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Norming operators for generalized domains of semistable attraction

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Abstract. It is shown that the norming operators of a measure belonging to the generalized domain of semistable attraction of a full operator semistable law can be embedded into a regularly varying sequence of linear operators. This powerful property is then used to show stochastic compactness results of the partial sum.

1. Introduction

Suppose that $X, X_1, X_2, \dots$ are independent and identically distributed random vectors on $\mathbb{R}^d$ with common distribution μ. Let $S_n = \sum_{i=1}^{n} X_i$ denote the partial sum. We say that μ is in the *generalized domain of semistable attraction* of a full (that is not concetrated on any proper hyperplane) measure ν, if there exists a sequence of positive integers $k_n \to \infty$ with $k_{n+1}/k_n \to c \geq 1$, linear operators A_n and nonrandom vectors a_n such that

$$A_n S_{k_n} - a_n \Rightarrow Y \tag{1.1}$$

where Y has distribution ν. Here $\Rightarrow$ denotes convergence in distribution. We write $\mu \in \mathrm{GDOSA}(\nu, c)$ if (1.1) holds.

If $c = 1$ then by a result of Jajte [5] ν is operator stable. In this case we can set $k_n = n$ in (1.1) and the in this case called generalized

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domains of attraction were characterized by Meerschaert [10]. On the other hand, if $c > 1$ then by [5] ν is (B, c) operator semistable, i.e. satisfies

$$\nu^c = B\nu * \delta(b) \tag{1.2}$$

for some invertible linear operator B and some vector $b \in \mathbb{R}^d$. Here ν^c is the c-fold convolution power of the (necessarely) infinitely divisible law ν, $(B\nu)(A) = \nu(B^{-1}A)$ for all Borel sets $A \subset \mathbb{R}^d$, $*$ denotes convolution and $\delta(b)$ denotes the point mass in b. We say that ν is (B, c) operator semistable if (1.2) holds.

In this case the generalized domains of semistable attraction were characterized in [11] and [14]. Earlier, the author [15] investigated the special case of norming by scalars, $A_n = a_n I$ for some $a_n > 0$, and the case of "vector norming" where every A_n is assumed to be diagonal.

In this paper we analyse the norming operators A_n in (1.1). We show that the sequence (A_n) can be embedded in a regularly varying sequence (B_n) of linear operators; that is $A_n = B_{k_n}$ and $B_{[\lambda n]}B_n^{-1} \to \lambda^{-E}$ where E is closely related to the limit distribution ν. This embedding property leads to a new proof of the spectral decomposition for generalized domains of attraction in [13].

Then we investigate the asymptotic behavior of the whole sequence of partial sums $(S_n)_n$. We show that for some linear operators B_n and nonrandom vectors b_n the sequence $(B_n S_n - b_n)_n$ is stochastically compact and describe its limit set. Furthermore, we show that the sum of the radial projections of the X_i onto a fixed direction θ is stochastically compact. All these results give additional information on the properties of random vectors attracted to operator semistable laws. They will enable us in [19] to prove large deviation type results as well as laws of the iterated logarithm.

2. The embedding property

Before we state the so-called embedding property for generalized domains of semistable attraction we first need to restate some results and notation.

A result in Chorny [3] states that the operator B in (1.2) can be chosen in the image of the exponential mapping of the Lie group $\mathrm{GL}(\mathbb{R}^d)$. Here $t^E = \exp(A \log t)$ for $t > 0$ and some $d \times d$ matrix A, where $\exp(A) = \sum_{k=0}^{\infty} A^k/k!$ denotes the usual exponential mapping on the Lie group $\mathrm{GL}(\mathbb{R}^d)$.

