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Commutativity of rings with variable constraints

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Abstract. Let $m > 1$, $r \geq 0$ be fixed non-negative integers and R a ring with unity 1 in which for each $x \in R$, there exists a polynomial $f(X,Y) = f_x(X,Y)$ in $R\langle X,Y\rangle$ satisfying the condition that for all y in R $f(x,y) = f(x,y+1) = f(x,x+y)$ so that either of the properties $y^r[x,y^m] = f(x,y)$ or $[x,y^m]y^r = f(x,y)$ for all y in R. The main result of the present paper asserts that R is commutative if it satisfies the property $Q(m)$ (for all $x, y \in R$, $m[x,y] = 0$ implies $[x,y] = 0$). Finally, some results have been extended to one-sided s-unital rings.

1. Introduction

Throughout, R will be an associative ring (maybe without unity 1), $Z(R)$ the center of R, $C(R)$ the commutator ideal of R, $N(R)$ the set of all nilpotent elements of R, $N'(R)$ the set of all zero-divisors in R. The symbol $[x,y]$ stands for the commutator $xy - yx$ of two elements x and y in R. As usual, $\mathbb{Z}[X,Y]$ the ring of polynomials in two commuting indeterminates and $\mathbb{Z}\langle X,Y\rangle$ the ring of polynomials in two non-commuting indeterminates over the ring $\mathbb{Z}$ of integers. For a ring R and a positive integer m we say that R has the property $Q(m)$ if $m[x,y] = 0$ implies that $[x,y] = 0$ for all $x, y \in R$.

Obviously, any m-torsion-free ring R has the property $Q(m)$ and if R has the property $Q(m)$, then R has the property $Q(n)$ for any factor m of n.

For fixed integers $m > 1$ and $r \geq 0$, consider the following ring properties.

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(P) For each $x \in R$, there exists a polynomial $f(X, Y) = f_x(X, Y)$ in $R\langle X, Y\rangle$ satisfying the condition that for all y in R $f(x, y) = f(x, y+1) = f(x, x+y)$ so that

$$y^r[x, y^m] = f(x, y) \quad \text{for all } y \text{ in } R.$$

(P_1) For each $x \in R$, there exists a polynomial $f(X, Y) = f_x(X, Y)$ in $R\langle X, Y\rangle$ satisfying the condition that for all y in R $f(x, y) = f(x, y+1) = f(x, x+y)$ so that

$$[x, y^m]y^r = f(x, y) \quad \text{for all } y \text{ in } R.$$

(P_2) For each x in R, there exist polynomials (depending on x) $n(X) = n_x(X)$, $p(X) = p_x(X)$, $q(X) = q_x(X)$ in $Z(R)[X]$ so that

$$y^r[x, y^m] = p(x)[n(x), y]q(x), \quad \text{for all } y \text{ in } R.$$

(P_3) For each x in R, there exist polynomials (depending on x) $n(X) = n_x(X)$, $p(X) = p_x(X)$, $q(X) = q_x(X)$ in $Z(R)[X]$ so that

$$[x, y^m]y^r = p(x)[n(x), y]q(x), \quad \text{for all } y \text{ in } R.$$

(P_4) For each x in R, there exist integers $n = n(x) \geq 0$, $p = p(x) \geq 0$ and $q = q(x) \geq 0$ such that

$$y^r[x, y^m] = \pm x^p[x^n, y]x^q, \quad \text{for all } y \text{ in } R.$$

(P_5) For each x in R, there exist integers $n = n(x) \geq 0$, $p = p(x) \geq 0$ and $q = q(x) \geq 0$ such that

$$[x, y^m]y^r = \pm x^p[x^n, y]x^q, \quad \text{for all } y \text{ in } R.$$

$Q(m)$ For all $x, y \in R$, $m[x, y] = 0$ implies that $[x, y] = 0$, where m is some positive integer.

Properties (P_2) and (P_3), as well as the properties (P_4) and (P_5) all follow from (P) and (P_1). There are several results in the existing literature concerning the commutativity of rings satisfying special cases of the properties (P) and (P_1).

In [3, Theorems 2 and 4], Abujabal has shown that a ring with unity 1 is commutative if, for every x, y in R, R satisfies any one of the polynomial identities $y^s[x, y^m] = \pm x^t[x^n, y]$ and $[x, y^m]y^s = \pm x^t[x^n, y]$, where $m > 1$, $n \geq 1$ and s, t are fixed non-negative integers with the property $Q(m)$.

