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Uncertainty inequalities and order of magnitude of Hankel transforms

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Abstract. In this paper we study the behaviour of the Hankel integral transform at the infinity in a type Cesàro sense. Also we establish uncertainty inequalities for Hankel transforms and Laguerre expansion. Finally an entropy inequality for Hankel transforms is obtained.

1. Introduction

Hankel integral transformation h_μ is usually defined by (see [19] and [22], for instance)

$$h_\mu(f)(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(y) y^{2\mu+1} dy,$$

where f is a nice function. Here J_μ represents the Bessel function of the first kind and order μ and we assume that $\mu > -1/2$. If $\mu = \frac{n-2}{2}$, $n = 2, 3, \dots$, the Hankel transform h_μ replaces the Fourier transform of radial functions in $\mathbb{R}^n$.

Let $1 \le p \le \infty$. We denote by $L_{p,\mu}$ the Lebesgue space $L_p((0,\infty), x^{2\mu+1}dx)$ and the usual norm in $L_{p,\mu}$ is represented by $\|\cdot\|_{p,\mu}$. It is well-known that if $f \in L_{1,\mu}$ then $h_\mu(f)$ is a bounded continuous function on

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$(0,\infty)$. Also, according to a version of the Riemann–Lebesgue Lemma for Hankel transforms ([31, p. 457, 14.41]),

$$h_\mu(f)(x) \to 0, \quad \text{as } x \to \infty, \tag{1}$$

provided that $f \in L_{1,\mu}$. However the above convergence can be arbitrarily slow. Indeed, let ϕ be a nonnegative function defined on $(0,\infty)$ such that $\phi(x) \to \infty$, as $x \to \infty$, and let $(x_n)_{n\in\mathbb{N}}$ be an increasing and unbounded sequence of positive real numbers. Assume that $n \in \mathbb{N}$. We define the functional T_n on $L_{1,\mu}$ by

$$T_n f = \phi(x_n) h_\mu(f)(x_n), \quad f \in L_{1,\mu}.$$

Since the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, T_n is continuous from $L_{1,\mu}$ into $\mathbb{C}$. Moreover, by [22, Theorem 2.a] the norm of T_n is given by

$$\|T_n\| = \phi(x_n) \sup_{y\in(0,\infty)} |(x_n y)^{-\mu} J_\mu(x_n y)| = \frac{\phi(x_n)}{2^\mu \Gamma(\mu+1)}.$$

By invoking now uniform boundedness theorem we deduce that there exists $f \in L_{1,\mu}$ such that the sequence $\{T_n f\}_{n\in\mathbb{N}}$ is not bounded. Thus we have found $f \in L_{1,\mu}$ for which

$$\limsup_{x\to\infty} \phi(x)|h_\mu(f)(x)| = \infty.$$

Our first objective in this paper, that is inspired in [2], is to analyze the behaviour on $h_\mu(f)(x)$, as $x \to \infty$, by considering convergence in a weaker sense. We establish in Section 2 sufficient conditions on a function f and on real numbers α and β in order that the following equality

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta t^\alpha h_\mu(f)(t) t^{2\mu+1} dt = 0 \tag{2}$$

holds. Note that (2) can be interpreted as a limit in the Cesàro sense.

Herz [19, Theorem 3] proved that the Hankel transformation h_μ can be extended to $L_{p,\mu}$ as a bounded operator from $L_{p,\mu}$ into $L_{p',\mu}$, where as usual by p' we denote the conjugated exponent of p, provided that $1 \le p \le 2$. Moreover, if $\frac{4(\mu+1)}{2\mu+3} < p \le 2$, then

$$\lim_{n\to\infty} \int_0^n (xy)^{-\mu} J_\mu(xy) h_\mu(f)(y) y^{2\mu+1} dy = f(x), \tag{3}$$

in the sense of convergence in $L_{p,\mu}$ ([19, Theorem 5]) and almost everywhere $x \in (0,\infty)$ ([23, Corollary 2]). However if $p = \frac{4(\mu+1)}{2\mu+3}$ there exists a function $f \in L_{p,\mu}$ having compact support such that the sequence $\{\int_0^n (xy)^{-\mu} J_\mu(xy) h_\mu(f)(y) y^{2\mu+1} dy\}_{n\in\mathbb{N}}$ diverges for almost all $x \in (0,\infty)$ ([23, Theorem 2]).

Also, the convergence in (3) does not hold a.e. $(0,\infty)$, in general, when $f \in L_{1,\mu}$. The equality $h_\mu(h_\mu f) = f$ holds a.e. $(0,\infty)$ provided that f and $h_\mu(f)$ are in $L_{1,\mu}$ ([22, Corollary 2.e]).

On the other hand, Stempak [28, p. 17] stated that if $\delta > \mu + 1/2$ and $f \in L_{p,\mu}$, with $1 \le p \le 2$, then

$$\lim_{\lambda\to\infty} \int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\delta (xt)^{-\mu} J_\mu(xt) h_\mu(f)(t) t^{2\mu+1} dt = f(x), \tag{4}$$

for almost all $x \in (0,\infty)$. This result can be extended to $p > 2$ involving Hankel convolution ([7]).

Note that the Bochner–Riesz type multiplier in (4) improves the convergence in (3). This fact leads to investigate the convergence in (2). Observe that the integral in (4) for $x = 0$, reduces, except a constant, to the integral in (2) for $\alpha = 0$, but, in general, (4) is not true for $x = 0$.

In Section 3 we present uncertainty inequalities for the Hankel transform and Laguerre expansions. Bowie [9] and recently Rösler and Voit [27] have obtained uncertainty inequalities for Hankel transforms. We establish a new uncertainty inequality for h_μ by using Laguerre expansions. Our procedure is inspired in the one developed by Pati, Sitaram, Sundari and Thangavelu [25] and we use that Laguerre functions are eigenfunctions for the Hankel transformation. Also, after stating a version of the celebrated Hardy's theorem for Hankel transform, we obtain an uncertainty principle associated to Laguerre expansions.

Heinig and Smith [18] established a number of generalizations of the classical Heisenberg–Weyl uncertainty inequality. From a general weighted form of the Hausdorff–Young theorem they proved weighted uncertainty inequalities for Fourier transforms. Here, by using weighted forms of the Hausdorff–Young theorem for Hankel transforms ([13], [14] and [20]) we obtain weighted uncertainty inequalities in the Hankel setting.

Throughout this paper by C we always denote a positive constant not necessarily the same in each ocurrence.