Theorem 2.1 (Chorny [3]). *Assume that ν is a full (B, c') operator semistable law for some $c' > 1$. Then there exists an* exponent E *and a* $c \in \{c', {c'}^2\}$ *such that ν is (c^E, c) operator semistable. All eigenvalues of E lie in the half plane $\{\operatorname{Re} z \geq \frac{1}{2}\}$. Furthermore, the measure ν can be decomposed into a convolution product $\nu = \nu_1 * \nu_2$ of two measures ν_1 and ν_2, concentrated on E-invariant subspaces W_1 and W_2, respectively, and such that $\mathbb{R}^d = W_1 \oplus W_2$, ν_1 is a full normal law on W_1 and ν_2 is a full operator semistable law on W_2 having no normal component. The eigenvalues of $E|_{W_1}$ have real part equal to $\frac{1}{2}$ whereas the eigenvalues of $E|_{W_2}$ are contained in $\{\operatorname{Re} z > \frac{1}{2}\}$.*

Since in view of Theorem 2.1 we might have to square c to get an exponent E one may ask whether this affects the corresponding generalized domains of semistable attraction. The following Lemma shows that $\mathrm{GDOSA}(\nu, c) = \mathrm{GDOSA}(\nu, c^2)$, so when dealing with operator semistable laws and their generalized domains of semistable attraction we can assume without loss of generality that ν is (c^E, c) operator semistable. Unless otherwise stated, we will assume this throughout this paper.

Lemma 2.2. *Assume that ν is a full (B, c) operator semistable law for some $c > 1$. Then* $\mathrm{GDOSA}(\nu, c) = \mathrm{GDOSA}(\nu, c^2)$.

Proof. Note that by passing to the subsequence of even numbers one inclusion is obvious. Assume now that $\mu \in \mathrm{GDOSA}(\nu, c^2)$. Then there exists a sequence (k_n) of natural numbers tending to infinity with $k_{n+1}/k_n \to c^2$, linear operators A_n and nonrandom vectors a_n such that $A_n \mu^{k_n} * \delta(-a_n) \Rightarrow \nu$. Note that in view of (1.2) we have
$B^{-1}\nu^c * \delta(-B^{-1}b) = \nu$. Now we put

$$\bar{k}_n = \begin{cases} k_\ell & \text{if } n = 2\ell \\ [ck_\ell] & \text{if } n = 2\ell + 1 \end{cases}$$

$$\bar{A}_n = \begin{cases} A_\ell & \text{if } n = 2\ell \\ B^{-1}A_\ell & \text{if } n = 2\ell + 1 \end{cases}$$

$$\bar{a}_n = \begin{cases} a_\ell & \text{if } n = 2\ell \\ \dfrac{[ck_\ell]}{k_\ell} a_\ell + B^{-1}b & \text{if } n = 2\ell + 1. \end{cases}$$

Then, by considering the subsequences of even and odd numbers separately, on easily gets that $\bar{k}_{n+1}/\bar{k}_n \to c$ and

$$\bar{A}_n \mu^{\bar{k}_n} * \delta(-\bar{a}_n) \Rightarrow \nu$$

showing $\mu \in \mathrm{GDOSA}(\nu, c)$. This concludes the proof. □

Suppose now that $\mu \in \mathrm{GDOSA}(\nu, c)$ for some $c > 1$ where ν is a full (c^E, c) operator semistable law. Let $X_1, X_2, \dots$ be independent and identically distributed according to μ such that (1.1) holds for some random vector Y with distribution ν. Throughout this section we will always assume that these assumptions hold. We show that the norming operators A_n in (1.1) can always be embedded in a regularly varying sequence $(B_n)_{n\geq 1} \subset \mathrm{GL}(\mathbb{R}^d)$ of index $(-E)$:

A sequence $(B_n) \subset \mathrm{GL}(\mathbb{R}^d)$ is said to be regularly varying with index $(-E)$ if

$$B_{[\lambda n]} B_n^{-1} \to \lambda^{-E} \quad \text{as } n \to \infty \tag{2.1}$$

for all $\lambda > 0$. The theory of regularly varying functions and sequences was developed my Meerschaert [8]. Note that by Theorem 2.2 of [8] the convergence in (2.1) is uniform on compact subsets of $\{\lambda > 0\}$. For further information on regular variation on $\mathrm{GL}(\mathbb{R}^d)$ see [12]. We call the above proberty the *embedding property* of the A_n.

