Publ. Math. Debrecen
58 / 3 (2001), 325–335

Extended Jacobson density theorem for Lie ideals of rings with automorphisms

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Abstract. We prove a version of the Chevalley–Jacobson density theorem for Lie ideals of rings with automorphisms and present some applications of the obtained results.

1. Introduction

In the present paper we continue the project initiated recently in [7] and developed further in [3], [4]; its main idea is to connect the concept of a dense action on modules with the concept of outerness of derivations and automorphisms. In [3] an extended version of Chevalley–Jacobson density theorem has been proved for rings with automorphisms and derivations. In the present paper we consider a Lie ideal of a ring acting on simple modules via multiplication. Our goal is to extend to this context results obtained in [3]. We confine ourselves with the case of automorphisms. We note that Chevalley–Jacobson density theorem has been generalized in various directions [1], [10], [14], [12], [13], [17], [19]–[22] (see also [18, 15.7, 15.8] and [9, Extended Jacobson Density Theorem]).

As an application we generalize results of Drazin on primitive rings with pivotal monomial to primitive rings whose noncentral Lie ideal has a pivotal monomial with automorphisms. Here we note that while Martindale's results on prime rings with generalized polynomial identity were extended to prime rings with generalized polynomial identities involving derivations and automorphisms, the corresponding program for results of

Mathematics Subject Classification: 16W20, 16D60, 16N20, 16N60, 16R50.
Key words and phrases: ring, Lie ideal, automorphism, density, simple module.

Amitsur and Drazin on primitive rings with (generalized) pivotal monomials has not been done (see [2] for related results and references). In the present paper we make the first step in this direction.

There are many results in the literature concerning automorphisms of rings (most often prime rings) satisfying certain identities of polynomial type. They are a part of the theory of rings with generalized identities which is treated in the book [2]. The technique used in the proofs of these results is well-established; usually one combines Kharchenko's theory of X-inner automorphsisms with some elementary but tricky calculations. In [3], [4] an alternative unified approach to such problems has been presented. In the present paper we continue to develop this approach by generalizing a result of Bergen [6].

2. Density theorems for Lie ideals

Given a left module $\mathcal{M}$ over a ring $\mathcal{A}$, we set

$$l(\mathcal{A};\mathcal{M}) = \{r \in \mathcal{A} \mid r\mathcal{M} = 0\}.$$

Clearly $l(\mathcal{A};\mathcal{M})$ is an ideal of $\mathcal{A}$. Recall that an additive subgroup $\mathcal{U}$ of a ring $\mathcal{A}$ is called a Lie ideal if $[\mathcal{A},\mathcal{U}] \subseteq \mathcal{U}$. First we recall some known results.

Theorem 2.2 ([15, Theorem 4]). *Let $\mathcal{A}$ be a prime ring and $\mathcal{U}$ a Lie ideal of $\mathcal{A}$ such that $[\mathcal{U},\mathcal{U}] = 0$. Then $\mathcal{U} \subseteq Z(\mathcal{A})$, the center of $\mathcal{A}$, unless* $\mathrm{char}(\mathcal{A}) = 2$ *and $\mathcal{A}$ satisfies the standard identity St_4 of degree* 4.

The following result is contained in the proof of [11, Lemma 1.3].

Lemma 2.2. *Let $\mathcal{A}$ be a semiprime ring with Lie ideal $\mathcal{U}$. Suppose that $\mathcal{U}$ is a subring of $\mathcal{A}$. Then $\mathcal{A}[\mathcal{U},\mathcal{U}]\mathcal{A} \subseteq \mathcal{U}$.*

We are now in a position to prove a density theorem for Lie ideals of rings, our first main result, which is a generalization of [9, Extended Jacobson Density Theorem].