In most of the cases, the underlying polynomials in (P) and (P_1) are particularly assumed to be monomials [1], [2], [4]–[8], [10]–[14], [16]–[18]. The object of the present paper is to investigate commutativity of rings satisfying one of the properties (P) and (P_1) together with the property $Q(m)$.

2. Main result

The main result of the present paper is the following:

Theorem 1. *Let R be a ring with unity 1 satisfying either of the properties* (P) *or* (P_1)*. If R satisfies the property $Q(m)$, then R is commutative.*

In the preparation for the proof of the above theorem, we start by stating without proof the following well-known results.

Lemma 1 [9, p. 221]**.** *If $[[x, y], x] = 0$ and $p(X)$ in $Z(R)[X]$, then $[p(x), y] = p'(x)[x, y]$ for all x, y in R.*

Lemma 2 [10, Theorem]**.** *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(a) *For any ring satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.*

(b) *For every prime p, $(GF(p))_2$ the ring of all 2×2 matrices over $GF(p)$, fails to satisfy $f = 0$.*

Following is a special case of a result which was proved by Streb [19, Hauptsatz 3].

Lemma 3. *Let R satisfy a polynomial identity of the form $[x, y] = p(x, y)$, where $p(X, Y)$ in $\mathbb{Z}\langle X, Y\rangle$ has the following properties:*

(i) *$p(X, Y)$ is in the kernel of the natural homomorphism from $\mathbb{Z}\langle X, Y\rangle$ to $\mathbb{Z}[X, Y]$;*

(ii) *each monomial of $p(X, Y)$ has total degree at least 3;*

(iii) *each monomial of* $p(X,Y)$ *has* X*-degree at least 2, or each monomial of* $p(X,Y)$ *has* Y*-degree at least 2.*

Then R *is commutative.*

Here, we shall prove the following lemma, which is proved in [15, Lemma 4] for a fixed exponent n, but with a slight modification in the proof it can be obtained for variable exponent n.

Lemma 4. *Let* R *be a ring with unity 1 and let* $f : R \to R$ *be any polynomial function of two variables with the property* $f(x+1,y) = f(x,y)$*, for all* x*,* y *in* R*. If for all* x,y *in* R *there exists an integer* $n = n(x,y) \geq 1$ *such that* $x^n f(x,y) = 0$*, then necessarily* $f(x,y) = 0$*.*

Proof. Given that $x^n f(x,y) = 0, n = n(x,y) \geq 1$. Choose an integer $n_1 = n(1+x,y)$ such that $(1+x)^{n_1} f(x,y) = 0$. If $k = \max\{n, n_1\}$, then $x^k f(x,y) = 0$ and $(1+x)^k f(x,y) = 0$. We have,

$$f(x,y) = \{(1+x) - x\}^{2k+1} f(x,y).$$

Expanding the expression on the right-hand side by the binomial theorem gives that $f(x,y) = 0$. □

We establish the following steps to prove Theorem 1.

Step 1. Let R be a ring satisfying either of the properties (P) or (P_1). Then $C(R) \subseteq N(R)$.

Proof. Let R satisfy the property (P), that is,

$$y^r[x, y^m] = f(x,y). \tag{1}$$

Replace y by $y+x$ in (1) to get

$$(y+x)^r[x, (y+x)^m] = f(x, x+y) = f(x,y). \tag{2}$$

Combining (1) and (2), we get

$$(y+x)^r[x, (y+x)^m] - y^r[x, y^m] = 0 \quad \text{for all } x, y \in R \tag{3}$$

and some fixed integers $r \geq 0$, $m > 1$. Equation (3) is a polynomial identity and we see that $x = e_{11} + e_{12}$ and $y = -e_{12}$ fail to satisfy this equality in $(GF(p))_2$, p a prime. Hence by Lemma 2, $C(R) \subseteq N(R)$.

