Publ. Math. Debrecen
58 / 3 (2001), 337–351

A note on the weak subalgebra lattice of a unary algebra with constants

By KONRAD PIÓRO (Warsaw)

Abstract. We prove that for a locally finite (total) algebra **A** having finitely many nullary and unary operations only, its weak subalgebra lattice uniquely determines its (strong) subalgebra lattice. More precisely, we show that for every partial algebra **B** of the same type, if the weak subalgebra lattices of **A** and **B** are isomorphic (with **A** as above), then the (strong) subalgebra lattices of **A** and **B** are isomorphic, and **B** is also total and locally finite.

1. Introduction

Investigations of relationships between (total) algebras or varieties of algebras and their subalgebra lattices are an important part of universal algebra. For instance, characterizations of subalgebra lattices for algebras in a given variety or of a given type are this kind of problems (see e.g. [11]). Moreover, several results (see e.g. [7], [14], [17], [18]) describe algebras or varieties of algebras which have special subalgebra lattices (i.e. modular, distributive, etc.). For example, T. Evans and B. Ganter proved in [7] that an arbitrary subalgebra modular variety (i.e. a variety in which every algebra has a modular subalgebra lattice) is Hamiltonian (i.e. any subalgebra is a congruence class of a suitable congruence); hence and by [12], it is Abelian. Moreover, J. Shapiro showed in [17] that every subalgebra distributive variety (i.e. each of its algebras has a distributive subalgebra lattice) is strongly Abelian. Note that some such results concern also classical algebras–Boolean algebras, groups, modules (see e.g. [13], [16] or [9],

Mathematics Subject Classification: 05C20, 05C90, 08A60, 08A30, 05C99, 08A55.
Key words and phrases: digraph, strong and weak subalgebras, subalgebra lattices, unary algebra, partial algebra.

[10]). For example, D. Sachs showed in [16] that two Boolean algebras are isomorphic iff their lattices of subalgebras are isomorphic; E. Lukács and P.P. Palfy proved in [13] that the modularity of the subgroup lattice of the direct square of any group implies that $\mathbf{G}$ is commutative.

The theory of partial algebras provides additional tools for such investigation, since several different structures may be considered in this case (see e.g. [4], [6]). In this paper we consider only two types of partial subalgebras. First, we have subalgebras defined as in the total case (such subalgebras will here be called strong as opposed to the other kinds of partial subalgebras). The second notion is that of the weak subalgebra: a partial algebra $\mathbf{B} = \langle B, (k^{\mathbf{B}})_{k\in K}\rangle$ is a weak subalgebra of a partial algebra $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ of the same type iff $B \subseteq A$ and $k^{\mathbf{B}} \subseteq k^{\mathbf{A}}$ for $k \in K$; the set of all weak subalgebras of $\mathbf{A}$ forms an algebraic lattice under (weak subalgebra) inclusion $\leq_w$, which will be denoted by $\mathbf{S}_w(\mathbf{A})$. Analogously, the strong subalgebra lattice of $\mathbf{A}$ will be denoted by $\mathbf{S}_s(\mathbf{A})$.

It seems that the weak subalgebra lattice alone, and also together with the strong subalgebra lattice, yields a lot of interesting information on an algebra, also in the case of total algebras. Several results of this kind are already known (see e.g. [2], [3] and [15]). Moreover, in [15] we introduce a graph-algebraic language which is very useful in the solution of some problems of subalgebra lattices of partial unary algebras. For instance, we characterized in [15] arbitrary two partial unary algebras having isomorphic weak subalgebra lattices. In the present paper we use this graph language and other (little generalized) results from [15] to prove the following result: Let $\mathbf{A}$ be a total and locally finite algebra having finitely many constants and unary operations only. Then for every partial algebra $\mathbf{B}$ of the same type, if the weak subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ are isomorphic, then their strong subalgebra lattices are isomorphic, and $\mathbf{B}$ is also total and locally finite.

2. Basic definitions

For basic notions and facts concerning partial algebras and partial subalgebras and lattices of such subalgebras see e.g. [4] or [6]; concerning (total) algebras and lattices of (total) subalgebras see e.g. [8] or [11]; concerning digraphs (directed graphs) and (undirected) graphs see e.g. [5].

We will use digraphs and graphs to represent partial unary algebras with constants. Therefore first, we consider digraphs with constants, where constants are defined as in algebras, i.e. they can be identified with some fixed vertices (of course two different constants may be represented by the same vertex); constants of a digraph $\mathbf{D}$ will be denoted by $c_1^{\mathbf{D}}, c_2^{\mathbf{D}}, \ldots$, and the set of all constants by $\mathrm{Cons}(\mathbf{D})$. Secondly, we consider also infinite digraphs and graphs, i.e. the sets $V^{\mathbf{D}}$ and $E^{\mathbf{D}}$ of vertices and edges of a digraph $\mathbf{D}$ may be infinite and of arbitrary cardinalities. Recall (see [15], where this construction is given for unary algebras) that each partial algebra $\mathbf{A} = \langle A, (c^{\mathbf{A}})_{c\in C}, (k^{\mathbf{A}})_{k\in K}\rangle$ of type (C, K) (where C is a set of constant symbols and K is a set of unary operation symbols) can be represented by the digraph with constants $D(\mathbf{A})$ as follows[1]: $V^{\mathbf{D}(\mathbf{A})} := A$, $E^{\mathbf{D}(\mathbf{A})} := \{(a, k, b) \in A \times K \times A : (a, b) \in k^{\mathbf{A}}\}$ and for each edge (a, k, b), a is its initial vertex and b is its final vertex, and moreover, all defined constants of $\mathbf{A}$ are constants of $\mathbf{D}(\mathbf{A})$, i.e. $c^{\mathbf{D}(\mathbf{A})} := c^{\mathbf{A}}$ for each $c \in C$ such that $c^{\mathbf{A}}$ is defined in $\mathbf{A}$ (recall that in a partial algebra some, and even all, of its constants can be undefined). Note that this construction for partial unary algebras (without constants) is a very particular case of the Grothendieck construction (see [1], Section 4.2 and 11.2). Observe also that with every digraph $\mathbf{D}$ with constants we can associate the (undirected) graph $\mathbf{D}^*$ by omitting the orientation of all edges of $\mathbf{D}$ and by replacing each constant $c^{\mathbf{D}} \in \mathrm{Cons}(\mathbf{D})$ by a new loop, say (v, c, v), in its vertex v. Thus with each partial unary algebra $\mathbf{A}$ with constants we can associate the graph $\mathbf{D}^*(\mathbf{A}) := (\mathbf{D}(\mathbf{A}))^*$.