Using a general spectral decomposition theorem for regular varying sequences of linear operators proved in [12], we then show that the A_n can be decomposed even further which yields to sharp bounds on the growth rates of $\|A_n\theta\|$ depending on the direction θ. This will lead to a new proof of the spectral decomposition for generalized domains of semistable attraction in [13].

Definition 2.3. The symmetry group of a probability distribution ρ on $\mathbb{R}^d$ is defined by

$$\mathcal{S}(\rho) = \{A \in \mathrm{GL}(\mathbb{R}^d) : A\rho = \rho * \delta(a) \text{ for some } a \in \mathbb{R}^d\}.$$

A result of Billingsley [1] states that for full probability measures ρ the set $\mathcal{S}(\rho)$ is a compact subgroup of $\mathrm{GL}(\mathbb{R}^d)$.

The following result is well known (see e.g. [20] and [7]). We include it here for sake of completeness.

Proposition 2.4. *Let ν be a full (c^E, c) operator semistable law. Then*

$$c^E \mathcal{S}(\nu) = \mathcal{S}(\nu) c^E.$$

Since the symmetry group $\mathcal{S}(\nu)$ is a compact subgroup of $\mathrm{GL}(\mathbb{R}^d)$, for the remainder of this section we can choose a norm on $\mathbb{R}^d$ which makes every element of $\mathcal{S}(\nu)$ orthogonal and we denote by $\|A\|$ the corresponding operator norm. Then if $G \in \mathcal{S}(\nu)$ and $A \in \mathrm{GL}(\mathbb{R}^d)$ we have $\|GA\| = \|AG\| = \|A\|$. If A is an element of $\mathrm{GL}(\mathbb{R}^d)$ and $\mathcal{C}$ is a compact subset of $\mathrm{GL}(\mathbb{R}^d)$ we will denote the distance between A and $\mathcal{C}$ by

$$\|A - \mathcal{C}\| = \min\{\|A - G\| : G \in \mathcal{C}\}. \tag{2.2}$$

Then (1.1) and the convergence of types theorem (see [1]) imply

Proposition 2.5. *Let A_n be the norming operators in* (1.1) *for some full (c^E, c) operator semistable law ν. Then as $n \to \infty$*

$$\|A_{n+1}A_n^{-1} - c^{-E}\mathcal{S}(\nu)\| \to 0.$$

Proof. Since $A_n\mu^{k_n} * \delta(-a_n) \Rightarrow \nu$ by (1.1), a straight forward computation using characteristic functions shows that for a suitable sequence of shifts (a_n') we have $A_{n+1}\mu^{k_n} * \delta(-a_n') \Rightarrow \nu^{1/c}$. Since $\nu^c = c^E\nu * \delta(b)$ for some $b \in \mathbb{R}^d$ it follows that $\nu^{1/c} = c^{-E}\nu * \delta(-c^{-E}b/c)$. Then convergence of types yields that $\{A_{n+1}A_n^{-1}\}_{n\geq 1}$ is relatively compact with all limit points in the compact set $c^{-E}\mathcal{S}(\nu)$. Then every subsequence contains a further subsequence (n_2) and a $G \in \mathcal{S}(\nu)$ such that $A_{n_2+1}A_{n_2}^{-1} \to c^{-E}G$. But then $\|A_{n_2+1}A_{n_2}^{-1} - c^{-E}\mathcal{S}(\nu)\| \leq \|A_{n_2+1}A_{n_2}^{-1} - c^{-E}G\| \to 0$, which concludes the proof.

By convergence of types [1], we may replace the sequence A_n in (1.1) by any other norming sequence of the form $C_n = G_nA_n$, where $G_n \in \mathcal{S}(\nu)$ for all n. Then for some choice of shifts a_n', we still get weak convergence to the same limiting distribution ν.

Before we state the main theorem of this section we first need an auxiliary result which is also of independent interest.