2. Order of magnitude of Hankel transforms

Our objective in this section is to determine conditions on a function f and on two real numbers α and β in order that the following

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^\beta t^\alpha h_\mu(f)(t) t^{2\mu+1} dt = 0$$

holds. Our results can be seen as Hankel versions of the ones obtained in [2] for Fourier transforms.

Propisition 2.1. (i) *Assume that $f \in L_{1,\mu} \cap L_{p,\mu}$, with $1 < p \leq 2$, and that $0 < \alpha + \frac{2(\mu+1)}{p} < 2$. Then*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^{-1/p} t^\alpha h_\mu(f)(t) t^{2\mu+1} dt = 0.$$

(ii) *Let $1 < p \leq 2$, $\beta > -1/p$ and $\mu \leq \frac{p-1}{2}$. If $f \in L_{p,\mu}$ then*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^\beta t^{1/p'} h_\mu(f)(t) t^{2\mu+1} dt = 0.$$

Proof. (i) We define

$$I(\lambda) = \frac{1}{\lambda^2} \int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^{-1/p} t^\alpha h_\mu(f)(t) t^{2\mu+1} dt, \quad \lambda > 0,$$

and

$$K(x,\lambda) = \frac{1}{\lambda^2} \int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^{-1/p} t^\alpha (xt)^{-\mu} J_\mu(xt) t^{2\mu+1} dt, \quad \lambda, x > 0.$$

By interchanging the order of integration we can write

$$\begin{aligned} I(\lambda) &= \frac{1}{\lambda^2} \int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^{-1/p} t^\alpha \int_0^\infty (xt)^{-\mu} J_\mu(xt) f(x) x^{2\mu+1} dx t^{2\mu+1} dt \\ &= \int_0^\infty f(x) K(x,\lambda) x^{2\mu+1} dx, \qquad \lambda > 0. \end{aligned}$$

According to [19, Theorem 3], it has

$$\|K(\cdot,1)\|_{q',\mu} = \left\|h_\mu[\chi_{(0,1)}(t)(1-t^2)^{-1/p}t^\alpha]\right\|_{q',\mu}$$
$$\leq C\|\chi_{(0,1)}(t)(1-t^2)^{-1/p}t^\alpha\|_{q,\mu} = C\left\{\int_0^1 (1-y)^{-q/p}y^{\mu+\alpha q/2}dy\right\}^{1/q},$$

provided that $1 \leq q \leq 2$. Note that the last integral is finite when $q < p$ and $\alpha q + 2\mu + 2 > 0$. Here χ represents the characteristic function of the interval $(0,1)$.

Also a straightforward manipulation leads to

$$\|K(\cdot,\lambda)\|_{q',\mu} \leq \|\lambda^{2\mu+\alpha}K(\cdot\lambda,1)\|_{q',\mu} = \lambda^{2(\mu+1)/q+\alpha-2}\|K(\cdot,1)\|_{q',\mu}, \quad \lambda > 0.$$

On the other hand, Hölder's inequality implies that

$$|I(\lambda)| \leq \|f\|_{q,\mu}\,\|K(\cdot,\lambda)\|_{q',\mu} \leq C\lambda^{2(\mu+1)/q+\alpha-2}(\|f\|_{1,\mu} + \|f\|_{p,\mu}^p)^{1/q},$$

provided that $1 \leq q < p$ and $\alpha q + 2\mu + 2 > 0$.

Hence, since we can find q such that $1 \leq q < p$ and $0 < \alpha q + 2\mu + 2 < 2q$, we conclude that $I(\lambda) \to 0$, as $\lambda \to \infty$.

(ii) Let $f \in L_{p,\mu}$. We now consider

$$I(\lambda) = \frac{1}{\lambda^2}\int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta t^{1/p'}h_\mu(f)(t)t^{2\mu+1}dt, \quad \lambda > 0.$$

Let $\varepsilon > 0$. By invoking [19, Theorem 3], $h_\mu(f) \in L_{p',\mu}$. Hence, there exists $\lambda_0 > 0$ such that

$$\int_{\lambda_0}^\infty |h_\mu(f)(t)|^{p'}t^{2\mu+1}dt < \varepsilon.$$

We divide the integral in $I(\lambda)$ as follows

$$I(\lambda) = \frac{1}{\lambda^2}\left(\int_0^{\lambda_0} + \int_{\lambda_0}^\lambda\right)\left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta t^{1/p'}h_\mu(f)(t)t^{2\mu+1}dt$$
$$= I_1(\lambda) + I_2(\lambda), \quad \lambda > \lambda_0.$$

Now, each integral is estimated.

Firstly, we have

$$|I_1(\lambda)| \leq \frac{1}{\lambda^2} \max\left\{1, \left(1-\left(\frac{\lambda_0}{\lambda}\right)^2\right)^\beta\right\} \int_0^{\lambda_0} t^{1/p'} |h_\mu(f)(t)| t^{2\mu+1} dt < \varepsilon,$$

when λ is sufficiently large.

On the other hand, Hölder's inequality leads to

$$\begin{aligned}|I_2(\lambda)| &\leq \frac{1}{\lambda^2}\left\{\int_0^\lambda \left(1-\left(\frac{t}{\lambda}\right)^2\right)^{\beta p} t^{p/p'} t^{2\mu+1} dt\right\}^{1/p} \left\{\int_{\lambda_0}^\lambda |h_\mu(f)(t)|^{p'} t^{2\mu+1} dt\right\}^{1/p'} \\ &\leq C\varepsilon^{1/p'} \lambda^{\frac{1}{p'}+\frac{2\mu+2}{p}-2} \left\{\int_0^1 (1-u)^{\beta p} u^{\frac{1}{2}(p/p'+2\mu)} du\right\}^{1/p} \leq C\varepsilon^{1/p'},\end{aligned}$$

provided that $\beta > -1/p$ and $\mu \leq \frac{p-1}{2}$.

Hence, under the imposed conditions $\lim_{\lambda\to\infty} I(\lambda) = 0$. □

An immediate consequence of Proposition 2.1 (ii), is the following.

Corollary 2.1. *If* $1 < p \leq 2$ *and* $\mu \leq \frac{p-1}{2}$ *then*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \int_0^\lambda t^{1/p'} h_\mu(f)(t) t^{2\mu+1} dt = 0,$$

provided that $f \in L_{p,\mu}$.