Theorem 2.3. *Let $m, n_1, n_2, \ldots, n_m$ be positive integers, let $\mathcal{A}$ be a ring with Lie ideal $\mathcal{U}$ and simple left modules $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$, let $\mathcal{D}_i = \mathrm{End}_{\mathcal{A}}\,\mathcal{M}_i)$, $i = 1, 2, \ldots, m$, let $x_{i1}, x_{i2}, \ldots, x_{in_i} \in \mathcal{M}_i$ be linearly independent over $\mathcal{D}_i$ and let $y_{i1}, y_{i2}, \ldots, y_{in_i} \in \mathcal{M}_i$, $i = 1, 2, \ldots, m$. Suppose that the following conditions are fulfilled:*

(1) *$\mathcal{M}_i \ncong \mathcal{M}_j$ for all $i \neq j$.*

(2) *$\sum_{j=1}^{n_i} x_{ij}\mathcal{D}_i \neq \mathcal{M}_i$ for all $i = 1, 2, \ldots, m$.*

(3) *$[\mathcal{A},\mathcal{U}]\mathcal{M}_i \neq 0$ for all $i = 1, 2, \ldots, m$.*

(4) *Either $\mathcal{U}$ generates the ring $\mathcal{A}$, or for each i, either $\mathrm{char}(\mathcal{A}/l(\mathcal{A};\mathcal{M}_i)) \neq 2$, or $\mathcal{A}/l(\mathcal{A};\mathcal{M}_i)$ does not satisfy the standard identity of degree 4.*

Then there exists $u \in \mathcal{U}$ such that $ux_{ij} = y_{ij}$ for all $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, m$.

Proof. It is enough to show that there exists $u \in \mathcal{U}$ such that $ux_{11} = y_{11}$ and $ux_{ij} = 0$ if $(i, j) \neq (1, 1)$. To this end, we denote by $\mathcal{W}$ the subring of $\mathcal{A}$ generated by $\mathcal{U}$ and set $I = \mathcal{A}[\mathcal{W}, \mathcal{W}]\mathcal{A}$. By [2, Lemma 9.1.2], $[\mathcal{W}, \mathcal{W}] \subseteq [\mathcal{W}, \mathcal{A}] = [\mathcal{U}, \mathcal{A}]$. Since $[\mathcal{U}, \mathcal{A}] \subseteq \mathcal{U} \subseteq \mathcal{W}$, we conclude that $\mathcal{W}$ is a Lie ideal of $\mathcal{A}$ and

$$[\mathcal{W}, \mathcal{W}] \subseteq \mathcal{U}. \tag{1}$$

Let $1 \leq i \leq m$ and let $\pi : \mathcal{A} \to \mathcal{A}/l(\mathcal{A};\mathcal{M}_i) = \overline{\mathcal{A}}$ be a canonical projection of rings. It follows from (3) that $\mathcal{U}^\pi$ is a noncentral Lie ideal of $\overline{\mathcal{A}}$. Therefore $\mathcal{W}^\pi$ is a noncentral Lie ideal and a subring of $\overline{\mathcal{A}}$. It follows now from (4) and Theorem 2.1, that $\mathcal{W}^\pi$ is not commutative and so $I^\pi \neq 0$ (recall that $\overline{\mathcal{A}}$ is a primitive ring). By Lemma 2.2, $I^\pi \subseteq \mathcal{W}^\pi$. Therefore $\mathcal{M}_i$ is a faithful simple left $\mathcal{W}^\pi$-module and $\mathrm{End}({}_{\mathcal{W}^\pi}\mathcal{M}_i) = \mathcal{D}_i$. We may now assume without loss of generality that $\mathcal{A} = \mathcal{W}$ and respectively $[\mathcal{A}, \mathcal{A}] \subseteq \mathcal{U}$ by (1).

By (2) there exists $x \in \mathcal{M}_1$ with $x \notin \sum_{j=1}^{n_1} x_{1j}\mathcal{D}_1$. Next, by [3, Proposition 2.1], there exist elements $a, b \in \mathcal{A}$ such that

$$\begin{aligned}
&ax_{ij} = 0 = bx_{ij} && \text{for all } j = 1, 2, \ldots, n_i,\ i = 2, 3, \ldots, m,\\
&ax = y_{11},\ ax_{1j} = 0 && \text{for all } j = 1, 2, \ldots, n_1 \quad \text{and}\\
&bx_{11} = x,\ bx_{1j} = 0 && \text{for all } j = 2, 3, \ldots, n_1.
\end{aligned}$$