Proposition 2.6. *We can choose the norming operators C_n such that*

$$C_{n+1}C_n^{-1} \to c^{-E} \quad \text{as } n \to \infty.$$

Proof. Let $C_1 = A_1$ and assume that $C_1, \dots, C_{n-1}$ have been constructed. Choose $G_n' \in \mathcal{S}(\nu)$ such that $\|A_nC_{n-1}^{-1} - c^{-E}\mathcal{S}(\nu)\| = \|A_nC_{n-1}^{-1} - c^{-E}G_n'\|$. By Proposition 2.4 there exists a $G_n \in \mathcal{S}(\nu)$ with $c^{-E}G_n' =$

$G_n c^{-E}$. Define $C_n = G_n^{-1} A_n$. Then (C_n) is a suitable sequence of norming operators, and we also have

$$
\begin{aligned}
\|C_n C_{n-1}^{-1} - c^{-E}\| &= \|G_n^{-1} A_n C_{n-1}^{-1} - c^{-E}\| \\
&= \|G_n^{-1}(A_n C_{n-1}^{-1} - G_n c^{-E})\| \\
&= \|A_n C_{n-1}^{-1} - c^{-E} G_n'\| \\
&= \|A_n C_{n-1}^{-1} - c^{-E} \mathcal{S}(\nu)\| \\
&= \|A_n A_{n-1}^{-1} G_{n-1} - c^{-E} \mathcal{S}(\nu)\| \\
&= \min\{\|A_n A_{n-1}^{-1} G_{n-1} - c^{-E} G\| : G \in \mathcal{S}(\nu)\} \\
&= \min\{\|A_n A_{n-1}^{-1} G_{n-1} - c^{-E} G G_{n-1}\| : G \in \mathcal{S}(\nu)\} \\
&= \min\{\|(A_n A_{n-1}^{-1} - c^{-E} G) G_{n-1}\| : G \in \mathcal{S}(\nu)\} \\
&= \min\{\|A_n A_{n-1}^{-1} - c^{-E} G\| : G \in \mathcal{S}(\nu)\} \\
&= \|A_n A_{n-1}^{-1} - c^{-E} \mathcal{S}(\nu)\| \to 0
\end{aligned}
$$

as $n \to \infty$ by Proposition 2.5. This concludes the proof. □

After these preliminary results we are now in position to prove the main result of this section, the *embedding property* of the norming operators A_n in (1.1).

Theorem 2.7. *Let $\mu \in \mathrm{GDOSA}(\nu, c)$ for some $c > 1$, where ν is full and (c^E, c) operator semistable. Then there exists a sequence (A_n) of norming operators such that* (1.1) *holds and a sequence (B_n) regularly varying with index $(-E)$, such that $A_n = B_{k_n}$, where (k_n) is the sampling sequence in* (1.1).

Proof. In view of Proposition 2.6 we can choose norming operators (A_n) such that $A_{n+1} A_n^{-1} \to c^{-E}$ as $n \to \infty$. Now for any $n \geq k_1$ write $n = \lambda(n) k_{p(n)}$ where $k_{p(n)} \leq n < k_{p(n)+1}$ and define

$$B_n = \lambda(n)^{-E} A_{p(n)}. \tag{2.3}$$

Since $\lambda(k_n) = 1$ and $p(k_n) = n$ we have $B_{k_n} = A_n$. Furthermore, since $k_{n+1}/k_n \to c$ it follows that $(\lambda(n))$ is relatively compact with all limit points in the set $[1, c]$.

It remains to show that (B_n) is regularly varying with index $-E$. We show that

$$B_{[\alpha_n n]} B_n^{-1} \to \alpha^{-E} \quad \text{as } n \to \infty \tag{2.4}$$

whenever $\alpha_n \to \alpha > 0$.