On the other hand, if R satisfies the property (P_1), then by using a similar technique of replacing y by $y+x$, we find that R satisfies the polynomial identity $[x,(y+x)^m](y+x)^r = [x,y^m]y^r$ for all $x, y \in R$ and some fixed integers $r \geq 0$, $m > 1$. But $x = e_{22} + e_{12}$ and $y = -e_{12}$ fail to satisfy this equality in $(GF(p))_2$, p a prime. Hence, Lemma 2 gives $C(R) \subseteq N(R)$. $\square$

Step 2. Let R be a ring with unity 1 satisfying either of the properties (P) or (P_1). If R has the property $Q(m)$, then $N(R) \subseteq Z(R)$.

Proof. Let R satisfy the property (P) and $a \in N(R)$. Then there exists an integer $t \geq 1$ such that

$$a^k \in Z(R), \quad \text{for all } k \geq t,\ t \text{ minimal}. \tag{4}$$

Suppose that $t > 1$. Replacing y by a^{t-1} in (P), we get

$$a^{r(t-1)}\big[x, a^{m(t-1)}\big] = f(x, a^{t-1}).$$

In view of (4) and the fact that $m(t-1) \geq t$, for $m > 1$, we get

$$f\big(x, a^{t-1}\big) = 0. \tag{5}$$

Replacing y by $1 + a^{t-1}$ in (P), we get

$$\big(1 + a^{(t-1)}\big)^r\big[x, (1 + a^{(t-1)})^m\big] = f\big(x, 1 + a^{t-1}\big) = f\big(x, a^{t-1}\big).$$

Using (5) gives $(1 + a^{(t-1)})^r[x, (1 + a^{(t-1)})^m] = 0$, for all x in R. Since $(1 + a^{t-1})$ is invertible, the last equation implies that

$$\big[x, (1 + a^{(t-1)})^m\big] = 0 \quad \text{for all } x \quad \text{in } R. \tag{6}$$

Combining (4) and (6), we get

$$0 = \big[x, (1 + a^{(t-1)})^m\big] = \big[x, 1 + ma^{t-1}\big] = m\big[x, a^{t-1}\big].$$

Applying the property $Q(m)$, it follows that $[x, a^{t-1}] = 0$ for all $x \in R$, i.e., $a^{t-1} \in Z(R)$. This contradicts the minimality of t in (4). Hence $t = 1$ and $a \in Z(R)$. So $N(R) \subseteq Z(R)$. $\square$

Similar arguments may be used if R satisfies the property (P_1).

PROOF of Theorem 1. In view of Step 1 and Step 2, we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{7}$$

Properties (P) and (P_1) are equivalent and by Lemma 1 both can be written as

$$m[x,y]y^{m+r-1} = f(x,y). \tag{8}$$

Replacing $1+y$ for y in (8),we get

$$m[x,y](1+y)^{m+r-1} = f(x,1+y) = f(x,y). \tag{9}$$

From (8) and (9), we get

$$m[x,y]\{(1+y)^{m+r-1} - y^{m+r-1}\} = 0 \quad \text{for all } x, y \text{ in } R.$$

Now, by using the property $Q(m)$ in the last equation, we get

$$[x,y]\big\{(1+y)^{m+r-1} - y^{m+r-1}\big\} = 0. \tag{10}$$

For $m+r=2$ in (10), we get the commutativity of R.

For $m+r>2$, (10) implies that $[x,y]=[x,y]f(y)$ for all x, y in R and for some polynomial $f(Y)$ in $\mathbb{Z}[Y]$ is a polynomial such that all monomials of f have degree at least one. Hence R is commutative by Lemma 3. □

The following results are immediate consequences of Theorem 1.

Corollary 1. *Let R be a ring with unity 1 satisfying one of the properties* (P_2) *and* (P_3)*. If R satisfies the property $Q(m)$, then R is commutative.*

Corollary 2. *Let R be a ring with unity 1 satisfying one of the properties (P_4) and (P_5). If R satisfies the property $Q(m)$, then R is commutative.*

Corollary 3 [3, Theorem 3]**.** *Suppose that $n>1$ and m are positive integers and let s, t be non-negative integers. Let R be a ring with unity 1 satisfying the polynomial identity $[x,y^m]y^s = \pm[y,x^n]x^t$ for all x, y in R. If R has the property $Q(m)$, then R is commutative.*

Corollary 4 [17, Theorem 1]**.** *Let $n > 1$, $m > 1$ and let p, q be non-negative integers. Let R be a ring with unity 1 satisfying the polynomial identity $[x, y^m]y^q = x^p[x^n, y]$ for all x, y in R. If R is n-torsion-free, then R is commutative.*

Corollary 5 [1, Lemma 2(2)]**.** *Let R be a ring with unity 1 and $n > 1$ a fixed positive integer. If R is n-torsion-free and satisfies the identity $[x^n, y] = [x, y^n]$ for all x, y in R, then R is commutative.*

Remark 1. The following example strengthens the existence of the property $Q(m)$ in Theorem 1 and Corollaries 1, 2, 3, 4, 5.