Clearly $u = [a, b] \in \mathcal{U}$ is the desired element. □

Example. Let $\mathcal{F}$ be a field, let $n \geq 2$ be a positive integer, let $\mathcal{A} = M_n(\mathcal{F})$ be the $n \times n$ matrix ring over $\mathcal{F}$ and let $\mathcal{V}$ be a vector space over $\mathcal{F}$ with basis $\{x_1, x_2, \ldots, x_n\}$. Clearly $\mathcal{V}$ is a simple left $\mathcal{A}$ module canonically. Let $\mathcal{U} = [\mathcal{A}, \mathcal{A}]$. Take $y_1 = x_1$, $y_2 = y_2 \cdots = y_n = 0$. We claim that there exists no $u \in \mathcal{U}$ with $ux_i = y_i$, $i = 1, 2, \ldots, n$. Indeed, such element u would have trace 1 whereas every matrix in $\mathcal{U}$ has trace 0. We now see that the condition (2) of Theorem 2.3 can not be omitted.

The following result is a particular case of the above theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a left primitive ring with faithful simple left module $\mathcal{M}$, let $\mathcal{U}$ be a noncentral Lie ideal of $\mathcal{A}$, let $\mathcal{D} = \mathrm{End}_{\mathcal{A}}\,\mathcal{M})$, let n be a nonnegative integer, let $x_1, x_2, \ldots, x_n \in \mathcal{M}$ be linearly independent over $\mathcal{D}$ and let $y_1, y_2, \ldots, y_n \in \mathcal{M}$. Suppose that the following conditions are fulfilled:*

(1) $\sum_{i=1}^{n} x_i \mathcal{D} \neq \mathcal{M}$.

(2) *$\mathcal{U}$ generates the ring $\mathcal{A}$, or* $\mathrm{char}(\mathcal{A}) \neq 2$, *or $\mathcal{A}$ does not satisfy the standard identity of degree* 4.

Then there exists $u \in \mathcal{U}$ with $ux_i = y_i$ for all $i = 1, 2, \ldots, n$.

Corollary 2.5. *Let $\mathcal{A}$ be a left primitive ring with faithful simple left module $\mathcal{M}$, let $\mathcal{U}$ be a noncentral Lie ideal of $\mathcal{A}$, let $\mathcal{D} = \mathrm{End}_{\mathcal{A}}\,\mathcal{M})$, let n be a nonnegative integer, let $x_1, x_2, \ldots, x_n \in \mathcal{M}$ be linearly independent over $\mathcal{D}$ and let $y_1, y_2, \ldots, y_n \in \mathcal{M}$. Suppose that $\mathcal{A}$ is not a simple Artinian ring. Then there exists $u \in \mathcal{U}$ with $ux_i = y_i$ for all $i = 1, 2, \ldots, n$.*

Let $\mathcal{M}$ be a simple left $\mathcal{A}$-module and $\mathcal{D} = \mathrm{End}_{\mathcal{A}}\,\mathcal{M})$. Given $r \in \mathcal{A}$, we define a linear transformation $L_r : \mathcal{M} \to \mathcal{M}$ of the right vector space $\mathcal{M}$ over the skew field $\mathcal{D}$ as follows: $L_r x = rx$ for all $x \in \mathcal{M}$. We now recall definitions of $\mathcal{M}$-inner and $\mathcal{M}$-outer automorphisms [3].

Definition 2.6. An automorphism α of the ring $\mathcal{A}$ is called $\mathcal{M}$-*inner* if there exist an invertible element $T \in \mathrm{End}(\mathcal{M})$ such that

$$(2) \qquad TL_aT^{-1} = L_{a^\alpha} \quad \text{for all } a \in \mathcal{A};$$

otherwise it is called $\mathcal{M}$-*outer*.

We shall say that automorphisms α and β of $\mathcal{A}$ are $\mathcal{M}$-*independent* if the automorphism $\alpha^{-1}\beta$ (and hence $\beta^{-1}\alpha$) is $\mathcal{M}$-outer; otherwise they are called $\mathcal{M}$-dependent (see [3, Section 3] for details). We are now in a position to prove the the following generalization of [3, Thorem 3.6], our second main result.