Proof. It is sufficient to take $\beta = 0$ in Proposition 2.1 (ii). □

Hirschman [22] and Haimo [17] investigated the convolution operation and a translation operator associated to the Hankel transformation. For every $x \in (0,\infty)$ and $f \in L_{p,\mu}$, $1 \leq p \leq \infty$, the Hankel translated $\tau_x f$ of f is given by

$$(\tau_x f)(y) = \int_0^\infty D_\mu(x,y,z) f(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz, \qquad y \in (0,\infty),$$

where

$$D_\mu(x,y,z) = (2^\mu\Gamma(\mu+1))^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt) t^{2\mu+1} dt, \quad x,y,z \in (0,\infty).$$

The operator τ_x is a contraction in $L_{p,\mu}$, for each $x \in (0,\infty)$ and $1 \leq p \leq \infty$ ([28, p. 16]).

In [5] we introduced the h_μ-Lebesgue points of a function f as follows. Let f be a Lebesgue measurable function on $(0,\infty)$ such that $\int_0^a |f(x)|x^{2\mu+1}dx < \infty$, for every $a > 0$. An $x_0 \in [0,\infty)$ is an h_μ-Lebesgue point of f when

$$\int_0^h |(\tau_{x_0} f)(t) - f(x_0)|t^{2\mu+1}dt = o(h^{2\mu+2}), \qquad \text{as } h \to 0^+.$$

In [5, Proposition 3.1] it was established that almost everywhere point of $(0,\infty)$ is an h_μ-Lebesgue point of f. Recently, Bloom and Xu [8] have considered Lebesgue points in the setting of Chébli–Trimèche hypergroups.

The following result, concerning to the convergence of the Bochner–Riesz means of Hankel transform in the origin, will be useful in the sequel. This property can be stated in the more general setting of Chébli–Trimèche hypergroups (note that in [8, Lemma 4.4] the origin is not included).

Lemma 2.1. *Let $f \in L_{1,\mu}$ and $\beta > \mu + 1/2$. If 0 is an h_μ-Lebesgue point of f then*

$$\lim_{\lambda\to\infty} \frac{1}{2^\mu\Gamma(\mu+1)} \int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta h_\mu(f)(t)t^{2\mu+1}dt = f(0).$$

Proof. Dominated convergence theorem allows us to write, for each $\lambda > 0$,

$$\begin{aligned}
&\frac{1}{2^\mu\Gamma(\mu+1)} \int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta h_\mu(f)(t)t^{2\mu+1}dt \\
&= \lim_{x\to 0^+} \int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta h_\mu(f)(t)(xt)^{-\mu}J_\mu(xt)t^{2\mu+1}dt.
\end{aligned}$$

Moreover, according to [6, p. 3], it has,

$$\begin{aligned}
&\int_0^\lambda \left(1 - \left(\frac{t}{\lambda}\right)^2\right)^\beta h_\mu(f)(t)(xt)^{-\mu}J_\mu(xt)t^{2\mu+1}dt \\
&= \int_0^\infty \phi_\lambda(y)(\tau_x f)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy, \qquad \lambda, x > 0,
\end{aligned}$$

where $\phi_\lambda(y) = 2^\beta\Gamma(\beta+1)\lambda^{2\mu+2}(y\lambda)^{-\mu-\beta-1}J_{\mu+\beta+1}(y\lambda)$, $y, \lambda > 0$.

Hence, by taking into account that $\tau_x f \to f$, as $x \to 0^+$, in $L_{1,\mu}$, since the function $z^{-\nu} J_\nu(z)$, $\nu \geq -1/2$, is bounded on $(0, \infty)$, it follows

$$\int_0^\lambda \Big(1 - \Big(\frac{t}{\lambda}\Big)^2\Big)^\beta h_\mu(f)(t) t^{2\mu+1} dt = \int_0^\infty \phi_\lambda(y) f(y) y^{2\mu+1} dy, \qquad \lambda > 0.$$

Then the proof of lemma is complete when we see that

$$\lim_{\lambda \to \infty} \frac{1}{2^\mu \Gamma(\mu+1)} \int_0^\infty \phi_\lambda(y) f(y) y^{2\mu+1} dy = f(0). \tag{5}$$

Since $\int_0^\infty \phi_\lambda(y) y^{2\mu+1} dy = 2^\mu \Gamma(\mu+1)$, $\lambda > 0$, it has

$$\begin{aligned}\int_0^\infty \phi_\lambda(y) f(y) y^{2\mu+1} dy - 2^\mu \Gamma(\mu+1) f(0)\\ = \int_0^\infty \phi_\lambda(y)[f(y) - f(0)] y^{2\mu+1} dy, \qquad \lambda > 0.\end{aligned}$$

To finish the proof we proceed as in [30, Theorem 13]. Let $\varepsilon > 0$. Since 0 is an h_μ-Lebesgue point of f, there exists $\eta > 0$ such that

$$\delta^{-(2\mu+2)} \int_0^\delta |f(y) - f(0)| x^{2\mu+1} dx < \varepsilon, \qquad 0 < \delta \leq \eta.$$

Hence, if $\lambda > 1/\eta$, by taking into account that the function $z^{-\nu} J_\nu(z)$, $\nu \geq -1/2$, is bounded on $(0, \infty)$, it has

$$\left| \int_0^{1/\lambda} \phi_\lambda(y)[f(y) - f(0)] y^{2\mu+1} dy \right| < C\varepsilon.$$

On the other hand, since $\sqrt{z} J_\mu(z)$ is bounded on $(0, \infty)$, we have

$$\begin{aligned}&\left| \int_{1/\lambda}^\eta \phi_\lambda(y)[f(y) - f(0)] y^{2\mu+1} dy \right|\\ &\quad \leq C\lambda^{\mu+1/2-\beta} \int_{1/\lambda}^\eta y^{-\mu-\beta-3/2} |f(y) - f(0)| y^{2\mu+1} dy, \qquad \lambda > 1/\eta.\end{aligned}$$

Then, by partial integration it deduces

$$\int_{1/\lambda}^{\eta} y^{-\mu-\beta-3/2}|f(y)-f(0)|y^{2\mu+1}dy = g(y)y^{-\mu-\beta-3/2}\Big]_{y=1/\lambda}^{y=\eta}$$

$$-(\mu+\beta+3/2)\int_{1/\lambda}^{\eta} y^{-\mu-\beta-5/2}g(y)dy, \qquad \lambda > 1/\eta,$$

where $g(y)=\int_0^y |f(t)-f(0)|t^{2\mu+1}dt$, $y>0$.