Let $\mathbf{D}$ be a digraph with constants and $v \in V^{\mathbf{D}}$. Then $E_{sr}^{\mathbf{D}}(v)$ is the set of all regular edges (i.e. with different endpoints) starting from v; $E_{er}^{\mathbf{D}}(v)$ is the set of all regular edges ending in v; $E_{sl}^{\mathbf{D}}(v)$ is the set of all loops in v; and $\mathrm{cons}^{\mathbf{D}}(v)$ is the set of all constants in v. Moreover, we define the cardinal numbers $sr^{\mathbf{D}}(v) = |E_{sr}^{\mathbf{D}}(v)|$, $sl^{\mathbf{D}}(v) = |E_{sl}^{\mathbf{D}}(v)|$, $s^{\mathbf{D}}(v) = |E_{sr}^{\mathbf{D}}(v) \cup E_{sl}^{\mathbf{D}}(v)|$, $er^{\mathbf{D}}(v) = |E_{er}^{\mathbf{D}}(v)|$ and $cn^{\mathbf{D}}(v) = |\,\mathrm{cons}^{\mathbf{D}}(v)|$ (where $|A|$ denotes the cardinality of a set A). Now we can define the type of digraphs with constants (recall that the digraph type is introduced in [15]). Let $\mathbf{D}$ be a digraph with constants and (η_1, η_2) a pair of cardinal numbers. Then $\mathbf{D}$ is of type (η_1, η_2) iff $|\,\mathrm{Cons}(\mathbf{D})| \leq \eta_1$ and $s^{\mathbf{D}}(v) \leq \eta_2$ for each $v \in V^{\mathbf{D}}$. We say that $\mathbf{D}$ of type (η_1, η_2) is total iff $|\,\mathrm{Cons}(\mathbf{D})| = \eta_1$ and $s^{\mathbf{D}}(v) = \eta_2$ for $v \in V^{\mathbf{D}}$. A type (η_1, η_2) is finite iff $\eta_1, \eta_2 \in \mathbb{N}$ (where $\mathbb{N}$ is the set

[1]The concept of "digraph" used in this paper is often called "multidigraph with loops", or even "edge-coloured multidigraph with loops" by other authors. Note that the edge colouring is not essential for our proofs.

of all non-negative integers). It is not difficult to see (the precise proof in the case of unary algebras is given in [15]) that for any partial unary algebra with constants $\mathbf{A} = \langle A, (c^{\mathbf{A}})_{k\in C}, (k^{\mathbf{A}})_{k\in K}\rangle$, its digraph $\mathbf{D}(\mathbf{A})$ with constants is of (digraph) type $(|C|, |K|)$. Conversely, for an algebraic type (C, K) and each digraph $\mathbf{D}$ with constants of type $(|C|, |K|)$, it is easily shown (see also [15]) that there is a partial algebra $\mathbf{A}$ of the type (C, K) such that $\mathbf{D}(\mathbf{A}) \simeq \mathbf{D}$. Moreover, for each partial algebra $\mathbf{A}$ of finite type (C, K) (i.e. C and K are finite sets), $\mathbf{A}$ is total iff $\mathbf{D}(\mathbf{A})$ is total.

Recall (see [15]) that two kinds of subdigraphs can be defined which represent weak and strong subalgebras of a partial unary algebra; moreover, each of these two families forms an algebraic lattice. The case of algebras with constants is analogous. More precisely, the first kind is formed by usual subdigraphs (with constants) which will be called weak to stress its relation with weak subalgebras. We assume that the empty digraph is also a weak subdigraph (analogously as for partial algebras). Secondly, we say that a digraph $\mathbf{H}$ with constants is a strong subdigraph of a digraph $\mathbf{D}$ with constants iff $\mathbf{H}$ is a weak subdigraph of $\mathbf{D}$ and $\mathrm{Cons}(\mathbf{D}) = \mathrm{Cons}(\mathbf{H})$ and for each edge e of $\mathbf{D}$, if the initial vertex of e belongs to $\mathbf{H}$, then e belongs to $\mathbf{H}$ (in particular, its final vertex also belongs to $\mathbf{H}$). Note that the empty digraph is a strong subdigraph of $\mathbf{D}$ iff $\mathrm{Cons}(\mathbf{D}) = \emptyset$. Note also that we call a subdigraph "strong" when it represent a strong subalgebra. It can be proved, in an analogous way as for partial algebras (note also that the precise proof for digraphs without constants is given in [15] and moreover, our case is similar), that the sets of all weak and strong subdigraphs of $\mathbf{D}$ form complete lattices under (weak and strong subdigraph) inclusion $\leq_w$ and $\leq_s$, respectively; they will be denoted by $\mathbf{S}_w(\mathbf{D})$ and $\mathbf{S}_s(\mathbf{D})$. The operations of infimum $\bigwedge$ and supremum $\bigvee$ are defined as for partial algebras. In particular, for any subset $W \subseteq V^{\mathbf{D}}$ of vertices, there is the least strong subdigraph of $\mathbf{D}$ containing W, which will be denoted by $\langle W\rangle_{\mathbf{D}}$. Analogously as for algebras, we say that a digraph $\mathbf{D}$ is locally finite iff for any finite $W \subseteq V^{\mathbf{D}}$, the strong subdigraph $\langle W\rangle_{\mathbf{D}}$ generated by W is finite (i.e. the vertex set $V^{\langle W\rangle_{\mathbf{D}}}$ is finite). Moreover, it can be proved, in a similar way as in [15], that the function assigning to each weak or strong subalgebra $\mathbf{B}$ of a partial unary algebra $\mathbf{A}$ with constants its digraph $\mathbf{D}(\mathbf{B})$, which is, of course, a weak or strong subdigraph of $\mathbf{D}(\mathbf{A})$, respectively, forms lattice isomorphisms. Thus for each partial unary algebra $\mathbf{A}$ with constants, $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{D}(\mathbf{A}))$ and $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{D}(\mathbf{A}))$.

It easily follows from this result and its proof (see also [15]) that for each partial unary algebra $\mathbf{A}$ with constants and $B \subseteq A$, $\mathbf{D}\big(\langle B\rangle_{\mathbf{A}}\big) = \langle B\rangle_{\mathbf{D}(\mathbf{A})}$ (where $\langle B\rangle_{\mathbf{A}}$ denotes the strong subalgebra of $\mathbf{A}$ generated by B). Hence we infer that $\mathbf{A}$ is locally finite iff $\mathbf{D}(\mathbf{A})$ is a locally finite digraph.

Recall also (see [2], where it is proved for unary algebras, but the proof of this case is the same) that the weak subdigraph lattice of any digraph $\mathbf{D}$ with constants is uniquely determined by $\mathbf{D}^*$. More formally, for any digraphs $\mathbf{D}$ and $\mathbf{G}$ with constants, $\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{S}_w(\mathbf{G})$ iff $\mathbf{D}^* \simeq \mathbf{G}^*$. Hence we deduce that for each partial unary algebras $\mathbf{A}$ and $\mathbf{B}$ with constants (which can be even of different types), $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B})$ iff $\mathbf{D}^*(\mathbf{A}) \simeq \mathbf{D}^*(\mathbf{B})$.

Now observe that the above results and definitions reduce our algebraic problem to some digraph question. More precisely, partial algebras $\mathbf{A}$ and $\mathbf{B}$, with finitely many nullary and unary operations $c_1, \ldots, c_n$ and $f_1, \ldots, f_m$, can be replaced by digraphs $\mathbf{D}$ and $\mathbf{G}$ with constants of finite type (n, m); assumptions on $\mathbf{A}$ are translated into the digraph language as follows: $\mathbf{D}$ is a total digraph with constants and locally finite; moreover, the property that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B})$ is equivalent to the condition that $\mathbf{D}^*$ and $\mathbf{G}^*$ are isomorphic. Thus to prove our algebraic result we must only show the following graph fact: Let $\mathbf{D}$ be a total digraph with constants of finite type (n_1, n_2) and locally finite, and let $\mathbf{G}$ be a digraph of type (n_1, n_2) such that $\mathbf{D}^* \simeq \mathbf{G}^*$; then the strong subdigraph lattices of $\mathbf{D}$ and $\mathbf{G}$ are isomorphic, and $\mathbf{G}$ is also total and locally finite.