Assume first, that $\alpha \in [1, c]$. Write $[\alpha_n n] = \lambda([\alpha_n n]) k_{p([\alpha_n n])}$. We have to consider several cases separately:

Case 1: Assume $p(n) = p([\alpha_n n])$ along a subsequence (n'). Then we have $\lambda([\alpha_n n])/\lambda(n) = [\alpha_n n]/n \to \alpha$ and

$$B_{[\alpha_n n]} B_n^{-1} = \lambda([\alpha_n n])^{-E} A_{p([\alpha_n n])} A_{p(n)}^{-1} \lambda(n)^E = \Big(\frac{[\alpha_n n]}{n}\Big)^{-E} \to \alpha^{-E}$$

along (n').

Case 2: Assume $p([\alpha_n n]) = p(n) + 1$ along a subsequence (n'). Then, along that subsequence $\lambda([\alpha_n n]) = ([\alpha_n n]/n) \cdot (k_{p(n)}/k_{p(n)+1})\lambda(n)$. Since $(\lambda(n))$ is relatively compact every subsequence (n'') of (n') contains a further subsequence (n''') with $\lambda(n) \to \lambda \in [1, c]$ along (n''') and hence $\lambda([\alpha_n n]) \to \alpha c^{-1}\lambda$ and therefore

$$B_{[\alpha_n n]} B_n^{-1} = \lambda([\alpha_n n])^{-E} A_{p(n)+1} A_{p(n)}^{-1} \lambda(n)^E \to (\alpha c^{-1}\lambda)^{-E} c^{-E} \lambda^E = \alpha^{-E}$$

along (n'''). Since every subsequence (n'') of (n') contains a further subsequence with this property we get (2.4) along the whole subsequence (n').

Case 3: Assume $p([\alpha_n n]) = p(n) + 2$ along (n'). This case is similar to Case 2.

Case 4: Assume that $p([\alpha_n n]) \geq p(n) + 3$ for infinitely many n, say along (n'). Then for every $\varepsilon > 0$ the relative compactness of the sequence $(\lambda(n))$ yields for all large n'

$$\frac{n}{[\alpha_n n]} = \frac{\lambda(n) k_{p(n)}}{\lambda([\alpha_n n]) k_{p([\alpha_n n])}} \leq (c + \varepsilon) \frac{k_{p(n)}}{k_{p(n)+3}}.$$

But the left hand side of the inequality above tends to α^{-1} whereas the right hand side tends to $(c + \varepsilon)c^{-3}$ along (n'). Hence $\alpha \geq (1 + \varepsilon/c)^{-1} c^2$ which contradicts $\alpha \leq c$ if ε is chosen small enough.

Case 5: Assume $p([\alpha_n n]) = p(n) - 1$ along (n'). Then we have $\lambda([\alpha_n n]) = ([\alpha_n n]/n)(k_{p(n)}/k_{p(n)-1})\lambda(n)$ and it follows as in Case 2 that (2.4) holds along (n')

Case 6: Assume $p([\alpha_n n]) \leq p(n) - 2$ along (n'). Then, using the relative compactness of $(\lambda(n))$ we get for any $\varepsilon > 0$ and all large n' that

$$\frac{[\alpha_n n]}{n} = \frac{\lambda([\alpha_n n]) k_{p([\alpha_n n])}}{\lambda(n) k_{p(n)}} \leq (c + \varepsilon) \frac{k_{p(n)-2}}{k_{p(n)}}.$$

The left hand side of the inequality above tends to α whereas the right hand side tends to $c^{-1}+\varepsilon c^{-2}$ which is a contradiction to $\alpha \geq 1$ if ε is small enough.

Therefore every subsequence has a further subsequence (n') such that either Case 1, 2, 3 or 5 applies which shows that (2.4) holds for $\alpha \in [1,c]$.

