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Generalizations of the Cauchy determinant

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Abstract. By means of partial fraction expansions, several determinant evaluations are established. They may be considered as generalizations of the Cauchy determinant formula. Applications to the determinant identities of the matrices with trigonometric entries are presented.

0. Preliminaries

For two polynomials $P(y)$ and $Q(y)$, define a matrix of order $n \times n$ with variables $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$, and its determinant by

$$\Omega(P,Q) = \det\left[\frac{P(y_j) + x_i Q(y_j)}{1 - x_i y_j}\right]_{1\le i,j\le n}. \tag{0.1}$$

When $P(y) = 1$ and $Q(y) = 0$, it reduces to the Cauchy determinant [2, §1.4]

$$\det\left[\frac{1}{1 - x_i y_j}\right]_{1\le i,j\le n} = \frac{\Delta(x)\,\Delta(y)}{\prod_{1\le i,j\le n}(1 - x_i y_j)} \tag{0.2}$$

where $\Delta(x)$ is the evaluation of the Vandermonde determinant

$$\Delta(x) = \prod_{1\le i<j\le n}(x_i - x_j) = \det\left[x_i^{n-j}\right]_{1\le i,j\le n}. \tag{0.3}$$

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The difference between the i-th row and the first row of $\Omega(P,Q)$ reads as

$$\frac{(x_i-x_1)\{Q(y_j)+y_jP(y_j)\}}{(1-x_1y_j)\,(1-x_iy_j)}, \qquad i=2,3,\dots,n, \quad j=1,2,\dots,n.$$

Now Laplace expansion of the resulting determinant with respect to the first row may be stated as

$$\Omega(P,Q)=\sum_{k=1}^{n}(-1)^{k+1}\frac{P(y_k)+x_1Q(y_k)}{1-x_1y_k} \tag{0.4a}$$

$$\times\det\left[\frac{(x_i-x_1)\{Q(y_j)+y_jP(y_j)\}}{(1-x_1y_j)(1-x_iy_j)}\right]_{i\neq 1,\, j\neq k}. \tag{0.4b}$$

Extracting the common fractors from rows and columns, we may rewrite (0.4b) as

$$\det\left[\frac{(x_i-x_1)\{Q(y_j)+y_jP(y_j)\}}{(1-x_1y_j)(1-x_iy_j)}\right]_{i\neq 1,\, j\neq k}$$
$$=\prod_{j\neq k}\frac{Q(y_j)+y_jP(y_j)}{1-x_1y_j}\,\prod_{i=2}^{n}(x_i-x_1)\det\left[\frac{1}{1-x_iy_j}\right]_{\substack{i\neq 1\\ j\neq k}}$$

where the Cauchy determinant may be used to derive

$$\det\left[\frac{1}{1-x_iy_j}\right]_{i\neq 1,\, j\neq k}$$
$$=(-1)^{k-1}\frac{\prod_{i=2}^{n}(1-x_iy_k)}{\prod_{j\neq k}(y_j-y_k)}\,\frac{\Delta(x)\,\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\,\frac{\prod_{j=1}^{n}(1-x_1y_j)}{\prod_{i=2}^{n}(x_i-x_1)}.$$

Substituting these expressions into (0.4b), we find a finite summation formula for $\Omega(P,Q)$

Lemma.

$$\Omega(P,Q)=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\prod_{\ell=1}^{n}\{Q(y_\ell)+y_\ell P(y_\ell)\} \tag{0.5a}$$

$$\times\sum_{k=1}^{n}\frac{P(y_k)+x_1\,Q(y_k)}{Q(y_k)+y_k\,P(y_k)}\,\frac{\prod_{i\neq 1}(1-x_iy_k)}{\prod_{j\neq k}(y_j-y_k)}. \tag{0.5b}$$

By evaluating (0.5b) with partial fraction method [1], we will establish the main theorem in the first section. Several examples will be demonstrated in the second and the third sections. In the last section, two determinant identities of the matrices with trigonometric entries are displayed as applications.