Hence, we obtain

$$\Big|\int_{1/\lambda}^{\eta} \phi_\lambda(y)[f(y)-f(0)]y^{2\mu+1}dy\Big| \leq C\lambda^{\mu+1/2-\beta}\Big[g(\eta)\eta^{-\mu-\beta-3/2}$$

$$+\lambda^{\mu+\beta+3/2}g(1/\lambda)+\varepsilon(\mu+\beta+3/2)\int_{1/\lambda}^{\eta} y^{\mu-\beta-1/2}dy\Big] \leq C\varepsilon, \quad \lambda > 1/\eta.$$

Finally, by using again the boundedness of the function $\sqrt{z}J_\mu(z)$, it has

$$\Big|\int_{\eta}^{\infty} \phi_\lambda(y)[f(y)-f(0)]y^{2\mu+1}dy\Big|$$

$$\leq C\lambda^{\mu+1/2-\beta}\Big(\int_{\eta}^{\infty}|f(y)|y^{2\mu+1}dy+\int_{\eta}^{\infty} y^{\mu-1/2-\beta}dy\Big) \to 0, \quad \text{as } \lambda\to\infty.$$

Thus we conclude that (5) holds. □

Proposition 2.2. (i) *Let $f \in L_{1,\mu}\cap L_{\infty,\mu}$. If $-\mu-1/2<\alpha<-\mu+3/2$ then*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\int_0^{\lambda} t^\alpha h_\mu(f)(t)t^{2\mu+1}dt = 0.$$

(ii) *Let $f \in L_{1,\mu}$. If $\beta>\mu+3/2$ and 0 is an h_μ-Lebesgue point of f then*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\int_0^{\lambda}\Big(1-\Big(\frac{t}{\lambda}\Big)^2\Big)^{\beta-1}t^2 h_\mu(f)(t)t^{2\mu+1}dt = 0.$$

Proof.

(i) We have to prove that $\lim_{\lambda\to\infty} I(\lambda)=0$, where

$$I(\lambda)=\int_0^{\infty} f(x)K(x,\lambda)x^{2\mu+1}dx, \qquad \lambda>0,$$

and

$$K(x,\lambda)=\frac{1}{\lambda^2}\int_0^\lambda t^\alpha (xt)^{-\mu}J_\mu(xt)t^{2\mu+1}dt,\qquad \lambda,x>0.$$

According to [32, p. 129 (6)], by partial integration it obtains

$$\begin{aligned}K(x,\lambda)&=\frac{x^{-2\mu}}{\lambda^2}\int_0^\lambda t^{\alpha+1}(xt)^\mu J_\mu(xt)dt\\&=\frac{x^{-2\mu-2}}{\lambda^2}\int_0^\lambda t^\alpha\frac{d}{dt}[(xt)^{\mu+1}J_{\mu+1}(xt)]dt\\&=\frac{x^{-2\mu-2}}{\lambda^2}\Big[t^\alpha(xt)^{\mu+1}J_{\mu+1}(xt)\Big]_{t=0}^{t=\lambda}-\alpha\int_0^\lambda t^{\alpha-1}(xt)^{\mu+1}J_{\mu+1}(xt)dt\Big]\\&=\frac{x^{-2\mu-2}}{\lambda^2}\Big[\lambda^\alpha(x\lambda)^{\mu+1}J_{\mu+1}(x\lambda)-\alpha\int_0^\lambda t^{\alpha-1}(xt)^{\mu+1}J_{\mu+1}(xt)dt\Big],\\&\quad\lambda,x>0.\end{aligned}\tag{6}$$

The last equality is justified because $z^{-\nu}J_\nu(z)$, $\nu\geq -1/2$, is bounded on $(0,\infty)$ and $\alpha>-2\mu-2$, since $\mu>-1/2$.

Then, by taking into account again the boundedness of the function $z^{-\nu}J_\nu(z)$, $\nu\geq-1/2$, we have that

$$|K(x,\lambda)|\leq C\lambda^{\alpha+2\mu},\qquad \lambda,x>0.\tag{7}$$

Moreover, since the function $\sqrt{z}J_\nu(z)$ is bounded on $(0,\infty)$, it infers from (6) that

$$|K(x,\lambda)|\leq C\lambda^{\alpha+\mu-3/2}x^{-\mu-3/2},\qquad \lambda,x>0,\tag{8}$$

because $\alpha>-\mu-1/2$.

To estimate $I(\lambda)$ we divide the integral in three parts, namely

$$\begin{aligned}I(\lambda)&=\Big(\int_0^{1/\lambda}+\int_{1/\lambda}^1+\int_1^\infty\Big)f(x)K(x,\lambda)x^{2\mu+1}dx\\&=I_1(\lambda)+I_2(\lambda)+I_3(\lambda),\qquad \lambda>1.\end{aligned}$$

Since $f\in L_{\infty,\mu}$, from (7) it follows

$$|I_1(\lambda)|\leq C\|f\|_{\infty,\mu}\lambda^{\alpha-2},\qquad \lambda>1.$$

We use (8) to obtain

$$\begin{aligned}|I_2(\lambda)| &\leq C\lambda^{\alpha+\mu-3/2}\|f\|_{\infty,\mu}\int_{1/\lambda}^{1} x^{\mu-1/2}dx\\ &= C\|f\|_{\infty,\mu}\lambda^{\alpha+\mu-3/2}\Big(1-\frac{1}{\lambda^{\mu+1/2}}\Big), \quad \lambda>1,\end{aligned}$$

and

$$|I_3(\lambda)| \leq C\lambda^{\alpha+\mu-3/2}\int_1^\infty |f(x)|x^{2\mu+1}dx \leq C\|f\|_{1,\mu}\lambda^{\alpha+\mu-3/2}, \quad \lambda>1.$$

Hence $I(\lambda)\to 0$, as $\lambda\to\infty$, provided that $-\mu-1/2<\alpha<-\mu+3/2$.

(ii) Assume that 0 is an h_μ-Lebesgue point of f. Then, by Lemma 2.1, we have that

$$\lim_{\lambda\to\infty}\frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\lambda \Big(1-\Big(\frac{t}{\lambda}\Big)^2\Big)^\gamma h_\mu(f)(t)t^{2\mu+1}dt = f(0),$$

provided that $\gamma>\mu+1/2$. In particular, if $\beta>\mu+3/2$ we can write

$$\begin{aligned}0 &= \lim_{\lambda\to\infty}\int_0^\lambda \Big[\Big(1-\Big(\frac{t}{\lambda}\Big)^2\Big)^\beta - \Big(1-\Big(\frac{t}{\lambda}\Big)^2\Big)^{\beta-1}\Big]h_\mu(f)(t)t^{2\mu+1}dt\\ &= \lim_{\lambda\to\infty}\frac{1}{\lambda^2}\int_0^\lambda \Big(1-\Big(\frac{t}{\lambda}\Big)\Big)^{\beta-1} t^2 h_\mu(f)(t)t^{2\mu+1}dt.\end{aligned}$$

Thus the proof is finished. □

The next lemma presents sufficient conditions in order that our type Cesàro convergence implies ordinary convergence. To prove this result it can be proceed as in the proof of [2, Lemma 5] and hence we omit the proof.