Theorem 2.7. *Let $\mathcal{A}$ be a ring with simple left module $\mathcal{M}$, let $\mathcal{D} = \operatorname{End}_{\mathcal{A}} \mathcal{M})$, let $\mathcal{U}$ be a Lie ideal of $\mathcal{A}$, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be automorphisms of $\mathcal{A}$, let m be a positive integer, let $x_1, x_2, \ldots, x_m \in \mathcal{M}$ be linearly independent over $\mathcal{D}$ and let $y_{ij} \in \mathcal{M}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Suppose that the following conditions are fulfilled:*

(1) *α_i and α_j are $\mathcal{M}$-independent for all $i \neq j$.*

(2) *$\sum_{i=1}^{m} x_i\mathcal{D} \neq \mathcal{M}$.*

(3) *$[\mathcal{A}, \mathcal{U}^{\alpha_i}]\mathcal{M} \neq 0$ for all $i = 1, 2, \ldots, n$.*

(4) *$\mathcal{U}$ generates the ring $\mathcal{A}$, or $\operatorname{char}(\mathcal{A}/l(\mathcal{A}; \mathcal{M})) \neq 2$, or $\mathcal{A}/l(\mathcal{A}; \mathcal{M})$ does not satisfy the standard identity of degree 4.*

Then there exists $u \in \mathcal{U}$ such that $u^{\alpha_i} x_j = y_{ij}$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Proof. Let $1 \leq i \leq n$. We denote by $\mathcal{M}_i$ the left $\mathcal{A}$-module on additive abelian group $\mathcal{M}$ with multiplication $a * x = a^{\alpha_i} x$ for all $a \in \mathcal{A}$, $x \in \mathcal{M}$. By [3, Proposition 3.5], $\mathcal{M}_i \not\cong \mathcal{M}_j$ for all $i \neq j$ and so the first condition of Theorem 2.3 is fulfilled. As it was noted in [3, (3)], $\operatorname{End}_{\mathcal{A}} \mathcal{M}_i) = \mathcal{D}$ and so the second condition of Theorem 2.3 is satisfied by (2). The third condition follows from (3). Since $l(\mathcal{A}; \mathcal{M}_i) = l(\mathcal{A}; \mathcal{M})^{\alpha_i^{-1}}$, we conclude that α_i^{-1} induces an isomorphism of rings $\mathcal{A}/l(\mathcal{A}; \mathcal{M})$ and $\mathcal{A}/l(\mathcal{A}; \mathcal{M}_i)$. Therefore the fourth condition of Theorem 2.3 follows from (4). Thus all the conditions of Theorem 2.3 are fulfilled and so there exists $u \in \mathcal{U}$ such that $u^{\alpha_i} x_j = y_{ij}$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. $\square$

Corollary 2.8. *Let $\mathcal{A}$ be a left primitive ring with faithful simple left module $\mathcal{M}$, let $\mathcal{D} = \operatorname{End}_{\mathcal{A}} \mathcal{M})$, let $\mathcal{U}$ be a noncentral Lie ideal of $\mathcal{A}$, let $\alpha_1, \ldots, \alpha_n$ be automorphisms of $\mathcal{A}$. Suppose that $\mathcal{A}$ is not simple Artinian and α_i and α_j are $\mathcal{M}$-independent for all $i \neq j$. Then for all linearly independent over $\mathcal{D}$ elements $x_1, x_2, \ldots, x_m \in \mathcal{M}$ and all $y_{ij} \in \mathcal{M}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, there exists $u \in \mathcal{U}$ such that $u^{\alpha_i} x_j = y_{ij}$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.*