3. Preliminary results

Let $\mathbf{D}$ be a digraph with constants, $F \subseteq E^{\mathbf{D}}_{reg}$, $L \subseteq E^{\mathbf{D}}_{lo}$ and $C \subseteq \mathrm{Cons}(\mathbf{D})$ (where $E^{\mathbf{D}}_{reg}$ and $E^{\mathbf{D}}_{lo}$ are sets of all regular edges and loops of $\mathbf{D}$, respectively). Then $\mathbf{D}(F; L; C)$ is the new digraph with constants obtained from $\mathbf{D}$ by inverting the orientation of all the edges in F and by replacing all the loops in L by constants and by replacing all the constants in C by loops. This simple construction of new digraphs with constants is important for us, because it holds:

Lemma 1. *Let $\mathbf{D}$ and $\mathbf{H}$ be digraphs with constants such that $\mathbf{D}^* \simeq \mathbf{H}^*$. Then there are $F \subseteq E^{\mathbf{D}}_{reg}$, $L \subseteq E^{\mathbf{D}}_{lo}$ and $C \subseteq \mathrm{Cons}(\mathbf{D})$ such that $\mathbf{H} \simeq \mathbf{D}(F; L; C)$.*

Proof. Let φ be an isomorphism of graphs $\mathbf{D}^*$ and $\mathbf{H}^*$. Then it is sufficient to take the set F of all regular edges e such that the image of the initial vertex of e and the initial vertex of the image of e are different; and the set L of all loops such that their images are constants in $\mathbf{H}$; and the set C of all constants such that their images are loops in $\mathbf{H}$. $\square$

Before the next lemmas note that in this paper we assume that a path in a digraph has pairwise different and regular edges (however, its vertices need not be different); we also assume that a single vertex forms the trivial path (i.e. without edges). We say that a cycle is simple iff it has at least one edge and its edges are regular and pairwise different (but again, its vertices need not be different). The sets of all edges and vertices of a path or cycle p will be denoted by E^p and V^p, respectively. We say that a path or cycle p and a path or cycle r are disjoint (edge-disjoint) iff $V^p \cap V^r = \emptyset$ ($E^p \cap E^r = \emptyset$). Moreover, for each edge e of a digraph $\mathbf{D}$, by $I_1^{\mathbf{D}}(e)$ and $I_2^{\mathbf{D}}(e)$ we denote the initial and the final vertex of e, respectively.

Lemma 2. *Let $\mathbf{D}$ be a digraph with finitely many edges, $c^{\mathbf{D}}$ be a constant of $\mathbf{D}$, l be a loop, and p be a path going from $c^{\mathbf{D}}$ to $I_1^{\mathbf{D}}(l)$. Let $\mathbf{H}$ be the digraph obtained from $\mathbf{D}$ by omitting $c^{\mathbf{D}}$ and l and all edges of p. Then for any $v \in V^{\mathbf{D}}$,*

$$\text{if} \quad er^{\mathbf{D}}(v) + cn^{\mathbf{D}}(v) \leq sr^{\mathbf{D}}(v) + sl^{\mathbf{D}}(v),$$

$$\text{then} \quad er^{\mathbf{H}}(v) + cn^{\mathbf{H}}(v) \leq sr^{\mathbf{H}}(v) + sl^{\mathbf{H}}(v).$$

Proof. Take an arbitrary $v \in V^{\mathbf{D}}$ such that $er^{\mathbf{D}}(v) + cn^{\mathbf{D}}(v) \leq sr^{\mathbf{D}}(v) + sl^{\mathbf{D}}(v)$.

If $v \notin V^p$, then $sr^{\mathbf{H}}(v) = sr^{\mathbf{D}}(v)$, $er^{\mathbf{H}}(v) = er^{\mathbf{D}}(v)$, $sl^{\mathbf{H}}(v) = sl^{\mathbf{D}}(v)$ and $cn^{\mathbf{H}}(v) = cn^{\mathbf{D}}(v)$. Thus obviously $er^{\mathbf{H}}(v)+cn^{\mathbf{H}}(v) \leq sr^{\mathbf{H}}(v)+sl^{\mathbf{H}}(v)$.

If $v \in V^p \setminus \{c^{\mathbf{D}}, I_1^{\mathbf{D}}(l)\}$, then $sl^{\mathbf{H}}(v) = sl^{\mathbf{D}}(v)$ and $cn^{\mathbf{H}}(v) = cn^{\mathbf{D}}(v)$; and by the definition of path we have $|E^p \cap E_{sr}^{\mathbf{D}}(v)| = |E^p \cap E_{er}^{\mathbf{D}}(v)|$. Hence, $er^{\mathbf{H}}(v) + cn^{\mathbf{H}}(v) = er^{\mathbf{D}}(v) - |E^p \cap E_{er}^{\mathbf{D}}(v)| + cn^{\mathbf{D}}(v) \leq sr^{\mathbf{D}}(v) - |E^p \cap E_{er}^{\mathbf{D}}(v)| + sl^{\mathbf{D}}(v) = sr^{\mathbf{D}}(v) - |E^p \cap E_{sr}^{\mathbf{D}}(v)| + sl^{\mathbf{D}}(v) = sr^{\mathbf{H}}(v) + sl^{\mathbf{H}}(v)$.

If the constant $c_1^{\mathbf{D}}$ is defined in v (i.e. $c_1^{\mathbf{D}} \in \mathrm{cons}^{\mathbf{D}}(v)$) and $v \neq I_1^{\mathbf{D}}(l)$, then (again applying the concept of path) we obtain $|E^p \cap E_{sr}^{\mathbf{D}}(v)| = |E^p \cap E_{er}^{\mathbf{D}}(v)| + 1$. We have also $cn^{\mathbf{D}}(v) = cn^{\mathbf{D}}(v) - 1$ and $sl^{\mathbf{H}}(v) = sl^{\mathbf{D}}(v)$ (because $I_1^{\mathbf{D}}(l) \neq v$). Thus $er^{\mathbf{H}}(v) + cn^{\mathbf{H}}(v) = er^{\mathbf{D}}(v) - |E^p \cap E_{er}^{\mathbf{D}}(v)| + cn^{\mathbf{D}}(v) - 1 = er^{\mathbf{D}}(v) - \big(|E^p \cap E_{er}^{\mathbf{D}}(v)| + 1\big) + cn^{\mathbf{D}}(v) \leq sr^{\mathbf{D}}(v) - \big(|E^p \cap E_{er}^{\mathbf{D}}(v)| + 1\big) + sl^{\mathbf{D}}(v) = sr^{\mathbf{D}}(v) - |E^p \cap E_{sr}^{\mathbf{D}}(v)| + sl^{\mathbf{D}}(v) = sr^{\mathbf{H}}(v) + sl^{\mathbf{H}}(v)$.

