathbb{N}_0. \tag{18}$$

Proof. $N_{1,p}(n) = \operatorname{card}\{(x,y) \in \mathbb{N}_0^2 : x + py = n\}$. Now write $n = pq_p(n)+r_p(n)$ as in equation (17). Then $(r_p(n), q_p(n))$, $(r_p(n)+p, q_p(n)-1)$, $\dots, (n, 0)$ is the complete list of non-negative solutions (x, y) to $x+py = n$. There are $q_p(n) + 1$ distinct entries on the list. So $N_{1,p}(n) = q_p(n) + 1$. □

In turn we can use the corollary above to determine another expression for $N_{a,b}(n)$.

Proposition. *Let χ_S be the characteristic function of S: that is $\chi_S(n) \in \{0, 1\}$ and $\chi_S(n) = 1$ if, and only if, $n \in S$. Then*

$$N_{a,b}(n) = q_{ab}(n) + \chi_S(r_{ab}(n)); \qquad n \in \mathbb{N}_0. \tag{19}$$

Proof. N_{ab} is a regular (a, b)-Cauchy function by Lemma 4. So, using the corollary to Theorem 3 we have

$$\begin{aligned} N_{ab}(n) &= \sum_{k=1}^{ab-1} \chi_{ab}^k(n)\beta_k + \gamma_0 + N_{1,ab}(n)\gamma_{ab} \\ &= \sum_{k=1}^{ab-1} \chi_{ab}^k(n)\beta_k + \gamma_0 + \gamma_{ab} + q_{ab}(n)\gamma_{ab} \end{aligned}$$

using Lemma 5. We know that $\gamma_0 = 2N_{a,b}(0) - N_{a,b}(ab) = 0$, and $\gamma_{a,b} = N_{a,b} - N_{a,b}(0) = 2-1 = 1$, $\beta_k = N_{a,b}(k) - N_{a,b}(0)$. So $N_{a,b}(n) = q_{ab}(n) + \chi =_S (r_{ab}(n))$ if, and only if $\chi_S(r_{ab}(n)) = 1 + \sum_{k=1}^{ab-1} \chi_{ab}^k(n)[N_{a,b}(k) - 1]$. We see that both sides remain invariant under the transformation $n \mapsto n+ab$. So it suffices to prove the result for $0 \le n < ab$. Now $N_{a,b}(k)-1 = -\chi_T(k)$ since $1 \le k < ab$. So $\sum_{k=1}^{ab-1} \chi_{ab}^k(n)(-\chi_T(k)) = -\chi_T(n)$ ($n < ab$ used here). Finally $1 - \chi_T(n) = \chi_S(n)$ for $0 \le n < ab$. Thus the result follows. □

Equation (19) is well-known. (See [4, p. 65].) However the above proof uses our analysis of the solutions of a functional equation and not elementary number theory directly.

3. Concluding remarks

We mention briefly how to use our results to solve, for $f : \mathbb{Z} \to \Gamma$, $a, b \in \mathbb{N}$ relatively prime

$$(20) \qquad f(am + bn) + f(0) = f(am) + f(bn); \quad (m, n) \in \mathbb{Z}^2.$$

If f is an (a, b)-Cauchy function over $\mathbb{Z}$ then f is ab-quasi-periodic over $\mathbb{Z}$. (For now $S(a, b) = \mathbb{Z}$ and Lemma 1 still gives the result.) Hence f restricted to $\mathbb{N}_0$ is a regular (a, b)-Cauchy function. We can therefore state

Theorem. *$f : \mathbb{Z} \to \Gamma$ satisfies equation (20) if, and only if, there are elements $\alpha_1, \dots, \alpha_{a-1}$, $\beta_1, \dots, \beta_{b-1}$, δ_0, δ_{ab} in Γ such that*

$$f(n) = \sum_{j=1}^{a-1} \chi_a^{jb}(n)\alpha_j + \sum_{k=1}^{b-1} \chi_b^{ka}(n)\beta_k + \delta_0 + \left[q_{ab}(n) + \chi_S(r_{ab}(n))\right]\delta_{ab}$$

for all $n \in \mathbb{Z}$.

Here, of course $q_p : \mathbb{Z} \to \mathbb{Z}$ is the quotient function extended to $\mathbb{Z}$:

$$q_p(n) := q_p(n + |n|p) - |n|; \qquad n \in \mathbb{Z}.$$

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T. M. K. DAVISON
MCMASTER UNIVERSITY
HAMILTON, ONTARIO
CANADA

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