1. The main theorem and proof

Proposition. *For two polynomials $P(y)$ and $Q(y)$ with $\deg(Py+Q) > \deg(P)$, let*

$$\varepsilon(P,Q) = \begin{cases} 0, & \deg(P) \geq \deg(Q) \\ \dfrac{L_c(Q)}{L_c(Py+Q)}, & \deg(P) < \deg(Q) \end{cases}$$

where $L_c(Q)$ and $L_c(Py+Q)$ denote the leading coefficients of $Q(y)$ and $P(y)\,y + Q(y)$ respectively. Suppose that $P(y)y + Q(y)$ has m distinct roots $\{\beta_\lambda\}_{\lambda=1}^m$ such that

$$Q(y) + yP(y) = L_c(Py+Q)\prod_{\lambda=1}^{m}(y-\beta_\lambda).$$

We may express (0.5b) as

$$\sum_{k=1}^{n}\frac{P(y_k)+x_1Q(y_k)}{Q(y_k)+y_kP(y_k)}\frac{\prod_{i\neq 1}(1-x_iy_k)}{\prod_{j\neq k}(y_j-y_k)} = \varepsilon(P,Q)x_1x_2\dots x_n + L_c^{-1}(Py+Q)\sum_{\ell=1}^{m}\frac{P(\beta_\ell)}{\prod_{j\neq\ell}(\beta_\ell-\beta_j)}\prod_{i=1}^{n}\frac{(1-x_i\beta_\ell)}{(y_i-\beta_\ell)}. \tag{1.1}$$

Replacing (0.5b) by the formula in the proposition, we get

Theorem. *Assume the condition and notation from the proposition. We have determinant evaluation formula*

$$\Omega(P,Q) = \frac{\Delta(x)\Delta(y)}{\prod_{1\leq i,j\leq n}(1-x_iy_j)}\prod_{k=1}^{n}\{Q(y_k)+y_kP(y_k)\} \times\left\{\varepsilon(P,Q)\prod_{\iota=1}^{n}x_\iota + L_c^{-1}(Py+Q)\sum_{\ell=1}^{m}\frac{P(\beta_\ell)}{\prod_{j\neq\ell}(\beta_\ell-\beta_j)}\prod_{i=1}^{n}\frac{(1-x_i\beta_\ell)}{(y_i-\beta_\ell)}\right\}. \tag{1.2}$$

Proof. Under the condition of the proposition, we may make the expansion in partial fractions

$$f(y) := \frac{P(y)+x_1Q(y)}{Q(y)+yP(y)}\,\frac{\prod_{i\neq 1}(1-x_iy)}{\prod_{j\neq k}(y_j-y)} = A+\sum_{\kappa\neq k}\frac{B_\kappa}{y_\kappa-y}+\sum_{\ell=1}^{m}\frac{C_\ell}{y-\beta_\ell} \tag{1.3}$$

with the coefficients determined by

$$A = \lim_{y\to\infty} f(y) = \varepsilon(P,Q)\,x_1x_2\dots x_n \tag{1.4a}$$

$$B_\kappa = \lim_{y\to y_\kappa}(y_\kappa-y)f(y) = \frac{P(y_\kappa)+x_1Q(y_\kappa)}{Q(y_\kappa)+y_\kappa P(y_\kappa)}\,\frac{\prod_{i\neq 1}(1-x_iy_\kappa)}{\prod_{j\neq \kappa,k}(y_j-y_\kappa)} \tag{1.4b}$$

$$C_\ell = \lim_{y\to\beta_\ell}(y-\beta_\ell)f(y) = L_c^{-1}(Py+Q)\frac{P(\beta_\ell)+x_1Q(\beta_\ell)}{\prod_{j\neq\ell}(\beta_\ell-\beta_j)}\,\frac{\prod_{i\neq 1}^{n}(1-x_i\beta_\ell)}{\prod_{i\neq k}^{n}(y_i-\beta_\ell)}. \tag{1.4c}$$

By means of series-rearrangement and

$$P(\beta_\ell)+x_1\,Q(\beta_\ell) = (1-x_1\beta_\ell)\,P(\beta_\ell), \quad \ell=1,2,\dots,m$$