Lemma 2.2. *Suppose that $f\in L_{1,\mu}$ and that $h_\mu(f)(x)$ is nonincreasing when $x\geq x_0$, for some $x_0>0$. If*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\int_0^\lambda \Big(1-\Big(\frac{t}{\lambda}\Big)^2\Big)^\beta t^\alpha h_\mu(f)(t)t^{2\mu+1}dt = 0,$$

for some $\alpha>-2\mu-2$ and $\beta>-1$, then

$$\lim_{t\to\infty} t^{\alpha+2\mu}h_\mu(f)(t) = 0.$$

An immediate consequence of Propositions 2.1 (ii) and 2.2 (ii) and Lemma 2.2 is the following.

Corollary 2.2. *Assume that $f \in L_{1,\mu}$ and that $h_\mu(f)(x)$ is nonincreasing when $x \geq x_0$, for some $x_0 > 0$.*

(i) *If $f \in L_{p,\mu}$, where $1 < p \leq 2$ and $\mu \leq \frac{p-1}{2}$ then*

$$\lim_{t\to\infty} t^{1/p'+2\mu} h_\mu(f)(t) = 0.$$

(ii) *If 0 is an h_μ-Lebesgue point of f then*

$$\lim_{t\to\infty} t^{2\mu+2} h_\mu(f)(t) = 0.$$

Note that from Corollary 2.2 it deduces conditions on a function f that imply integrability properties for the Hankel transform $h_\mu f$ of f. The next results complete the ones established in [6, Theorems 3.1 and 3.2].

Corollary 2.3. *Let $f \in L_{1,\mu}$. Suppose that there exists $x_0 > 0$ for which $h_\mu(f)(x)$ is nonincreasing when $x \geq x_0$.*

(i) *If $f \in L_{p,\mu}$, where $1 < p \leq 2$ and $\mu \leq \frac{p-1}{2}$ then $t^\alpha h_\mu(f) \in L_{1,\mu}$ provided that $-2\mu - 2 < \alpha < -1 - 1/p$.*

(ii) *If 0 is an h_μ-Lebesgue point of f then $t^\alpha h_\mu(f) \in L_{1,\mu}$, provided that $-2\mu - 2 < \alpha < 0$.*

Proof. It is sufficient to take into account that if $f \in L_{1,\mu}$ then $h_\mu(f)$ is a bounded function on $(0, \infty)$ and to use Corollary 2.2. □

3. Uncertainty inequalities for Hankel transforms

In this section we obtain new uncertainty inequalities for Hankel transforms. Our results are different to the uncertainty principle for h_μ established by Bowie [9, Theorem 2.2] and Rsler and Voit [27, Theorem 1.1].

To prove our first uncertainty property we use, inspired in the procedure developed by [26], Laguerre expansions.

For every $n \in \mathbb{N}$, we denote by ϕ_n^μ the function

$$\phi_n^\mu(x) = \left(\frac{2n!}{\Gamma(n+\mu+1)}\right)^{1/2} e^{-x^2/2} L_n^\mu(x^2), \qquad x \in (0, \infty).$$

Here L_n^μ represents the Laguerre polynomial of degree n and order μ. The sequence $\{\phi_n^\mu\}_{n\in\mathbb{N}}$ forms an orthonormal basis in $L_{2,\mu}$.

We now collect some properties of ϕ_n^μ, $n \in \mathbb{N}$, that will be useful in the sequel.

Let $n \in \mathbb{N}$. ϕ_n^μ is an eigenfunction for the Hankel transformation h_μ associated to the eigenvalue $(-1)^n$, that is,

$$h_\mu(\phi_n^\mu) = (-1)^n \phi_n^\mu \tag{9}$$

([15, 8.9 (3)]).

A straightforward manipulation in [29, (5.1.14)] leads to

$$x^2 \phi_{n-1}^{\mu+1}(x) = -n^{1/2}\phi_n^\mu(x) + (\mu+n)^{1/2}\phi_{n-1}^\mu(x), \quad x \in (0,\infty). \tag{10}$$

Also, from [29, Theorem 5.1] it infers that

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_n^\mu(x)\phi_n^\mu(y) w^{2n} &= \frac{2}{1-w^2} \exp\Big\{ -\frac{1}{2}(x^2+y^2)\frac{1+w^2}{1-w^2}\Big\} e^{-\frac{\pi i \mu}{2}} (xyw)^{-\mu} \\ &\times J_\mu\Big(\frac{2ixyw}{1-w^2}\Big), \quad |w|<1,\ x,y>0. \end{aligned} \tag{11}$$

We now prove a Parseval type equality involving Hankel transfom and Laguerre expansions. By $\langle \cdot, \cdot \rangle_\mu$ we denote the usual inner product in the Hilbert space $L_{2,\mu}$.

Proposition 3.1. *Let $f \in L_{2,\mu} \cap L_{2,\mu+2}$. Then*

$$\int_0^\infty x^6(|f(x)|^2 + |h_\mu(f)(x)|^2) x^{2\mu+1} dx = 2\sum_{n=0}^{\infty} |\left\langle x^2 f, \phi_n^\mu \right\rangle_\mu|^2 (\mu+2n+1).$$

Proof. Parseval's equality for Laguerre expansions and (10) allow us to write

$$\begin{aligned} \int_0^\infty (x^2|f(x)|)^2 x^{2\mu+3} dx &= \sum_{n=0}^{\infty} |\left\langle x^2 f, \phi_n^{\mu+1}\right\rangle_{\mu+1}|^2 \\ &= \sum_{n=0}^{\infty} |\left\langle f, x^2\phi_n^{\mu+1}\right\rangle_{\mu+1}|^2 \\ &= \sum_{n=0}^{\infty} |\langle f, -(n+1)^{1/2}\phi_{n+1}^\mu + (\mu+n+1)^{1/2}\phi_n^\mu\rangle_{\mu+1}|^2. \end{aligned}$$