If $c^{\mathbf{D}}$ is defined in v and $v = I_1^{\mathbf{D}}(l)$, then p is a trivial path. Thus $sr^{\mathbf{D}}(v) = sr^{\mathbf{H}}(v)$, $er^{\mathbf{D}}(v) = er^{\mathbf{H}}(v)$, and $cn^{\mathbf{D}}(v) = cn^{\mathbf{H}}(v) - 1$, $sl^{\mathbf{D}}(v) = sl^{\mathbf{H}}(v) - 1$. Hence we get $er^{\mathbf{H}}(v) + cn^{\mathbf{H}}(v) = er^{\mathbf{D}}(v) + cn^{\mathbf{D}}(v) - 1 \leq sr^{\mathbf{D}}(v) + sl^{\mathbf{D}}(v) - 1 = sr^{\mathbf{H}}(v) + sl^{\mathbf{H}}(v)$.

Finally, assume $v = I_1^{\mathbf{D}}(l)$ and $I_1^{\mathbf{D}}(l) \neq c^{\mathbf{D}}$. Then, $sl^{\mathbf{H}}(v) = sl^{\mathbf{D}}(v) - 1$ and $cn^{\mathbf{H}}(v) = cn^{\mathbf{D}}(v)$ and moreover, we obtain (in an analogous way as previously) $|E^p \cap E_{er}^{\mathbf{D}}(v)| = |E^p \cap E_{sr}^{\mathbf{D}}(v)| + 1$. Thus $er^{\mathbf{H}}(v) + cn^{\mathbf{H}}(v) = er^{\mathbf{D}}(v) - |E^p \cap E_{er}^{\mathbf{D}}(v)| + cn^{\mathbf{D}}(v) \leq sr^{\mathbf{D}}(v) - |E^p \cap E_{er}^{\mathbf{D}}(v)| + sl^{\mathbf{D}}(v) = sr^{\mathbf{D}}(v) - \big(|E^p \cap E_{sr}^{\mathbf{D}}(v)| + 1\big) + sl^{\mathbf{D}}(v) = \big(sr^{\mathbf{D}}(v) - |E^p \cap E_{sr}^{\mathbf{D}}(v)|\big) + \big(sl^{\mathbf{D}}(v) - 1\big) = sr^{\mathbf{H}}(v) + sl^{\mathbf{H}}(v)$.

These cases complete the proof. □

Lemma 3. *Let* $\mathbf{D}$ *be a digraph with finitely many edges such that*

(∗) $\mathbf{D}$ *has exactly* m *constants* $c_1^{\mathbf{D}}, \ldots, c_m^{\mathbf{D}}$,

(∗∗) *for each* $v \in V^{\mathbf{D}}$, $er^{\mathbf{D}}(v) + cn^{\mathbf{D}}(v) \leq sr^{\mathbf{D}}(v) + sl^{\mathbf{D}}(v)$.

Then there are m *pairwise different loops* $l_1, \ldots, l_m$ *and* m *pairwise edge-disjoint paths* $p_1, \ldots, p_m$ *such that* p_i *goes from* $c_i^{\mathbf{D}}$ *to* $I_1^{\mathbf{D}}(l_i)$, *for each* $1 \leq i \leq m$.

Proof. Take $c_1^{\mathbf{D}}$ and observe that if there is a loop l in $c_1^{\mathbf{D}}$, then we must only take $l_1 = l$ and the trivial path in this vertex. Thus we can assume that there is no loop in $c_1^{\mathbf{D}}$. Then by (∗∗) there is a regular edge e_1 starting from $c_1^{\mathbf{D}}$.

Now assume that $p := (e_1, \ldots, e_i)$ is a path starting from $c_1^{\mathbf{D}}$. Then, of course, there is a loop in $I_2^{\mathbf{D}}(e_i)$ or not. In the second case, using (∗∗), we obtain (because $e_1, \ldots, e_i$ are pairwise distinct) $|E^p \cap E_{sr}^{\mathbf{D}}(I_2^{\mathbf{D}}(e_i))| + 1 = |E^p \cap E_{er}^{\mathbf{D}}(I_2^{\mathbf{D}}(e_i))| \leq er^{\mathbf{D}}(I_2^{\mathbf{D}}(e_i)) \leq er^{\mathbf{D}}(I_2^{\mathbf{D}}(e_i)) + cn^{\mathbf{D}}(I_2^{\mathbf{D}}(e_i)) \leq sr^{\mathbf{D}}(I_2^{\mathbf{D}}(e_i))$, which implies that there is a regular edge e_{i+1} starting from $I_2^{\mathbf{D}}(e_i)$ and $e_{i+1} \notin \{e_1, \ldots, e_i\}$, so $(e_1, \ldots, e_i, e_{i+1})$ is a path starting from $c_1^{\mathbf{D}}$. Since $\mathbf{D}$ has only finitely many edges, the above facts imply that there is a loop l_1 and there is a path p_1 going from $c_1^{\mathbf{D}}$ to $I_1^{\mathbf{D}}(l_1)$. Thus the lemma holds for $m = 1$.

For $m \geq 2$ we apply induction on m. Take the digraph $\mathbf{H}$ obtained from $\mathbf{D}$ by omitting $c_1^{\mathbf{D}}$ and l_1 and all the edges of p_1. Then $\mathbf{H}$ has $m - 1$ constants, and it satisfies (∗∗), by Lemma 2.

Now by the induction hypothesis, there are pairwise different loops $l_2, \ldots, l_m$ and pairwise edge-disjoint paths $p_2, \ldots, p_m$ in $\mathbf{H}$ (and thus also in $\mathbf{D}$) such that p_i goes from $c_i^{\mathbf{H}} = c_i^{\mathbf{D}}$ to $I_1^{\mathbf{D}}(l_i)$ for $i = 2, \dots, m$. Then $l_1, l_2, \dots, l_m$ and $p_1, p_2, \dots, p_m$ are the desired loops and paths in $\mathbf{D}$. □

Let $\mathbf{D}$ be a digraph with constants and P a family of paths and cycles of $\mathbf{D}$. In the sequel we will use the following notations: $E^P = \bigcup_{p \in P} E^p$ and $V^P = \bigcup_{p \in P} V^p$, and $\mathbf{D}(P; L; C) := \mathbf{D}(E^P; L; C)$ for any $L \subseteq E_{lo}^{\mathbf{D}}$ and $C \subseteq \mathrm{Cons}(\mathbf{D})$. Families P and R of paths and cycles are disjoint (edge-disjoint) iff $V^P \cap V^R = \emptyset$ ($E^P \cap E^R = \emptyset$).