3. Applications

Let $\mathcal{A}$ be a left primitive ring with faithful simple left $\mathcal{A}$-module $\mathcal{M}$, with Lie ideal $\mathcal{U}$ and with automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\mathcal{D} = \operatorname{End}_{\mathcal{A}} \mathcal{M})$ and let $\mathcal{E} = \operatorname{End}(\mathcal{M}_{\mathcal{D}})$. Let X be an infinite set, let Z be the ring of integers and let $\mathcal{F}$ be the free Z-algebra on the set $\{x^{\alpha_i} \mid x \in X, i = 1, 2, \ldots, n\}$. For a monomial

$$\pi = x_{i_m}^{\alpha_{r_m}} x_{i_{m-1}}^{\alpha_{r_{m-1}}} \ldots x_{i_1}^{\alpha_{r_1}} \in \mathcal{F}$$

of "length" m, the complement P_π is defined to be the set of all monomials

$$\tau = x_{j_l}^{\alpha_{s_l}} x_{j_{l-1}}^{\alpha_{s_{l-1}}} \ldots x_{j_1}^{\alpha_{s_1}} \in \mathcal{F}$$

subject to the following condition: if $l \leq m$, then either $j_k \neq i_k$, or $s_k \neq r_k$ for some $1 \leq k \leq l$. We shall say that π is a *pivotal monomial with automorphisms on* $\mathcal{U}$ (abbreviated PMA on $\mathcal{U}$) if for any homomorphism of rings $\phi : \mathcal{F} \to \mathcal{A}$ such that $X^\phi \subseteq \mathcal{U}$ and $(x^{\alpha_s})^\phi = (x^\phi)^{\alpha_s}$ for all $x \in X$, $s = 1, 2, \ldots, n$, we have that $\pi^\phi \in \sum_{\tau \in P_\pi} \mathcal{E}\tau^\phi$. We are now in a position to prove our third main result.

Theorem 3.1. *Let $\mathcal{A}$ be a left primitive ring with faithful simple left module $\mathcal{M}$, with Lie ideal $\mathcal{U}$ and with automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are pairwise $\mathcal{M}$-independent and $\mathcal{U}$ has a PMA*

$$\pi = x_{i_m}^{\alpha_{r_m}} x_{i_{m-1}}^{\alpha_{r_{m-1}}} \ldots x_{i_1}^{\alpha_{r_1}}$$

of length $m \geq 2$. Then $\dim_{\mathcal{D}}(\mathcal{M}) \leq m+1$ and $\mathcal{A}$ is a simple Artinian ring.

Proof. Assume to the contrary that $\dim_{\mathcal{D}}(\mathcal{M}) > m+1$. Let $v_1, \ldots, v_{m+1} \in \mathcal{M}$ be linearly independent over $\mathcal{D}$. Since $\dim_{\mathcal{D}}(\mathcal{M}) > 2$, it follows from Kaplansky's theorem on primitive PI rings that $\mathcal{A}$ does not satisfy St_4 (see [16]). Therefore all the conditions of Theorem 2.7 are fulfilled and so there exist elements $a_{i_1}, a_{i_2}, \ldots, a_{i_m} \in \mathcal{U}$ such that

$$a_{i_k}^{\alpha_s} v_t = \begin{cases} v_{t+1} & \text{if } k = t,\ s = r_t \text{ and } t \leq m \\ 0 & \text{if } k \neq t,\ \text{or } s \neq r_t,\ \text{or } t = m+1 \end{cases}$$

for all $k = 1, 2, \ldots, m,\ s = 1, 2, \ldots, n$. We define a homomorphism $\phi : \mathcal{F} \to \mathcal{A}$ as follows: ϕ maps $x_{i_k}^{\alpha_s}$ to $a_{i_k}^{\alpha_s}$ for all $k = 1, 2, \ldots, m$, $s = 1, 2, \ldots, n$ and ϕ maps all other variables to 0. It follows that $\pi^\phi v_1 = v_{m+1}$ while $\tau^\phi v_1 = 0$ for all $\tau \in P_\pi$, a contradiction. The proof is complete. □

We note that the case $\mathcal{U} = \mathcal{A}$, $n = 1$ and $\alpha_1 = 1$ of the above theorem is due to Drazin [8]. We are now in a position to prove our fourth main result.