we can deduce

$$\sum_{k=1}^{n}\sum_{\kappa\neq k}\frac{B_\kappa}{y_\kappa-y_k} = -\sum_{\kappa=1}^{n}\sum_{k\neq\kappa}\frac{P(y_\kappa)+x_1Q(y_\kappa)}{Q(y_\kappa)+y_\kappa P(y_\kappa)}\,\frac{\prod_{i\neq 1}(1-x_iy_\kappa)}{\prod_{j\neq\kappa}(y_j-y_\kappa)} = (1-n)\sum_{\kappa=1}^{n}f(y_\kappa)$$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{m}\frac{C_\ell}{y_k-\beta_\ell}=L_c^{-1}(Py+Q)\sum_{\ell=1}^{m}\sum_{k=1}^{n}\frac{P(\beta_\ell)+x_1Q(\beta_\ell)}{(1-x_1y_\ell)\prod_{j\neq\ell}(\beta_\ell-\beta_j)}$$
$$\times\prod_{i=1}^{n}\frac{1-x_i\beta_\ell}{y_i-\beta_\ell}$$
$$=\frac{n}{L_c(Py+Q)}\sum_{\ell=1}^{m}\frac{P(\beta_\ell)}{\prod_{j\neq\ell}(\beta_\ell-\beta_j)}\prod_{i=1}^{n}\frac{1-x_i\beta_\ell}{y_i-\beta_\ell}$$

which permit us to sum (1.3a–1.3b) for $y=y_k$, $k=1,2,\dots,n$. After some trivial modification, the resulting relation reads as

$$\text{(1.5)}\quad \sum_{k=1}^{n}f(y_k)=\varepsilon(P,Q)x_1x_2\dots x_n$$
$$+L_c^{-1}(Py+Q)\sum_{\ell=1}^{m}\frac{P(\beta_\ell)}{\prod_{j\neq\ell}(\beta_\ell-\beta_j)}\prod_{i=1}^{n}\frac{1-x_i\beta_\ell}{y_i-\beta_\ell}.$$

It is a restatement of the formula in the proposition. This completes the proof of the theorem. □

2. Determinant evaluations

It is clear that the Cauchy determinant (0.2) is a special case of the theorem for $P(y)=1$ and $Q(y)=0$. Now we will derive, by means of (1.2), several determinant evaluations. For the computations involved are quite routine, it is not necessary to show in detail.

Example 2.1 (Determinant evaluation).

For $\quad P(y)=a, \qquad Q(y)=b+cy$

$\quad\varepsilon(P,Q)=c/(a+c), \qquad L_c(Py+Q)=a+c$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a+bx_i+cx_iy_j}{1-x_iy_j}\right]=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\prod_{k=1}^{n}\{b+(a+c)y_k\}$$
$$\times\left\{\frac{c}{a+c}\prod_{\iota=1}^{n}x_\iota+\frac{a}{a+c}\prod_{\ell=1}^{n}\frac{a+c+bx_\ell}{b+(a+c)y_\ell}\right\}.$$

Example 2.2 (Determinant evaluation).

For
$$P(y)=a+cy, \qquad Q(y)=b$$
$$\varepsilon(P,Q)=0, \qquad L_c(Py+Q)=c$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a+bx_i+cy_j}{1-x_iy_j}\right]=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\prod_{k=1}^{n}\{b+ay_k+cy_k^2\}$$
$$\times\left\{\frac{a+c\alpha}{c(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{1-x_\ell\alpha}{y_\ell-\alpha}-\frac{a+c\beta}{c(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{1-x_\ell\beta}{y_\ell-\beta}\right\}$$

provided that the quadratic polynomial

$$P(y)y+Q(y)=b+ay+cy^2$$

has two distinct zeros $\{\alpha,\beta\}$.

Example 2.3 (Determinant evaluation).

For
$$P(y)=a+cy, \qquad Q(y)=b+dy$$
$$\varepsilon(P,Q)=0, \qquad L_c(Py+Q)=c$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a+bx_i+cy_j+dx_iy_j}{1-x_iy_j}\right]$$
$$=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\prod_{k=1}^{n}\{b+(a+d)y_k+cy_k^2\}$$
$$\times\left\{\frac{a+c\alpha}{c(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{1-x_\ell\alpha}{y_\ell-\alpha}-\frac{a+c\beta}{c(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{1-x_\ell\beta}{y_\ell-\beta}\right\}$$

provided that the quadratic polynomial

$$P(y)\,y+Q(y)=b+(a+d)y+cy^2$$

has two distinct zeros $\{\alpha,\beta\}$.

Example 2.4 (Determinant evaluation).