Lemma 4. *Let digraphs $\mathbf{D}$ and $\mathbf{G}$ with constants satisfy the following conditions (where $m, n \in \mathbb{N}$):*

- $(*)$ *$\mathbf{D}$ is a total digraph with constants of finite type (m, n) and locally finite,*
- $(**)$ *$\mathbf{G}$ is a digraph with constants of finite type (m, n),*
- $(***)$ *$\mathbf{D}^* \simeq \mathbf{G}^*$.*

Then there is a family R of pairwise disjoint simple cycles and there are k pairwise different constants $c_1^{\mathbf{D}}, \dots, c_k^{\mathbf{D}}$ and k pairwise different loops $l_1, \dots, l_k$ and k pairwise edge-disjoint paths $p_1, \dots, p_k$ such that:

- (i) *for each $1 \leq i \leq k$, p_i goes from $c_i^{\mathbf{D}}$ to $I_1^{\mathbf{D}}(l_i)$,*
- (ii) *families R and $\{p_1, \dots, p_k\}$ are edge-disjoint,*
- (iii) *$\mathbf{G} \simeq \mathbf{D}(R \cup \{p_1, \dots, p_k\}; \{l_1, \dots, l_k\}; \{c_1^{\mathbf{D}}, \dots, c_k^{\mathbf{D}}\})$.*

Proof. By $(***)$ and Lemma 1 there are sets $F \subseteq E_{reg}^{\mathbf{D}}$ and $L \subseteq E_{lo}^{\mathbf{D}}$ and $C = \{c_1^{\mathbf{D}}, \dots, c_k^{\mathbf{D}}\} \subseteq \mathrm{Cons}(\mathbf{D})$ such that $\mathbf{G} \simeq \mathbf{D}(F; L; C)$.

Take $v \in V^{\mathbf{D}}$ and observe $E_{sr}^{\mathbf{D}(F;L;C)}(v) \cup E_{sl}^{\mathbf{D}(F;L;C)}(v) = \big((E_{sr}^{\mathbf{D}}(v) \setminus F) \cup (E_{er}^{\mathbf{D}}(v) \cap F)\big) \cup \big((E_{sl}^{\mathbf{D}}(v) \setminus L) \cup (\mathrm{cons}^{\mathbf{D}}(v) \cap C)\big)$. Since $\mathbf{D}$ is total and of finite type and $\mathbf{D}(F; L; C) \simeq \mathbf{G}$ with $\mathbf{G}$ of finite type (m, n), we have $|\big((E_{sr}^{\mathbf{D}}(v) \setminus F) \cup (E_{er}^{\mathbf{D}}(v) \cap F)\big) \cup \big((E_{sl}^{\mathbf{D}}(v) \setminus L) \cup (\mathrm{cons}^{\mathbf{D}}(v) \cap C)\big)| = s^{\mathbf{D}(F;L;C)}(v) \leq n = s^{\mathbf{D}}(v) = |E_{sr}^{\mathbf{D}}(v)| + |E_{sl}^{\mathbf{D}}(v)|$, so $|E_{sr}^{\mathbf{D}}(v)| - |E_{sr}^{\mathbf{D}}(v) \cap F| + |E_{er}^{\mathbf{D}}(v) \cap F| + |E_{sl}^{\mathbf{D}}(v)| - |E_{sl}^{\mathbf{D}}(v) \cap L| + |\mathrm{cons}^{\mathbf{D}}(v) \cap C| \leq |E_{sr}^{\mathbf{D}}(v)| + |E_{sl}^{\mathbf{D}}(v)|$. Thus

$$(1)\qquad |E_{er}^{\mathbf{D}}(v) \cap F| + |\mathrm{cons}^{\mathbf{D}}(v) \cap C| \leq |E_{sr}^{\mathbf{D}}(v) \cap F| + |E_{sl}^{\mathbf{D}}(v) \cap L| \quad \text{for each } v \in V^{\mathbf{D}}.$$

Now let $\mathbf{H} \leq_w \mathbf{D}$ be the weak subdigraph of $\mathbf{D}$ with $V^{\mathbf{H}} = V^{\mathbf{D}}$ and $E^{\mathbf{H}} = F \cup L$ and $\mathrm{Cons}(\mathbf{H}) = C = \{c_1^{\mathbf{D}}, \dots, c_k^{\mathbf{D}}\}$. Then by (1),

$$(2)\qquad er^{\mathbf{H}}(v) + cn^{\mathbf{H}}(v) \leq sr^{\mathbf{H}}(v) + sl^{\mathbf{H}}(v) \quad \text{for each } v \in V^{\mathbf{H}}.$$

Obviously $\mathbf{H}$ is locally finite (by $(*)$), so $\langle C\rangle_{\mathbf{H}}$ is a finite digraph. Moreover, $\langle C\rangle_{\mathbf{H}}$ is of finite type (m,n), because $\langle C\rangle_{\mathbf{H}}$ is also a weak subdigraph of $\mathbf{D}$. These two facts imply that $\langle C\rangle_{\mathbf{H}}$ has only finitely many edges (more precisely, it is well-known that $E^{\langle C\rangle_{\mathbf{H}}} = \bigcup_{v\in V^{\langle C\rangle_{\mathbf{H}}}} E_s^{\langle C\rangle_{\mathbf{H}}}(v)$, which implies $|E^{\langle C\rangle_{\mathbf{H}}}| = \sum_{v\in V^{\langle C\rangle_{\mathbf{H}}}} s^{\langle C\rangle_{\mathbf{H}}} \leq |V^{\langle C\rangle_{\mathbf{H}}}|\cdot n$, so $E^{\langle C\rangle_{\mathbf{H}}}$ is finite, because $V^{\langle C\rangle_{\mathbf{H}}}$ is finite). Thus by Lemma 3 and (2) there are k pairwise different loops $l_1,\dots,l_k \in L$ (because L is the set of all loops in $\mathbf{H}$) and k pairwise edge-disjoint paths $p_1,\dots,p_k$ of $\mathbf{H}$ (thus also of $\mathbf{D}$) such that p_i goes from $c_i^{\mathbf{H}} = c_i^{\mathbf{D}}$ to $I_1^{\mathbf{H}}(l_i) = I_1^{\mathbf{D}}(l_i)$ for $i=1,\dots,k$. Hence, in particular $|C|\leq|L|$. On the other hand, we know that $\mathrm{Cons}(\mathbf{D}(F;L;C)) = (\mathrm{Cons}(\mathbf{D})\setminus C)\cup L$ and $\mathbf{D}$ has exactly m constants and $\mathbf{D}(F;L;C)$ has at most m constants. Thus $|\,\mathrm{Cons}(\mathbf{D})| = m \geq |\,\mathrm{Cons}(\mathbf{D}(F;L;C))| = |\,\mathrm{Cons}(\mathbf{D})| - |C| + |L|$, so $|C|\geq|L|$. These two inequalities imply $|C|=|L|$, so $L=\{l_1,\dots,l_k\}$. Hence we infer that C, L and $p_1,\dots,p_k$ satisfy the desired conditions of our lemma.

Now observe that if $p_1,\dots,p_k$ contain all the edges of F (i.e. $\bigcup\limits_{i=1}^{i=k} E^{p_i} = F$), then to end the proof we must only take the empty family R. Thus we can assume that F is not equal to $\bigcup_{i=1}^{i=k} E^{p_i}$.

Take the weak subdigraph $\mathbf{K}\leq_w \mathbf{H}$ with $V^{\mathbf{K}} = V^{\mathbf{H}}$ and $E^{\mathbf{K}} = F\setminus\bigcup_{i=1}^{i=k} E^{p_i}$, i.e. we remove all the edges in $\bigcup_{i=1}^{i=k} E^{p_i}$ and all the loops and all the constants of $\mathbf{H}$.