Also, Parseval's equality for Hankel transformation leads, by taking into account (9) and (10), to

$$\begin{aligned}\int_0^\infty (x^2|h_\mu(f)(x)|)^2 x^{2\mu+3}dx &= \sum_{n=0}^\infty |\left\langle x^2 h_\mu(f), \phi_n^{\mu+1}\right\rangle_{\mu+1}|^2 \\ &= \sum_{n=0}^\infty |\left\langle h_\mu(f), x^2\phi_n^{\mu+1}\right\rangle_{\mu+1}|^2 \\ &= \sum_{n=0}^\infty |\left\langle h_\mu(f), -(n+1)^{1/2}\phi_{n+1}^\mu + (\mu+n+1)^{1/2}\phi_n^\mu\right\rangle_{\mu+1}|^2 \\ &= \sum_{n=0}^\infty |\left\langle f, (n+1)^{1/2}\phi_{n+1}^\mu + (\mu+n+1)^{1/2}\phi_n^\mu\right\rangle_{\mu+1}|^2.\end{aligned}$$

Hence, it obtains that

$$\begin{aligned}&\int_0^\infty x^6(|f(x)|^2 + |h_\mu(f)(x)|^2)x^{2\mu+1}dx \\ &= 2\left(\sum_{n=0}^\infty |\left\langle x^2 f, \phi_{n+1}^\mu\right\rangle_\mu|^2(n+1) + \sum_{n=0}^\infty (\mu+n+1)|\left\langle x^2 f, \phi_n^\mu\right\rangle_\mu|^2\right) \\ &= 2\sum_{n=0}^\infty |\left\langle x^2 f, \phi_n^\mu\right\rangle_\mu|^2(\mu+2n+1). \qquad \square\end{aligned}$$

An immediate consequence of Proposition 3.1 is the following.

Corollary 3.1. *For every $f \in L_{2,\mu} \cap L_{2,\mu+2}$, it has*

$$\int_0^\infty x^6(|f(x)|^2 + |h_\mu(f)(x)|^2)x^{2\mu+1}dx \geq 2(\mu+1)\|f\|_{2,\mu+2}^2. \tag{12}$$

Moreover, equality holds if, and only if, $f(x) = c\frac{e^{-x^2/2}}{x^2}$, $x \in (0,\infty)$, for a certain $c \in \mathbb{C}$.

Note that if the equality in (12) holds then $\mu > 1$.

We now deduce from Corollary 3.1 our first uncertainty inequality involving Hankel transforms.

Proposition 3.2. *If* $f \in L_{2,\mu} \cap L_{2,\mu+2}$ *then*

$$\|x^3 f\|_{2,\mu}^5 \, \|x^3 h_\mu(f)\|_{2,\mu} \geq \frac{5^{5/2}}{27}(\mu+1)^3 \|f\|_{2,\mu+2}^6. \tag{13}$$

Moreover, the equality holds if, and only if, $f(x) = c\frac{e^{-(\alpha x)^2}}{x^2}$, *for certain* $c \in \mathbb{C}$ *and* $\alpha \in \mathbb{R}$.

Proof. Suppose that $\|f\|_{2,\mu+2} = 1$ and define, for every $\alpha > 0$, f_α by

$$f_\alpha(x) = \alpha^{-\mu-3} f\left(\frac{x}{\alpha}\right), \quad x \in (0, \infty).$$

It is not hard to see that $\|f_\alpha\|_{2,\mu+2} = 1$ and that $h_\mu(f_\alpha)(x) = \alpha^{\mu-1} h_\mu(f)(\alpha x)$, $x, \alpha > 0$.

Corollary 3.1 implies that

$$\int_0^\infty x^6 (|f_\alpha(x)|^2 + |h_\mu(f_\alpha)(x)|^2) x^{2\mu+1} dx \geq 2(\mu+1).$$

We now define the function $F_f(\alpha)$ by

$$F_f(\alpha) = \int_0^\infty x^6 (|f_\alpha(x)|^2 + |h_\mu(f_\alpha)(x)|^2) x^{2\mu+1} dx, \quad \alpha > 0.$$

It is clear that

$$F_f(\alpha) = \alpha^2 \|x^3 f\|_{2,\mu}^2 + \frac{1}{\alpha^{10}} \|x^3 h_\mu(f)\|_{2,\mu}^2, \quad \alpha > 0.$$

The function F_f obtains the minimum in $\alpha_0 = \left(\frac{\sqrt{5}\|x^3 h_\mu(f)\|_{2,\mu}}{\|x^3 f\|_{2,\mu}}\right)^{1/6}$, being

$$F_f(\alpha_0) = \frac{6}{5^{5/6}} \|x^3 f\|_{2,\mu}^{5/3} \|x^3 h_\mu(f)\|_{2,\mu}^{1/3}.$$

Hence, we conclude

$$\|x^3 f\|_{2,\mu}^5 \, \|x^3 h_\mu(f)\|_{2,\mu} \geq \frac{5^{5/2}}{27}(\mu+1)^3.$$

To determinate the functions f for which the equality in (13) holds it is sufficient to use Corollary 3.1. □

A celebrated result due to Hardy ([11, §3.2], for instance) established that if f is a measurable function on $\mathbb{R}^n$, $n \geq 1$, for which

$$|f(x)| \leq Ce^{-\alpha|x|^2}, \quad |\mathcal{F}(f)(x)| \leq Ce^{-\beta|x|^2}, \qquad x \in \mathbb{R}^n,$$

where α, β are positive real numbers such that $\alpha\beta > 1/4$ and $\mathcal{F}(f)$ denotes the Euclidean Fourier transform of f defined by

$$\mathcal{F}(f)(x) = \int_{\mathbb{R}^n} e^{-ixy} f(y)dy, \qquad x \in \mathbb{R}^n,$$

then $f = 0$. A version of this property in symmetric spaces of non compact type was established in [25, Theorem 3].

We now state a Hardy type result for Hankel transforms.

Proposition 3.3. *Let $\alpha, \beta > 0$ and let f be a measurable function on $(0,\infty)$. If $\alpha\beta > 1/4$ and, for every $x > 0$,*

$$|f(x)| \leq Ce^{-\alpha x^2} \quad \textit{and} \quad |h_\mu(f)(x)| \leq Ce^{-\beta x^2},$$

then $f = 0$.

Proof. By taking into account that $|z^{-\mu}J_\mu(z)| \leq Ce^{|\operatorname{Im} z|}$, $z \in \mathbb{C}$, ([24, Lemma 4]), it is sufficient to proceed as in the proof of the Hardy's result ([11, §3.2] or [25, Theorem 3]). $\square$

As a consequence of Hardy theorem for Hankel transforms (Proposition 3.3) we prove, in the spirit of [25], an uncertainty principle associated to Laguerre expansions.