Example 1. Let $R = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$, where $\alpha, \beta, \gamma \in GF(4)$, the finite Galois field, be the set of all matrices. It is readily verified that R (with the usual matrix addition and multiplication) is a non-commutative local ring with unity I, the identity matrix. Further, R satisfies

$$x^{48} \in Z(R) \quad \text{for all } x \in R. \tag{11}$$

Since $N'(R)$ consists of all matrices x in R with zero diagonal elements, and thus, contains exactly 16 elements. For any $x \in N'(R)$, $x^2 = 0$ and hence $x^{48} = 0 \in Z(R)$. The set $R \setminus N'(R)$ is a multiplicative group of order 48 and hence $x^{48} = I \in Z(R)$ for all $x \in R \setminus N'(R)$. In view of (11) it follows that R satisfies the properties (P) or (P_1). This shows that the assumption that R has the property $Q(m)$ in Theorem 1 and above corollaries cannot be eliminated.

The following result demonstrates that Corollary 2 is still valid if the property "$Q(m)$" is replaced by the condition that "m and n are relatively prime positive integers".

Theorem 2. *Let $m > 1$ and $r \geq 0$ be fixed integers and let R be a ring with unity 1 in which for every x in R there exist integers $n = n(x) > 1$, $p = p(x) \geq 0$ and $q = q(x) \geq 0$ such that m and n are relatively prime and R satisfies one of the properties* (P_4) *and* (P_5). *Then R is commutative.*

Proof. Let R satisfy the property (P_4) and let a be an arbitrary element in $N(R)$. Then there exists a positive integer t such that $a^k \in Z(R)$, for all $k \geq t$, t minimal.

Using the same arguments as used to prove Step 2, we have

$$m[x, a^{t-1}] = 0 \quad \text{for all } x \text{ in } R. \tag{12}$$

Further, choose integers $n' = n(a^{t-1}) > 1$ relatively prime to m and $p' = p(a^{t-1}) \geq 0$ and $q' = q(a^{t-1}) \geq 0$ such that $y^r[a^{t-1}, y^m] = \pm a^{p'(t-1)}[a^{n'(t-1)}, y]a^{q'(t-1)}$. Using (12) and the fact that $n'(t-1) \geq t$ for $n' > 1$, we have

$$y^r[a^{t-1}, y^m] = 0, \quad \text{for all } y \text{ in } R. \tag{13}$$

Again, choose integer $n'' = n(1 + a^{t-1}) > 1$ relatively prime to m and $p'' = p(1 + a^{t-1}) \geq 0$, $q'' = q(1 + a^{t-1}) \geq 0$ such that

$$y^r[a^{t-1}, y^m] = \pm(1 + a^{(t-1)})^{p''}[(1 + a^{(t-1)})^{n''}, y](1 + a^{t-1})^{q''}. \tag{14}$$

Hence, in view of (13) and the fact that $1 + a^{t-1}$ is invertible, (14) yields

$$\left[(1 + a^{(t-1)})^{n''}, y\right] = 0, \quad \text{for all } y \text{ in } R. \tag{15}$$

Combining (4) and (15), we obtain

$$0 = [(1 + a^{(t-1)})^{n''}, y] = [1 + n''a^{t-1}, y] = n''[a^{t-1}, y].$$

This implies that $n''[x, a^{t-1}] = 0$, for all x in R, and in view of (12), the relative primeness of n'' and m gives that $a^{t-1} \in Z(R)$. This contradicts the minimality of t and thus $t = 1$ and $a \in Z(R)$. Hence by Step 1, we get $C(R) \subseteq N(R) \subseteq Z(R)$ and Lemma 1 gives that

$$my^{m+r-1}[x, y] = \pm nx^{p+q+n-1}[x, y]. \tag{16}$$

Let $m[x, y] = 0$. Then equation (16) gives that

$$nx^{p+q+n-1}[x, y] = x^{p+q+n-1}n[x, y] = 0. \tag{17}$$

Using Lemma 4, (17) becomes $n[x, y] = 0$, for all x, y in R, and the relative primeness of m and n implies that $[x, y] = 0$. This shows that R also has the property $Q(m)$. Hence, commutativity of R follows from Theorem 1. $\square$