Obviously $\mathbf{K}$ has only regular edges and no constants and no loops. Hence and by $(*)$, because $\mathbf{K}$ is a weak subdigraph of $\mathbf{D}$, $\mathbf{K}$ is of finite type $(0,n)$ and is locally finite. Moreover, it follows from (2), by a simple verification, that

$$er^{\mathbf{K}}(v) \leq sr^{\mathbf{K}}(v) \quad \text{for each } v\in V^{\mathbf{K}}. \tag{3}$$

More precisely, since $p_1,\dots,p_k$ are pairwise edge-disjoint, $\mathbf{K}$ can be obtained from $\mathbf{H}$ in k steps in such a way that we omit only p_i (more precisely, all the edges of p_i), $c_i^{\mathbf{D}}$ and l_i in i-th step. By Lemma 2 we have that the digraph with constants obtained from $\mathbf{H}$ in the first step satisfies (2). Thus by a simple induction on i we deduce that the digraph obtained in the i-th step satisfies (2), in particular this equality holds for $\mathbf{K}$.

Take $v\in V^{\mathbf{K}}$ and the strong subdigraph $\mathbf{M} := \langle v\rangle_{\mathbf{K}}$ generated by v. Then $\mathbf{M}$ is a finite digraph of finite type $(0,n)$, so $\mathbf{M}$ has also only finitely many edges. Moreover, $sr^{\mathbf{M}}(w) = sr^{\mathbf{K}}(w)$ for $v \in V^{\mathbf{M}}$ (by the definition of strong subdigraphs) and $er^{\mathbf{M}}(w) \leq er^{\mathbf{K}}(w)$. Thus by (3),

$$er^{\mathbf{M}}(w) \leq sr^{\mathbf{M}}(w) \quad \text{for each } w \in V^{\mathbf{M}}. \tag{4}$$

Since $\mathbf{M}$ has no constants, we have from [15] that a vertex u belongs to $\mathbf{M}$ iff $u = v$ or there is a non-trivial path going from v to u. This is a graph-theoretical generalization of the algebraic result on the generation of strong subalgebras and its proof is similar. Hence, in particular, $\mathbf{M}$ is a connected digraph. Thus, having the above inequality we can show, in an analogous way as in the proof of Euler's Theorem from [5] (Chapter 11, §1, Theorem 1), that $\mathbf{M}$ is a single vertex or there is a simple cycle r_v containing all the edges of $\mathbf{M}$ (recall that $\mathbf{K}$, and thus also $\mathbf{M}$, has no loops). More precisely, we can, of course, assume that $\mathbf{M}$ is not a single vertex. Then there is a non-trivial path p starting from v (because $\mathbf{M}$ is connected). Let w be the final vertex of p and assume that $w \neq v$. Then $|E^p \cap E^{\mathbf{M}}_{sr}(w)| + 1 = |E^p \cap E^{\mathbf{M}}_{er}(w)|$ (since all edges of p are pairwise different). Hence and by (4) there is a regular edge starting from w and does not belong to p. This fact easily implies, since $\mathbf{M}$ has only finitely many edges, that there is a simple cycle r starting from v. If r does not contain all the edges of $\mathbf{M}$, then after removing all the edges of r, we obtain a new digraph $\mathbf{C}$ (without loops) which also satisfies the inequality $er^{\mathbf{C}}(u) \leq sr^{\mathbf{C}}(u)$ for $u \in V^{\mathbf{C}}$. Let $\mathbf{C}_1, \dots, \mathbf{C}_d$ be all the non-trivial (i.e. having at least one edge) connected components of $\mathbf{C}$. By the induction hypothesis (note that if a digraph has at least one regular edge and satisfies (4), then it has at least two regular edges; moreover, if a digraph has exactly two regular edges and satisfies (4), then these edges form a simple cycle), there are simple cycles $r_1, \dots, r_d$ containing all edges of $\mathbf{C}_1, \dots, \mathbf{C}_d$, respectively. Since $\mathbf{M}$ is connected, r has common vertices with each of these cycles. Thus now we must only insert cycles $r_1, \dots, r_d$ in suitable places of r to obtain a simple cycle r_v containing all the edges of $\mathbf{M} = \langle v\rangle_{\mathbf{K}}$.

Now observe that for any two different vertices v, w of $\mathbf{K}$, $\langle v\rangle_{\mathbf{K}} = \langle w\rangle_{\mathbf{K}}$ or they are disjoint (i.e. $V^{\langle v\rangle_{\mathbf{K}}} \cap V^{\langle w\rangle_{\mathbf{K}}} = \emptyset$). Assume that $V^{\langle v\rangle_{\mathbf{K}}} \cap V^{\langle w\rangle_{\mathbf{K}}} \neq \emptyset$. Then, since all the edges of $\langle v\rangle_{\mathbf{K}}$ and of $\langle w\rangle_{\mathbf{K}}$ lie on cycles r_v and r_w, respectively, we deduce that there is a path from v to w and there is a path from w to v. Thus $v \in V^{\langle w\rangle_{\mathbf{K}}}$ and $w \in V^{\langle v\rangle_{\mathbf{K}}}$, so $\langle v\rangle_{\mathbf{K}} = \langle w\rangle_{\mathbf{K}}$ (because $\langle v\rangle_{\mathbf{K}}$ and $\langle w\rangle_{\mathbf{K}}$ are the least strong subdigraphs of $\mathbf{K}$ containing v and w, respectively).

Now let $\{\langle v_i\rangle_{\mathbf{K}} : \ i \in I\}$ be the family of all pairwise different and non-trivial (i.e. having at least two vertices) strong subdigraphs of $\mathbf{K}$ generated by single vertices; of course it is a non-empty family, because $\mathbf{K}$ has regular edges. Then first, these digraphs are pairwise disjoint and together contain all the edges of $\mathbf{K}$. Secondly, for each $i \in I$, there is a simple cycle r_i containing all the edges of $\langle v_i\rangle_{\mathbf{K}}$. These facts imply that $R := \{r_i : \ i \in I\}$ is a family of pairwise disjoint simple cycles of $\mathbf{K}$ (thus also of $\mathbf{D}$) and R contains all the edges of $\mathbf{K}$ (i.e. $E^R = F \setminus \bigcup_{i=1}^{i=k} E^{p_i}$). Hence we infer that R and $\{p_1, \dots, p_k\}$ are edge-disjoint and moreover, $E^{R\cup\{p_1,\dots,p_k\}} = F$. Thus $\mathbf{D}(R \cup \{p_1, \dots, p_k\}; L; C) = \mathbf{D}(F; L; C)$, which completes the proof. □

4. The main result

Theorem 5. *Let digraphs $\mathbf{D}$ and $\mathbf{G}$ with constants satisfy the following conditions (where $m, n \in \mathbb{N}$)*

(∗) $\mathbf{D}^* \simeq \mathbf{G}^*$,

(∗∗) $\mathbf{D}$ *is a total digraph with constants of finite type* (m, n) *and locally finite,*

(∗ ∗ ∗) $\mathbf{G}$ *is a digraph with constants of finite type* (m, n).

Then $\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{S}_s(\mathbf{D})$, *and moreover,* $\mathbf{G}$ *is also total and locally finite.*

Remark. Observe that (∗) of the above Theorem 5 can be replaced by the following condition: $\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{S}_w(\mathbf{G})$ (see remarks at the beginning of the paper).