Proposition 3.4. *Let f be a measurable function on $(0,\infty)$ such that*

$$|f(x)| \leq Ce^{-\alpha x^2}, \qquad x \in (0,\infty),$$

and

$$|\langle f, \phi_n^\mu \rangle_\mu| \leq Ce^{-\beta n}, \qquad n \in \mathbb{N}, \tag{14}$$

for some $\alpha, \beta > 0$. If $\alpha \operatorname{tgh} \frac{\beta}{4} > \frac{1}{4}$ then $f = 0$.

PROOF. It is clear that $f \in L_{1,\mu} \cap L_{2,\mu}$. Then we have that

$$f = \sum_{n=0}^{\infty} \langle f, \phi_n^\mu \rangle_\mu \phi_n^\mu, \tag{15}$$

in the sense of convergence in $L_{2,\mu}$. Moreover, according to (9) and [19, Theorem 3], it has

$$h_\mu(f) = \sum_{n=0}^{\infty} \langle f, \phi_n^\mu \rangle_\mu (-1)^n \phi_n^\mu, \tag{16}$$

where the convergence of the series is also understood in $L_{2,\mu}$. On the other hand, (14) and [10, Corollary 2.2] imply that the series in (15) and (16) converge absolute and uniformly in $x \in [0, \infty)$. Hence, it follows that

$$h_\mu(f)(x) = \sum_{n=0}^{\infty} \langle f, \phi_n^\mu \rangle_\mu (-1)^n \phi_n^\mu(x), \qquad x \in (0, \infty).$$

Then, Cauchy–Schwartz's inequality leads to

$$\begin{aligned} |h_\mu(f)(x)| &\leq \left(\sum_{n=0}^{\infty} |\langle f, \phi_n^\mu \rangle_\mu| \right)^{1/2} \left(\sum_{n=0}^{\infty} |\langle f, \phi_n^\mu \rangle_\mu| \phi_n^\mu(x)^2 \right)^{1/2} \\ &\leq C \left(\sum_{n=0}^{\infty} e^{-\beta n} \right)^{1/2} \left(\sum_{n=0}^{\infty} e^{-\beta n} \phi_n^\mu(x)^2 \right)^{1/2}, \\ &\quad x \in (0, \infty). \end{aligned} \tag{17}$$

By letting $x = y$ and $w = e^{-\beta/2}$ in (11) it obtains

$$\begin{aligned} &\sum_{n=0}^{\infty} e^{-\beta n} \phi_n^\mu(x)^2 \\ &= \frac{2}{1 - e^{-\beta}} \exp\left\{ -x^2 \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right\} (x^2 e^{-\beta/2})^{-\mu} e^{-\pi i \mu/2} J_\mu\left(\frac{2x^2 i e^{-\beta/2}}{1 - e^{-\beta}} \right) \\ &= \frac{2^{\mu+1}}{(1 - e^{-\beta})^{\mu+1}} \exp\left\{ -x^2 \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right\} \left(\frac{2x^2 e^{-\beta/2}}{1 - e^{-\beta}} \right)^{-\mu} I_\mu\left(\frac{2x^2 e^{-\beta/2}}{1 - e^{-\beta}} \right), \\ &\qquad x \in (0, \infty), \end{aligned}$$

where I_μ denotes the modified Bessel function of the first kind and order μ ([31, p. 77]). According to [31, §3.71 (9)] we have

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-\beta n}\phi_n^\mu(x)^2 &\leq C\exp\left\{-x^2\frac{e^{\beta/2}+e^{-\beta/2}-2}{e^{\beta/2}-e^{-\beta/2}}\right\}\\ &\leq C\exp\left\{-x^2\operatorname{tgh}\frac{\beta}{4}\right\}, \quad x\in(0,\infty).\end{aligned} \tag{18}$$

Hence, from (17) and (18) it deduces that

$$|h_\mu(f)(x)| \leq C\exp\left\{-x^2\operatorname{tgh}\frac{\beta}{4}\right\}, \qquad x\in(0,\infty).$$

Proposition 3.3 allows now to conclude that $f=0$ provided that $\alpha\operatorname{tgh}\frac{\beta}{4}>\frac{1}{4}$. □

We now obtain weighted uncertainty principle for Hankel transforms. To establish these properties we use weighted Hausdorff–Young inequalities for Hankel transformation. We say that a pair (u,v) of nonnegative measurable functions on $(0,\infty)$ is in $HY_\mu(p,q)$, $1\leq p,\ q<\infty$, when there exists a constant $A>0$ such that

$$\begin{aligned}&\left\{\int_0^\infty u(x)|h_\mu(f)(x)|^q x^{2\mu+1}dx\right\}^{1/q}\\ &\leq A\left\{\int_0^\infty v(x)|f(x)|^p x^{2\mu+1}dx\right\}^{1/p},\end{aligned} \tag{19}$$

for every continuous and compactly supported functions on $(0,\infty)$. We collect some pairs (u,v) of functions in $HY_\mu(p,q)$ for some $1\leq p,q<\infty$. [19, Theorem 3] implies that $(\mathbf{1},\mathbf{1})\in HY_\mu(p,p')$, for every $1<p\leq 2$, where $\mathbf{1}$ represents the function $\mathbf{1}(x)=1$, $x\in(0,\infty)$. From [20, Lemma 1] it deduces that $(x^{(\mu+3/2-\lambda)q-2\mu-2}, x^{(\lambda+\mu+1/2)p-2\mu-2})\in HY_\mu(p,q)$, provided that $1<p\leq q<\infty$ and $\max\{1/p,1/q'\}\leq\lambda<\mu+3/2$. According to [14], we say that $(u,v)\in F^*_{p,q}$ if

$$\sup_{s>0}\left\{\int_0^{1/s} u^*(t)^q dt\right\}^{1/q}\left\{\int_0^s \left(\frac{1}{v}\right)^*(t)^{p'}dt\right\}^{1/p'}<\infty, \tag{20}$$

holds, for $1 < p \le q < \infty$, and, in the case $1 < q < p < \infty$ the conditions

$$\int_0^\infty \left[\Big(\int_0^{1/x} u^*(t)^q dt \Big)^{1/q} \left(\int_0^x \left(\frac{1}{v}\right)^* (t)^{p'} dt \right)^{1/q'} \right]^r \left[\left(\frac{1}{v}\right)^* (x) \right]^{p'} dx < \infty$$

and

$$\int_0^\infty \left[\left(\int_{1/x}^\infty [t^{-1/2} u^*(t)]^q dt \right)^{1/q} \left(\int_x^\infty \left[t^{-1/2} \left(\frac{1}{v}\right)^* (t) \right]^{p'} dt \right)^{1/q'} \right]^r \times \left[x^{-1/2} \left(\frac{1}{v}\right)^* (x) \right]^{p'} dx < \infty,$$

hold, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Here, if g is a measurable function on $(0, \infty)$, by g^* we understand, as usual, the equimeasurable decreasing rearrangements of g. By invoking [14, Theorem 3] or [13, Theorem 11] if $((ux^{2\mu+1})^{1/q}, (vy^{(2\mu+1)(1-p)})^{1/p}) \in F^*_{p,q}$, with $1 < p,\ q < \infty$, then $(u, v) \in HY_\mu(p, q)$. In [20, Theorem 1] other conditions on u and v analogous to (20) that do not involve equimeasurable rearrangements and which imply that $(u, v) \in HY_\mu(p, q)$ are presented.