For
$$P(y) = a, \qquad Q(y) = b + cy + dy^2$$
$$\varepsilon(P,Q) = 1, \qquad L_c(Py+Q) = d$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a + bx_i + cx_iy_j + dx_iy_j^2}{1 - x_iy_j}\right]$$
$$= \frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\prod_{k=1}^{n}\{b + (a+c)y_k + dy_k^2\}$$
$$\times\left\{x_1x_2\ldots x_n + \frac{a/d}{\alpha-\beta}\prod_{\ell=1}^{n}\frac{1-x_\ell\alpha}{y_\ell-\alpha} - \frac{a/d}{\alpha-\beta}\prod_{\ell=1}^{n}\frac{1-x_\ell\beta}{y_\ell-\beta}\right\}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = b + (a+c)y + dy^2$$

has two distinct zeros $\{\alpha,\beta\}$.

Example 2.5 (Determinant evaluation).

For
$$P(y) = a + cy, \qquad Q(y) = b + dy + ey^2$$
$$\varepsilon(P,Q) = e/(c+e), \qquad L_c(Py+Q) = c+e$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a + bx_i + cy_j + dx_iy_j + ex_iy_j^2}{1 - x_iy_j}\right]$$
$$= \frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(1-x_iy_j)}\prod_{k=1}^{n}\{b + (a+d)y_k + (c+e)y_k^2\}$$
$$\times\frac{1}{c+e}\left\{e\,x_1x_2\ldots x_n + \frac{a+c\alpha}{\alpha-\beta}\prod_{\ell=1}^{n}\frac{1-x_\ell\alpha}{y_\ell-\alpha} - \frac{a+c\beta}{\alpha-\beta}\prod_{\ell=1}^{n}\frac{1-x_\ell\beta}{y_\ell-\beta}\right\}$$

provided that the quadratic polynomial

$$P(y)\,y+Q(y)=b+(a+d)y+(c+e)y^2$$

has two distinct zeros $\{\alpha,\beta\}$.

3. Reformulations

Replacing x_k by $-1/x_k$ for $k=1,2,\ldots,n$, we may restate the theorem as

Corollary. *For two polynomials $P(y)$ and $Q(y)$, assume the condition and notation from the proposition. We have determinant evaluation formula*

$$\det_{1\le i,j\le n}\left[\frac{Q(y_j)-x_i\,P(y_j)}{x_i+y_j}\right]=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(x_i+y_j)}\prod_{k=1}^{n}\{P(y_k)\,y_k+Q(y_k)\}$$

$$\times\left\{\varepsilon(P,Q)+L_c^{-1}(Py+Q)\sum_{\ell=1}^{m}\frac{P(\beta_\ell)}{\prod_{j\neq\ell}(\beta_\ell-\beta_j)}\prod_{i=1}^{n}\frac{\beta_\ell+x_i}{\beta_\ell-y_i}\right\}. \tag{3.1}$$

When $P(y)=0$ and $Q(y)=1$, the formula in the corollary reduces to another determinant identity of Cauchy

$$\det_{1\le i,j\le n}\left[\frac{1}{x_i+y_j}\right]=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(x_i+y_j)}.$$

Similar to the last section, the following determinant evaluations may be established consequently.

Example 3.1 (Determinant evaluation).

For
$$P(y)=-b,\qquad Q(y)=a+cy$$
$$\varepsilon(P,Q)=c/(c-b),\qquad L_c(Py+Q)=c-b$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a+bx_i+cy_j}{x_i+y_j}\right]=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(x_i+y_j)}\prod_{k=1}^{n}\{a-(b-c)y_k\}$$
$$\times\left\{\frac{c}{c-b}+\frac{b}{b-c}\prod_{\ell=1}^{n}\frac{a+(b-c)x_\ell}{a-(b-c)y_\ell}\right\}.$$

Example 3.2 (Determinant evaluation).

For
$$P(y) = -b - cy, \qquad Q(y) = a$$
$$\varepsilon(P,Q) = 0, \qquad L_c(Py + Q) = -c$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a + bx_i + cx_iy_j}{x_i + y_j}\right] = \frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(x_i + y_j)}\prod_{k=1}^{n}\{a - by_k - cy_k^2\}$$
$$\times\left\{\frac{b + c\alpha}{c(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{\alpha + x_\ell}{\alpha - y_\ell} - \frac{b + c\beta}{c(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{\beta + x_\ell}{\beta - y_\ell}\right\}$$

provided that the quadratic polynomial

$$P(y)\,y + Q(y) = a - by - cy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 3.3 (Determinant evaluation).