Theorem 3.2. *Let $\mathcal{A}$ be a ring, let α be an automorphism of $\mathcal{A}$, let $J(\mathcal{A})$ be the Jacobson radical of $\mathcal{A}$ and let $\mathcal{U}$ be a Lie ideal of $\mathcal{A}$ such that for every $u \in \mathcal{U}$ there is a positive integer $n = n(u)$ with $(u - u^\alpha)^n \in J(\mathcal{A})$. Suppose that $\mathcal{U}$ generates the ring $\mathcal{A}$. Then $a - a^\alpha \in J(\mathcal{A})$ for all $a \in \mathcal{A}$.*

Proof. Let $\mathcal{M}$ be a simple left $\mathcal{A}$-module and let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$. It is enough to show that $a - a^\alpha \in l(\mathcal{A}; \mathcal{M})$ for all $a \in \mathcal{A}$.

First assume that

$$u - u^\alpha \in l(\mathcal{A}; \mathcal{M}) \quad \text{for all } u \in \mathcal{U}. \tag{3}$$

It follows from Zorn's lemma that we may assume without loss of generality that $\mathcal{U}$ is a maximal among the Lie ideals of $\mathcal{A}$ generating $\mathcal{A}$ and satisfying (3). Given $u, v \in \mathcal{U}$, we have that $u - u^\alpha, v - v^\alpha \in l(\mathcal{A}; \mathcal{M})$ and so

$$uv - (uv)^\alpha = u(v - v^\alpha) + (u - u^\alpha)v^\alpha \in l(\mathcal{A}; \mathcal{M}).$$

Setting $\mathcal{V} = \mathcal{U} + \mathcal{U}^2$, we see that $\mathcal{V}$ is a Lie ideal of $\mathcal{A}$ generating $\mathcal{A}$ and satisfying (3). Therefore $\mathcal{V} = \mathcal{U}$ forcing $\mathcal{U}^2 \subseteq \mathcal{U}$ and so $\mathcal{U} = \mathcal{A}$. We see that it is enough to show that (3) is fulfilled.

Next suppose that α is $\mathcal{M}$-outer. Moreover assume that $\dim(\mathcal{M}_{\mathcal{D}}) \geq 2$ and pick a nonzero $x \in \mathcal{M}$. By Theorem 2.7 there exists $u \in \mathcal{U}$ such $ux = x$ and $u^\alpha x = 0$. But then $(u - u^\alpha)^n x = x$ for every positive integer n, contradicting our assumption. Now assume that $\dim(\mathcal{M}_{\mathcal{D}}) = 1$. Then $\overline{\mathcal{A}} = A/l(\mathcal{A}; \mathcal{M})$ is a skew field and so it has no nonzero nilpotent elements. Denote by $\overline{a}$ the image of $a \in \mathcal{A}$ in $\overline{\mathcal{A}}$. Since $(u - u^\alpha)^{n(u)} \in J(\mathcal{A})$, $\overline{u - u^\alpha}$ is nilpotent element of $\overline{\mathcal{A}}$ forcing $u - u^\alpha \in l(\mathcal{A}; \mathcal{M})$ for all $u \in \mathcal{U}$ and whence (3) is satisfied.

Therefore we may assume without loss of generality that α is $\mathcal{M}$-inner. That is there exists an invertible element $T \in \mathrm{End}(\mathcal{M})$ such that $a^\alpha x = T^{-1}aTx$ for all $x \in \mathcal{M}$ and $a \in \mathcal{A}$. It is now easy to see that $l(\mathcal{A}; \mathcal{M})^\alpha = l(\mathcal{A}; \mathcal{M})$. Replacing $\mathcal{A}$ by $\mathcal{A}/l(\mathcal{A}; \mathcal{M})$ we reduce the proof to the case when $\mathcal{A}$ is a left primitive ring with simple faithful left module $\mathcal{M}$ and with Lie ideal $\mathcal{U}$ generating $\mathcal{A}$ such that $(u - u^\alpha)^{n(u)} = 0$ for all $u \in \mathcal{U}$. In view of (3) it is enough to show that $\alpha = 1$.

Suppose that there is $x \in \mathcal{M}$ with x and Tx are linearly independent over $\mathcal{D}$. There are two cases to consider.

Case 1. Suppose that $\dim(\mathcal{M}_{\mathcal{D}}) > 2$. Then according to Theorem 2.4 there exists $u \in \mathcal{U}$ with $ux = x$ and $uTx = 0$. It follows that $(u - u^\alpha)^n x = x$ for any positive integer n, a contradiction.