Proof. By Lemma 4 we have that $\mathbf{G} \simeq \mathbf{M}$, where $\mathbf{M} := \mathbf{D}(R \cup \{p_1, \dots, p_k\}; \{c_1^{\mathbf{D}}, \dots, c_k^{\mathbf{D}}\}; \{l_1, \dots, l_k\})$ and R is a family of pairwise disjoint simple cycles of $\mathbf{D}$ and $c_1^{\mathbf{D}}, \dots, c_k^{\mathbf{D}}$ are pairwise different constants and $l_1, \dots, l_k$ are pairwise different loops and $p_1, \dots, p_k$ are pairwise edge-disjoint paths such that p_i goes from $c_i^{\mathbf{D}}$ to $I_1^{\mathbf{D}}(l_i)$ for $1 \le i \le k$, and R and $\{p_1, \dots, p_k\}$ are edge-disjoint.

Since isomorphic digraphs with constants have isomorphic strong subdigraph lattices, it is sufficient to show that $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s(\mathbf{M})$. To this purpose take a strong subdigraph $\mathbf{H} \le_s \mathbf{D}$ and observe that by a simple induction (recall that for an edge e of $\mathbf{D}$, if $I_1^{\mathbf{D}}(e) \in V^{\mathbf{H}}$, then e and its final vertex $I_2^{\mathbf{D}}(v)$ belong to $\mathbf{H}$) we obtain that each cycle of $\mathbf{D}$ having common vertices with $\mathbf{H}$ must be contained in $\mathbf{H}$; and analogously each path of $\mathbf{D}$ starting from $\mathbf{H}$ is contained in $\mathbf{H}$. Moreover, we have $\mathrm{Cons}(\mathbf{D}) \subseteq \mathrm{Cons}(\mathbf{H})$. Hence, $\mathbf{H}$ contains $p_1, \dots, p_k$ and $l_1, \dots, l_k$. In particular, $\mathbf{H}$ contains all the constants of $\mathbf{M}$.

The above facts imply that for any strong subdigraph $\mathbf{H} \leq_s \mathbf{D}$, the weak subdigraph $\mathbf{K} \leq_w \mathbf{M}$ with $V^{\mathbf{K}} = V^{\mathbf{H}}$, $E^{\mathbf{K}} = E^{\mathbf{H}}$ and $\mathrm{Cons}(\mathbf{K}) = \mathrm{Cons}(\mathbf{M})$ is well-defined and is a strong subdigraph of $\mathbf{M}$.

Observe that for each cycle $(f_1, \dots, f_m) \in R$, the sequence $(f_m, \dots, f_1)$ is a cycle in $\mathbf{M}$; and analogously for paths $p_1, \dots, p_k$. Thus, in the same way (because $l_1, \dots, l_k$ are constants in $\mathbf{M}$), we obtain that for any $\mathbf{K} \leq_s \mathbf{M}$, the weak subdigraph $\mathbf{H} \leq_w \mathbf{D}$ with $V^{\mathbf{H}} = V^{\mathbf{K}}$, $E^{\mathbf{H}} = E^{\mathbf{K}}$ and $\mathrm{Cons}(\mathbf{H}) = \mathrm{Cons}(\mathbf{D})$ is well-defined and is a strong subdigraph of $\mathbf{D}$.

Having the above facts we can take a surjection φ of the set of all strong subdigraphs of $\mathbf{D}$ onto the set of all strong subdigraphs of $\mathbf{M}$ such that for each $\mathbf{H} \leq_s \mathbf{D}$, $\varphi(\mathbf{H})$ is the strong subdigraph of $\mathbf{M}$ with $V^{\varphi(\mathbf{H})} = V^{\mathbf{H}}$ and $E^{\varphi(\mathbf{H})} = E^{\mathbf{H}}$ (then, of course, $\mathrm{Cons}(\varphi(\mathbf{H})) = \mathrm{Cons}(\mathbf{M})$).

It can be also easily shown, using the definition of strong subdigraphs, that for every $\mathbf{H}_1, \mathbf{H}_2 \leq_s \mathbf{D}$; and analogously for $\mathbf{H}_1, \mathbf{H}_2 \leq_s \mathbf{M}$; the following equivalence holds:

$$\mathbf{H}_1 \leq_s \mathbf{H}_2 \ (\mathbf{H}_1 = \mathbf{H}_2) \quad \text{iff} \quad V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2} \ (V^{\mathbf{H}_1} = V^{\mathbf{H}_2}).$$

These facts imply that the surjection φ is also injective and that φ and its inverse φ^{-1} preserve (the strong subdigraph) inclusion $\leq_s$. More precisely, for any strong subdigraphs $\mathbf{H}_1, \mathbf{H}_2$ of $\mathbf{D}$ we obtain $\mathbf{H}_1 \leq_s \mathbf{H}_2$ $\big(\mathbf{H}_1 = \mathbf{H}_2\big)$ iff $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$ $\big(V^{\mathbf{H}_1} = V^{\mathbf{H}_2}\big)$ iff $V^{\varphi(\mathbf{H}_1)} \subseteq V^{\varphi(\mathbf{H}_2)}$ $\big(V^{\varphi(\mathbf{H}_1)} = V^{\varphi(\mathbf{H}_2)}\big)$ iff $\varphi(\mathbf{H}_1) \leq_s \varphi(\mathbf{H}_2)$ $\big(\varphi(\mathbf{H}_1) = \varphi(\mathbf{H}_2)\big)$. Thus φ is the desired isomorphism of lattices $\mathbf{S}_s(\mathbf{D})$ and $\mathbf{S}_s(\mathbf{M})$. Hence and by the definition of φ we obtain, in particular, that for each set $W \subseteq V^{\mathbf{D}}$, $\langle W \rangle_{\mathbf{D}}$ and $\langle W \rangle_{\mathbf{M}}$ have the same vertex and edge sets. By these facts and $(**)$ $\mathbf{M}$, and thus also $\mathbf{G}$, are locally finite.

Thus now it remains to show that $\mathbf{M}$ is total, since $\mathbf{G} \simeq \mathbf{M}$. First, $|\mathrm{Cons}(\mathbf{M})| = |\mathrm{Cons}(\mathbf{D})| = m$, by the definition of $\mathbf{M}$. Secondly, take $v \in V^{\mathbf{D}}$ and observe $|E^r \cap E^{\mathbf{D}}_{sr}(v)| = |E^r \cap E^{\mathbf{D}}_{er}(v)|$ for any $r \in R$ (because r is a simple cycle). Hence, $|E^R \cap E^{\mathbf{D}}_{sr}(v)| = |E^R \cap E^{\mathbf{D}}_{er}(v)|$, since cycles of R are pairwise disjoint. This implies that $s^{\mathbf{M}}(v) = s^{\mathbf{D}}(v) = n$, if $v \notin \bigcup_{i=1}^{i=k} V^{p_i}$.