Next we establish a weighted uncertainty inequality that can be seen as a modest extension of [9, Theorem 2.2]. By $\mathcal{S}_e$ we represent the subspace of $\mathcal{S}$ constituted by all those even functions belonging to the Schwartz space $\mathcal{S}$. The Hankel transformation is investigated on $\mathcal{S}_e$ by Altenburg [1] (see [12] also).

Proposition 3.5. *Let $f \in \mathcal{S}_e$. If $1 \le p,\ q < \infty$ and $(u^{1-q}, v) \in HY_{\mu+1}(p, q)$ then*

$$\left\{ \int_0^\infty u(x) |x^2 f(x)|^{q'} x^{2\mu+1} dx \right\}^{1/q'} \left\{ \int_0^\infty v(x) |x^2 h_\mu(f)(x)|^p x^{2\mu+1} dx \right\}^{1/p} \ge \frac{\mu+1}{A} \|f\|^2_{2,\mu},$$

being A the constant appearing in (19), when we replace u and μ by u^{1-q} and $\mu + 1$, respectively ($(u^{1-q}, v) \in HY_{\mu+1}(p, q)$).

Proof. Let $f \in \mathcal{S}_e$. Then as it is showed in the proof of [9, Theorem 2.2] we have that

$$\int_0^\infty x f(x) \frac{d}{dx}(f(x)) x^{2\mu+1} dx = (\mu+1) \int_0^\infty |f(x)|^2 x^{2\mu+1} dx.$$

Hence, Hölder's inequality and [1, §4.1 (3)] lead to

$$\begin{aligned}
(\mu+1)\|f\|_{2,\mu}^2 \leq & \left\{ \int_0^\infty u(x) |x^2 f(x)|^{q'} x^{2\mu+1} dx \right\}^{1/q'} \\
& \times \left\{ \int_0^\infty \left| \frac{1}{u(x)^{1/q'}} \left(\frac{1}{x} \frac{d}{dx} \right) f(x) \right|^q x^{2\mu+1} dx \right\}^{1/q} \\
& = \left\{ \int_0^\infty u(x) |x^2 f(x)|^{q'} x^{2\mu+1} dx \right\}^{1/q'} \\
& \times \left\{ \int_0^\infty \left| \frac{1}{u(x)^{1/q'}} h_{\mu+1}(y^2 h_\mu(f)(y))(x) \right|^q x^{2\mu+1} dx \right\}^{1/q} \\
\leq A & \left\{ \int_0^\infty u(x) |x^2 f(x)|^{q'} x^{2\mu+1} dx \right\}^{1/q'} \left\{ \int_0^\infty v(x) |x^2 h_\mu(f)(x)|^p x^{2\mu+1} dx \right\}^{1/p},
\end{aligned}$$

and the proof is completed. $\square$

If $g \in L_2(\mathbb{R})$ and $\int_{-\infty}^\infty |g(x)|^2 dx = 1$, the entropy $E[g]$ of g is defined by

$$E[g] = \int_{-\infty}^\infty g(x) \log[g(x)] dx.$$

Hirschman [21] established that

$$E\left[|g|^2\right] + E\left[|\mathcal{F}(g)|^2\right] \leq E_H \tag{21}$$

where $E_H = 0$ and $\mathcal{F}$ represents the Fourier transform on $\mathbb{R}$. Moreover, he conjetured that $E_H = -1 + \log 2$. This can be proved by invoking a sharp form for the Hausdorff–Young inequality for the Fourier transform ([3]).

Herz [19, Theorem 3] established a Hausdorff–Young inequality for Hankel transformation. That is, for every $1 < p \leq 2$ and $f \in L_{p,\mu}$ it has

$$\begin{aligned}&\int_0^\infty |h_\mu(f)(x)|^{p/(p-1)} x^{2\mu+1} dx \\ &\leq (2^\mu \Gamma(\mu+1))^{\frac{p-2}{p-1}} \left\{ \int_0^\infty |f(x)|^p x^{2\mu+1} dx \right\}^{1/(p-1)}.\end{aligned} \tag{22}$$

Later, Fitouhi [16] established a Hankel version of the Babenko inequality ([3, Theorem 1]). In [16, Theorem 4.1] it was proved that, for every $1 < p \leq 2$ and $f \in L_{p,\mu}$,

$$\begin{aligned}&\int_0^\infty |h_\mu(f)(x)|^{p/(p-1)} x^{2\mu+1} dx \\ &\leq \left[p^{(2-p)/(p-1)}(p-1)\right]^{\mu+1} \left[2^\mu \Gamma(\mu+1)\right]^{\frac{p-2}{p-1}} \left\{ \int_0^\infty |f(x)|^p x^{2\mu+1} dx \right\}^{1/(p-1)}.\end{aligned} \tag{23}$$

Equality in (23) holds when $f(x) = e^{-x^2}$, $x \in (0,\infty)$. Thus (23) is a sharp Hausdorff–Young inequality for Hankel transforms.

By using (23) we now obtain an entropy inequality for Hankel transforms that can be seen as a Hankel version of the Hirschman's inequality (21).

Proposition 3.6. *Let $f \in L_{1,\mu} \cap L_{2,\mu}$. If $\|f\|_{2,\mu} = 1$ then*

$$\begin{aligned}&\int_0^\infty |h_\mu(f)(x)|^2 \log |h_\mu(f)(x)| x^{2\mu+1} dx \\ &+ \int_0^\infty |f(x)|^2 \log |f(x)| x^{2\mu+1} dx \leq -(\mu+1) - \log \left[\frac{\Gamma(\mu+1)}{2}\right].\end{aligned}$$

Proof. To prove this inequality we can proceed in a standard way ([4]). It is sufficient to differentiate both sides in (23) and then to consider $p = 2$. □

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