Case 2. Assume that $\dim(\mathcal{M}_\mathcal{D}) = 2$. Then $\{x, Tx\}$ is a basis of $\mathcal{M}$ over $\mathcal{D}$. Clearly that $\mathcal{A} = M_2(\mathcal{D}) = \mathrm{End}(\mathcal{M}_\mathcal{D})$ is a simple ring. Since $\mathcal{U}$ generates $\mathcal{A}$, it follows from [2, Lemma 9.1.2] that $[\mathcal{A}, \mathcal{A}] \subseteq \mathcal{U}$. As $\mathcal{A} = M_2(\mathcal{D})$, $a^2 = 0$ for any nilpotent element $a \in \mathcal{A}$ and so $([x, y] - [x, y]^\alpha)^2 = 0$ for all $x, y \in \mathcal{A}$ is a generalized polynomial identity on $\mathcal{A}$ (see [2, Chapter 7]). If α is X-outer, then $([x, y] - [u, v])^2 = 0$ for all $x, y, u, v \in \mathcal{A}$ by Kharchenko's theorem [2, Theorem 7.5.9]. Taking $u = 0 = v$, we see that $[x, y]^2 = 0$ for all $x, y \in \mathcal{A}$ which is impossible. Therefore α is X-inner and so there exists an invertible element $t \in \mathcal{A}$ with $a^\alpha = t^{-1}at$ for all $a \in \mathcal{A}$. Clearly $t^{-1}T$ commutes with all elements of the ring $\mathcal{A}$ and so there exists $0 \neq \lambda \in \mathcal{D}$ with $t^{-1}Ty = y\lambda$ for all $y \in \mathcal{M}$. We conclude that $\{x, tx\}$ is a basis of $\mathcal{M}$ over $\mathcal{D}$.

Clearly $\{t, 1\}$ and $\{t^{-1}, 1\}$ are linearly independent over $\mathcal{C} = Z(\mathcal{A})$. Therefore $(x_1 - t^{-1}x_1t) \circ (x_2 - t^{-1}x_2t)$ is a nonzero generalized polynomial, where $u \circ v = uv + vu$. Linearizing $(a - t^{-1}at)^2 = 0$, we get that

$$(a - t^{-1}at) \circ (b - t^{-1}bt) = 0 \quad \text{for all } a, b \in \mathcal{U}$$

and so $([x_1, x_2] - t^{-1}[x_1, x_2]t) \circ ([x_3, x_4] - t^{-1}[x_3, x_4]t)$ is a nontrivial GPI on $\mathcal{A}$. By Martindale's theorem [2, Theorem 6.1.6], $\dim_\mathcal{C}(\mathcal{A}) < \infty$.

We claim that we may assume without loss of generality that $\mathcal{A} = M_n(\mathcal{C})$ for some $n \geq 2$. Indeed, if $|\mathcal{C}| < \infty$, then $\mathcal{D} = \mathcal{C}$ by Wedderburn's theorem on finite skew fields [2, Theorem 4.2.3].

Suppose that $|\mathcal{C}| = \infty$. Let $\overline{\mathcal{C}}$ be the algebraic closure of $\mathcal{C}$. Set $\overline{\mathcal{A}} = \mathcal{A} \otimes_\mathcal{C} \overline{\mathcal{C}}$ and $\overline{\mathcal{U}} = \mathcal{U} \otimes_\mathcal{C} \overline{\mathcal{C}}$. Clearly $\overline{\mathcal{U}} \supseteq [\overline{\mathcal{A}}, \overline{\mathcal{A}}]$ is a Lie ideal of $\overline{\mathcal{A}}$ generating $\overline{\mathcal{A}}$. According to [2, Corollary 4.2.2], $\overline{\mathcal{A}} = M_n(\overline{\mathcal{C}})$ where $n^2 = \dim_\mathcal{C}(\mathcal{A})$. Next, $\{a - (t \otimes 1)^{-1}a(t \otimes 1)\}^2 = 0$ for all $a \in \overline{\mathcal{U}}$ by [5, Lemma 2.3] which proves our claim.