Assume now that $v \in \bigcup_{i=1}^{i=k} V^{p_i}$ and $v \notin \{c^{\mathbf{D}}_1, \dots, c^{\mathbf{D}}_k, I^{\mathbf{D}}_1(l_1), \dots, I^{\mathbf{D}}_1(l_k)\}$. Then $|E^{p_i} \cap E^{\mathbf{D}}_{sr}(v)| = |E^{p_i} \cap E^{\mathbf{D}}_{er}(v)|$ for $1 \leq i \leq k$, because v is no endpoint of p_i. Hence, $\left|\left(\bigcup_{i=1}^{i=k} E^{p_i}\right) \cap E_{sr}^{\mathbf{D}}(v)\right| = \left|\left(\bigcup_{i=1}^{i=k} E^{p_i}\right) \cap E_{er}^{\mathbf{D}}(v)\right|$, since $p_1, \ldots, p_k$ are pairwise edge-disjoint. This equality and the above fact for R imply $s^{\mathbf{M}}(v) = s^{\mathbf{D}}(v) = n$ (because R and $\{p_1, \ldots, p_k\}$ are edge-disjoint).

At the end, assume that $v \in \{c_1^{\mathbf{D}}, \ldots, c_k^{\mathbf{D}}\}$ and let $c_{j_1}^{\mathbf{D}}, \ldots, c_{j_h}^{\mathbf{D}}$ be all the constants in v. Then $p_{j_1}, \ldots, p_{j_h}$ are all the paths in the family $\{p_1, \ldots, p_k\}$ starting from v. Thus $s^{\mathbf{M}}(v) = s^{\mathbf{D}}(v) - h + h = s^{\mathbf{D}}(v) = n$, because $c_{j_1}^{\mathbf{D}}, \ldots, c_{j_h}^{\mathbf{D}}$ form loops of $\mathbf{M}$ in the vertex v and $p_{j_1}, \ldots, p_{j_h}$ are pairwise edge-disjoint; and if p_{j_i} is trivial, then $I_1^{\mathbf{D}}(l_{j_i})$ forms a constant of $\mathbf{M}$ in the vertex v.

For $v \in \{I_1^{\mathbf{D}}(l_1), \ldots, I_1^{\mathbf{D}}(l_k)\}$, the proof that $s^{\mathbf{M}}(v) = s^{\mathbf{D}}(v) = n$ is analogous, and is therefore omitted. □

Theorem 6. *Let* $\mathbf{A}$ *be a total and locally finite algebra with* m *constants and* n *unary operations (where* $m, n \in \mathbb{N}$*), and let* $\mathbf{B}$ *be a partial algebra of the same type such that*

$$\mathbf{S}_w(\mathbf{B}) \simeq \mathbf{S}_w(\mathbf{A}).$$

Then $\mathbf{S}_s(\mathbf{B}) \simeq \mathbf{S}_s(\mathbf{A})$*, and moreover,* $\mathbf{B}$ *is also total and locally finite.*

Proof. Take the digraphs with constants $\mathbf{D}(\mathbf{A})$ and $\mathbf{D}(\mathbf{B})$. Then they are of finite type (m, n), and $\mathbf{D}(\mathbf{A})$ is total and locally finite. Moreover, $\mathbf{D}^*(\mathbf{A}) \simeq \mathbf{D}^*(\mathbf{B})$, because the weak subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ are isomorphic (see the beginning of this paper). Now we can apply Theorem 5 to obtain that $\mathbf{S}_s\big(\mathbf{D}(\mathbf{A})\big) \simeq \mathbf{S}_s\big(\mathbf{D}(\mathbf{B})\big)$, and also $\mathbf{D}(\mathbf{B})$ is a total digraph of type (m, n) and locally finite. Hence we easily deduce that $\mathbf{S}_s(\mathbf{B}) \simeq \mathbf{S}_s(\mathbf{A})$ and $\mathbf{B}$ is locally finite. Moreover, since the type is finite, since at each point $v \in B$ each unary operation can be defined at most once, and since n unary operations are defined at each point, each unary operation of $\mathbf{B}$ has to be defined at each point of $\mathbf{B}$. Since $\mathbf{D}(\mathbf{B})$ contains exactly m constants, and since between constants defined in $\mathbf{B}$ and constants of $\mathbf{D}(\mathbf{B})$ we have the bijective correspondence, all constants in $\mathbf{B}$ are also defined. By these two facts we obtain that $\mathbf{B}$ is a total algebra with m constants and n unary operations. □

Finally, observe that all the conditions of Theorem 5 (and thus also Theorem 6) are necessary. To this purpose take digraphs $\mathbf{D}$ and $\mathbf{G}$ with constants such that $V^{\mathbf{D}} = V^{\mathbf{G}} = \mathbb{N}$, $E^{\mathbf{D}} = \{\langle i, i+1\rangle : i \in \mathbb{N}\}$, $c^{\mathbf{D}} = 0$ and

$E^{\mathbf{G}} = \{\langle i{+}1, i\rangle : i \in \mathbb{N}\} \cup \{\langle 0,0\rangle\}$, $\mathrm{Cons}(\mathbf{G}) = \emptyset$. Then obviously $\mathbf{D}^* \simeq \mathbf{G}^*$. But $\mathbf{S}_s(\mathbf{D}) \not\simeq \mathbf{S}_s(\mathbf{G})$, since $\mathbf{D}$ has exactly one strong subdigraph, and $\mathbf{G}$ has infinitely many. Note also that $\mathbf{D}$ is a total digraph of type $(1,1)$, but not locally finite; and $\mathbf{G}$ is locally finite and of type $(1,1)$, but not total.

The above example shows that the digraphs in Theorem 5 must be indeed of the same type, because $\mathbf{G}$ is also of type $(0,1)$ and then it is total, and, of course, $\mathbf{D}$ is not of type $(0,1)$.

Observe that the condition on finiteness of type is also necessary. Take digraphs $\mathbf{D}$ and $\mathbf{G}$ such that $V^{\mathbf{D}} = V^{\mathbf{G}} = \mathbb{N}$ and $E^{\mathbf{D}} = \{\langle i+1, j, i\rangle : i, j \in \mathbb{N}\} \cup \{\langle 0, j, 0\rangle : j \in \mathbb{N}\}$ and $E^{\mathbf{G}} = \{\langle i+1, j, i\rangle : i, j \in \mathbb{N},\ j \neq 0\} \cup \{\langle i, 0, i+1\rangle : i \in \mathbb{N}\} \cup \{\langle 0, j, 0\rangle : j \in \mathbb{N} \setminus \{0\}\}$ (of course for $\langle i+1, j, i\rangle$, $i+1$ is its initial vertex and i is its final vertex). Then $\mathbf{D}$ and $\mathbf{G}$ are total digraphs of type $(0, |\mathbb{N}|)$ and $\mathbf{D}$ is locally finite and $\mathbf{D}^* \simeq \mathbf{G}^*$. On the other hand, $\mathbf{S}_s(\mathbf{D}) \not\simeq \mathbf{S}_s(\mathbf{G})$, because $\mathbf{D}$ has infinitely many strong subdigraphs, and $\mathbf{G}$ has exactly two (the empty digraph and $\mathbf{G}$). Note also that $\mathbf{G}$ is not locally finite.

Having the above digraphs we can, of course, construct counterexamples for unary partial algebras with constants.

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KONRAD PIÓRO
INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
UL. BANACHA 2
02–097 WARSAW
POLAND

E-mail: kpioro@mimuw.edu.pl

(Received February 28, 1999; revised November 16, 1999; received in final form December 15, 1999)