For
$$P(y) = -b - dy, \qquad Q(y) = a + cy$$
$$\varepsilon(P,Q) = 0, \qquad L_c(Py + Q) = -d$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a + bx_i + cy_j + dx_iy_j}{x_i + y_j}\right]$$
$$= \frac{\Delta(x)\,\Delta(y)}{\prod_{1\le i,j\le n}(x_i + y_j)}\prod_{k=1}^{n}\{a - (b-c)y_k - dy_k^2\}$$
$$\times\left\{\frac{b + d\alpha}{d(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{\alpha + x_\ell}{\alpha - y_\ell} - \frac{b + d\beta}{d(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{\beta + x_\ell}{\beta - y_\ell}\right\}$$

provided that the quadratic polynomial

$$P(y)y + Q(y) = a - (b-c)y - dy^2$$

has two distinct zeros $\{\alpha, \beta\}$.

Example 3.4 (Determinant evaluation).

For
$$P(y)=-b,\qquad Q(y)=a+cy+dy^2$$
$$\varepsilon(P,Q)=1,\qquad L_c(Py+Q)=d$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a+bx_i+cy_j+dy_j^2}{x_i+y_j}\right]$$
$$=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(x_i+y_j)}\prod_{k=1}^{n}\{a-(b-c)y_k+dy_k^2\}$$
$$\times\left\{1-\frac{b/d}{\alpha-\beta}\prod_{\ell=1}^{n}\frac{\alpha+x_\ell}{\alpha-y_\ell}+\frac{b/d}{\alpha-\beta}\prod_{\ell=1}^{n}\frac{\beta+x_\ell}{\beta-y_\ell}\right\}$$

provided that the quadratic polynomial

$$P(y)y+Q(y)=a-(b-c)y+dy^2$$

has two distinct zeros $\{\alpha,\beta\}$.

Example 3.5 (Determinant evaluation).

For
$$P(y)=-b-dy,\qquad Q(y)=a+cy+ey^2$$
$$\varepsilon(P,Q)=e/(e-d),\qquad L_c(Py+Q)=e-d$$

there holds determinant identity

$$\det_{1\le i,j\le n}\left[\frac{a+bx_i+cy_j+dx_iy_j+ey_j^2}{x_i+y_j}\right]$$
$$=\frac{\Delta(x)\Delta(y)}{\prod_{1\le i,j\le n}(x_i+y_j)}\prod_{k=1}^{n}\{a-(b-c)y_k-(d-e)y_k^2\}$$
$$\times\left\{\frac{e}{e-d}+\frac{b+d\alpha}{(d-e)\,(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{\alpha+x_\ell}{\alpha-y_\ell}-\frac{b+d\beta}{(d-e)\,(\alpha-\beta)}\prod_{\ell=1}^{n}\frac{\beta+x_\ell}{\beta-y_\ell}\right\}$$

provided that the quadratic polynomial

$$P(y)\,y + Q(y) = a - (b-c)y - (d-e)y^2$$

has two distinct zeros $\{\alpha, \beta\}$.

4. Applications

As applications, we will derive two interesting determinants of the matrices with trigonometric entries. Throughout this section, denote by λ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ for n given real numbers $\{\lambda_k\}_{k=1}^n$.

In Example 2.1, let

$$a = c = \sqrt{-1}, \quad b = 0$$
$$x_k = -y_k = e^{2\lambda_k\theta\sqrt{-1}}.$$

Then it is trivial to verify that

$$\det_{1\le i,j\le n}\left[\frac{a + cx_iy_j}{1 - x_iy_j}\right] = \det_{1\le i,j\le n} \tan(\lambda_i + \lambda_j)\theta$$

$$\Delta(x) = \prod_{1\le i<j\le n} \sin(\lambda_j - \lambda_i)\theta\Big\{-2\sqrt{-1}e^{(\lambda_i+\lambda_j)\theta\sqrt{-1}}\Big\}$$

$$\Delta(y) = \prod_{1\le i<j\le n} \sin(\lambda_j - \lambda_i)\theta\Big\{2\sqrt{-1}e^{(\lambda_i+\lambda_j)\theta\sqrt{-1}}\Big\}$$

$$\prod_{1\le i,j\le n}(1 - x_iy_j) = \prod_{1\le i,j\le n} \cos(\lambda_i + \lambda_j)\theta\Big\{2\,e^{(\lambda_i+\lambda_j)\theta\sqrt{-1}}\Big\}.$$