Corollary 6 [12, Theorem 2]. *Let $m > 1$, $n > 1$ be fixed relative prime positive integers and let p, r fixed non-negative integers. If R is a ring with unity 1 satisfying the polynomial identity $y^r[x, y^m] = \pm x^p[x^n, y]$ for all x, y in R, then R is commutative.*

Corollary 7 [17, Theorem 2]. *Suppose that $m > 1$, $n > 1$ be fixed relative prime positive integers. Let p, q be fixed non-negative integers and R a ring with unity 1 satisfying the polynomial identity $[x, y^m]y^q = x^p[x^n, y]$ for all x,y in R. Then R is commutative.*

Remark 2. The following example shows that R need not be commutative if "m and n are not relatively prime" in the hypothesis of Theorem 2 and Corollaries 6, 7.

Example 2. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}$. Then R is a non-commutative ring with unity 1 satisfying $y^r[x, y^4] = \pm x^p[x^4, y]x^q$ (or $[x, y^4]y^r = \pm x^p[x^4, y]x^q$), for any non-negative integers p, q and r.

3. Extension to s-unital rings

Since there are non-commutative rings with R^2 being central, neither of these conditions guarantees the commutativity of arbitrary rings. Before we go ahead with our task, we pause to recall a few preliminaries in order to make our paper self contained as possible. A ring R is said to be left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. As shown in [8], then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all x in F. Such an element e is called a pseudo-identity (resp. pseudo-left identity or pseudo-right identity) of F in R. The results proved in the preceding section can be extended to one-sided s-unital ring.

Theorem 3. *Let $m > 1$ and r be fixed non-negative integers. Let R be a left (resp. right) s-unital ring in which for every x in R there exist integers $n = n(x) \geq 0$, $p = p(x) \geq 0$ and $q = q(x) \geq 0$ such that R satisfies the property* (P_4) *(resp.* (P_5)*). Then R is commutative if one of the following conditions hold:*

(I) *R has the property $Q(m)$;*

(II) *$n > 1$ and $m > 1$ are relatively prime integers.*

Proof. Let R be a left (resp. right) s-unital ring satisfying the property (P_4) (resp. (P_5)) and x, y arbitrary elements of R. Choose an element e in R such that $ex = x$ and $ey = y$ (resp. $xe = x$ and $ye = y$). If $(n, p, q) \neq (1, 0, 0)$, then replace y by e in (P_4) (resp. (P_5)) we have

$$e^r[x, e^m] = \pm x^p[x^n, e]x^q \text{ (resp. } [x, e^m]e^r = \pm x^p[x^n, e]x^q).$$

$$x = xe^m \pm x^p e x^{n+q} \mp x^{n+p} e x^q \in xR$$
$$(\text{resp. } x = e^m x \mp x^p e x^{n+q} \pm x^{p+n} e x^q \in Rx).$$

Hence, R is right (resp. left) s-unital ring.

On the other hand, if $(n, p, q) = (1, 0, 0)$, then $(m, r) \neq (1, 0)$. Replace x by e in (P_4) (resp. (P_5)) to get

$$y = ye \pm y^{r+m} e \mp y^r e y^m \in yR \text{ (resp. } y = ey \mp ey^{m+r} \pm y^r e y^m \in Ry).$$

Hence, again R is right (resp. left) s-unital. Thus we observe that R is s-unital in both cases. Now, in view of [8, Proposition 1] we can assume that R has unity 1 and hence the commutativity of R follows from an application of Theorem 1 and Theorem 2. □

Remark 3. As a consequence of Theorem 3, we get the following corollary which includes [2, Theorem], [3, Theorems 1–4], [12, Theorems 2 and 3] and [18, Theorem].

Corollary 8. *Let $m > 1$, p, q, n and r be fixed non-negative integers and R a left (resp. right) s-unital ring satisfying* (P_4) *(resp.* (P_5)*). Then R is commutative in each of the following cases:*

(I) *R has the property $Q(m)$;*

(II) *$n > 1$ and $m > 1$ are relatively prime integers.*

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