Since $\mathcal{A} = \mathcal{C} + [\mathcal{A}, \mathcal{A}] = \mathcal{C} + \mathcal{U}$, we conclude that $(a - t^{-1}at)^2 = 0$ for all $a \in \mathcal{A}$. Recalling that $\{x, tx\}$ are linearly independent over $\mathcal{C}$, we pick $a \in \mathcal{A}$ with $ax = x$ and $atx = 0$. Then $(a - t^{-1}at)^2x = x \neq 0$, a contradiction.

This shows that x and Tx are linearly dependent over $\mathcal{D}$ for any $x \in \mathcal{M}$. Assume that $\dim(\mathcal{M}_\mathcal{D}) > 1$. Then by [3, Lemma 7.1] there exists

$\lambda \in \mathcal{D}$ such that $Tx = x\lambda$ for all $x \in \mathcal{M}$. Given $a \in \mathcal{A}$ and $x \in \mathcal{M}$, we now have that

$$a^\alpha x = T^{-1}aTx = T^{-1}a(x\lambda) = T^{-1}[(ax)\lambda] = ax$$

and so $(a - a^\alpha)\mathcal{M} = 0$ forcing $\alpha = 1$.

Finally, assume that $\dim(\mathcal{M}_\mathcal{D}) = 1$. Then $\mathcal{A}$ is a skew field. In particular it has no nonzero nilpotent elements and so $u = u^\alpha$ for all $u \in \mathcal{U}$. Since $\mathcal{U}$ generates $\mathcal{A}$, we conclude that $\alpha = 1$. The proof is thereby complete. □

Corollary 3.3 [6, Theorem 5]). *Let $\mathcal{A}$ be a semiprime ring and let α be an automorphism of $\mathcal{A}$. If $(a - a^\alpha)^n = 0$ for every $a \in \mathcal{A}$, where n is a fixed integer, then $\alpha = 1$.*

Proof. The reduction to the prime case is easy (see [6, p. 232]). Thus, assume that $\mathcal{A}$ is prime. Clearly $(x - x^\alpha)^n$ is a generalized identity on $\mathcal{A}$ and so it is a generalized identity on the maximal right ring of quotients Q_{mr} of $\mathcal{A}$ by [2, Theorem 7.8.7]. If α is not X-inner, then Q_{mr} is a primitive ring with nonzero socle by [2, Theorem 7.8.4]. Therefore $a - a^\alpha \in J(Q_{mr}) = 0$ by Theorem 3.2 and whence $\alpha = 1$, a contradiction. Consequently, α is X-inner and so there exists an invertible element $t \in Q_{mr}$ such that $a^\alpha = t^{-1}at$ for all $a \in \mathcal{A}$. We now have that $(x - t^{-1}xt)^n$ is a generalized polynomial identity on $\mathcal{A}$. Let C be the extended centroid of $\mathcal{A}$. Assume that $t \notin C$. Let $\mathcal{F} = Q_{mr} * C[x]$ be the free product of C-algebras Q_{mr} and $C[x]$. We denote by V the C-subspace of $\mathcal{F}$ generated by all the generalized monomials in x of the form

$$t^{\epsilon_1} x t^{\epsilon_2} \dots x t^{\epsilon_{n+1}}$$

where

$$\epsilon_i \in \begin{cases} \{-1, 0, 1\} & \text{if } \{1, t, t^{-1}\} \text{ are linearly independent} \\ \{0, 1\} & \text{if } \{1, t, t^{-1}\} \text{ are linearly dependent} \end{cases}$$

and $\sum_{i=1}^{n+1} |\epsilon_i| \neq 0$. Clearly $x^n \notin V$ and so $(x - t^{-1}xt)^n$ is a nontrivial GPI on $\mathcal{A}$. By Martindale Theorem [2, Corollary 6.17], Q is a primitive ring with nonzero socle and as before this contradicts Theorem 3.2. Thus $t \in C$ and so $\alpha = 1$. □

Acknowledgements. The paper has been written while the second-named author was visiting the National Cheng-Kung University. He would like to express his deep gratitude to the University for its hospitality and for the financial support of his visit.

The authors would also like to thank the referee for valuable suggestions.

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(Received December 21, 1998; revised April 25, 2000)