Substituting these relations into Example 2.1 and performing a little simplification, we obtain a general determinant identity

$$\text{(4.1)}\qquad \det_{1\le i,j\le n}[\tan(\lambda_i + \lambda_j)\theta] = (-1)^{\lfloor n/2\rfloor}\frac{\prod_{1\le i<j\le n}\sin^2(\lambda_j - \lambda_i)\theta}{\prod_{1\le i,j\le n}\cos(\lambda_i + \lambda_j)\theta} \times \begin{cases} \cos(2\lambda\theta), & n \equiv 0 \pmod 2 \\ \sin(2\lambda\theta), & n \equiv 1 \pmod 2. \end{cases}$$

For $\lambda_k = k$, $k = 1, 2, \ldots, n$, it reduces to a determinant evaluation

$$\det_{1\le i,j\le n}[\tan(i+j)\theta] = (-1)^{\lfloor n/2\rfloor} \prod_{k=1}^{n} \frac{\sin^{2n-2k} k\theta}{\cos^{k-1} k\theta \cos^{1+n-k}(n+k)\theta} \times \begin{cases} \cos n(n+1)\theta, & n \equiv 0 \pmod 2 \\ \sin n(n+1)\theta, & n \equiv 1 \pmod 2. \end{cases} \tag{4.2}$$

When $\lambda_k = k-1/2$, $k = 1, 2, \ldots, n$, we get another determinant evaluation

$$\det_{1\le i,j\le n}[\tan(i+j-1)\theta] = (-1)^{\lfloor n/2\rfloor} \prod_{k=1}^{n} \frac{\sin^{2n-2k} k\theta}{\cos^{k} k\theta \cos^{n-k}(n+k)\theta} \times \begin{cases} \cos n^2\theta, & n \equiv 0 \pmod 2 \\ \sin n^2\theta, & n \equiv 1 \pmod 2 \end{cases} \tag{4.3}$$

which is due to Stanley Rabinowitz [3].

Instead, if we put in Example 2.1

$$a = c = -\sqrt{-1}, \quad b = 0$$
$$x_k = y_k = e^{2\lambda_k\theta\sqrt{-1}}$$

then it is not hard to check that

$$\det_{1\le i,j\le n}\left[\frac{a + cx_iy_j}{1 - x_iy_j}\right] = \det_{1\le i,j\le n}[\cot(\lambda_i + \lambda_j)\theta]$$

$$\Delta(x) = \Delta(y) = \prod_{1\le i<j\le n} \sin(\lambda_j - \lambda_i)\theta\Big\{-2\sqrt{-1}e^{(\lambda_i+\lambda_j)\theta\sqrt{-1}}\Big\}$$

$$\prod_{1\le i,j\le n}(1 - x_iy_j) = \prod_{1\le i,j\le n} \sin(\lambda_i + \lambda_j)\theta\Big\{-2\sqrt{-1}e^{(\lambda_i+\lambda_j)\theta\sqrt{-1}}\Big\}.$$

Making these substitutions in Example 2.1, we get another general determinant identity

$$\det_{1\le i,j\le n}[\cot(\lambda_i + \lambda_j)\theta] = \cos(2\lambda\theta)\frac{\prod_{1\le i<j\le n}\sin^2(\lambda_j - \lambda_i)\theta}{\prod_{1\le i,j\le n}\sin(\lambda_i + \lambda_j)\theta}. \tag{4.4}$$

For $\lambda_k = k$ and $k-1/2$, $k = 1, 2, \dots, n$, we deduce from it two determinant evaluations

$$\det_{1\le i,j\le n}[\cot{(i+j)\theta}] = \cos n(n+1)\theta \prod_{k=1}^{n} \frac{\sin^{1+2n-3k} k\theta}{\sin^{1+n-k}(n+k)\theta} \tag{4.5a}$$

$$\det_{1\le i,j\le n}[\cot{(i+j-1)\theta}] = \cos(n^2\theta) \prod_{k=1}^{n} \frac{\sin^{2n-3k} k\theta}{\sin^{n-k}(n+k)\theta}. \tag{4.5b}$$

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