Now if $\alpha > c$ write $\alpha = \bar{\alpha} c^j$ for some natural $j \geq 1$ and $\bar{\alpha} \in [1,c)$. Furthermore, write $\alpha_n = \bar{\alpha}_n c^{j_n}$ where eventually $j_n \geq 1$ and $\bar{\alpha}_n \in [1,c)$. Again we have to consider several cases separately:

Case A: If $j_n = j$ along a subsequence (n') then $\bar{\alpha}_n \to \bar{\alpha}$ along that subsequence. Write

$$B_{[\alpha_n n]} B_n^{-1} = \big(B_{[\bar{\alpha}_n c^j n]} B_{[c^j n]}^{-1}\big)\big(B_{[c^j n]} B_{[c^{j-1} n]}^{-1}\big)\cdots\big(B_{[cn]} B_n^{-1}\big)$$

and note that by the already proved uniform convergence in (2.4) for $\bar{\alpha} \in [1,c]$ we get

$$B_{[\bar{\alpha}_n c^j n]} B_{[c^j n]}^{-1} = B_{\left[\frac{\bar{\alpha}_n c^j n}{[c^j n]}[c^j n]\right]} B_{[c^j n]}^{-1} \to \bar{\alpha}^{-E}$$

and for $k = 1, \dots, j$

$$B_{[c^k n]} B_{[c^{k-1} n]}^{-1} = B_{\left[\frac{[c^k n]}{[c^{k-1} n]}[c^{k-1} n]\right]} B_{[c^{k-1} n]}^{-1} \to c^{-E}$$

along (n') and hence (2.4) holds along (n').

Case B: If $j_n = j-1$ along (n') then $\alpha_n = \bar{\alpha}_n c^{j-1} \to \bar{\alpha} c^j = \alpha$ and hence $\bar{\alpha}_n \to c\bar{\alpha}$ along (n') and since $\bar{\alpha}_n < c$ and $\bar{\alpha} \geq 1$ this can only happen if $\bar{\alpha} = 1$, so $\alpha = c^j$. Then

$$B_{[\alpha_n n]} B_n^{-1} = \big(B_{[\bar{\alpha}_n c^{j-1} n]} B_{[c^{j-1} n]}^{-1}\big)\big(B_{[c^{j-1} n]} B_{[c^{j-2} n]}^{-1}\big)\cdots\big(B_{[cn]} B_n^{-1}\big)$$

and as in Case A we see that then along (n')

$$B_{[\alpha_n n]} B_n^{-1} \to c^{-E} \cdot (\underbrace{c^{-E} \cdots c^{-E}}_{j-1 \text{ times}}) = (c^j)^{-E} = \alpha^{-E}.$$

Case C: If $j_n = j-k$ along (n') and $k \notin \{0,1\}$ then $\bar{\alpha}_n \to c^k \bar{\alpha} \in [c^k, c^{k+1})$ along (n') which contradicts $\bar{\alpha}_n \in [1,c)$ for all n so this case is void.

These are all possible cases, hence it follows that (2.4) holds for any $\alpha \geq 1$.

Finally assume $0 < \alpha < 1$ and let $m(n) = [\alpha_n n]$. Then $n/m(n) = n/[\alpha_n n] \to \alpha^{-1} > 1$ and then we get from (2.4) that

$$\left(B_{[\alpha_n n]} B_n^{-1}\right)^{-1} = B_n B_{m(n)}^{-1} = B_{\left[\frac{n}{m(n)} m(n)\right]} B_{m(n)}^{-1} \to (\alpha^{-1})^{-E} = \alpha^E.$$

Taking the inverse on both sides (2.4) follows in this case too and the proof is complete. □

It will be seen in the following sections that the embedding property of the norming operators for GDOSA is a powerful tool. Not only to derive technical results like the spectral decomposition, but also to analyze the asymptotic properties of the whole partial sum $S_n = X_1 + \cdots + X_n$ for $\mu \in \mathrm{GDOSA}(\nu, c)$ in contrast to previously known results which only consider the subsequence (S_{k_n}) for a sampling sequence (k_n).

3. The spectral decomposition for GDOSA

Regular variation on $\mathrm{GL}(\mathbb{R}^d)$, proved for the embedding sequence (B_n) of norming operators (A_n) for GDOSA, is the most natural extension of the one variable theory. It is also the key to the theory of regularly varying functions and measures, which we discuss in [17] and [18]. The main purpose of this section is to give a new proof of a structure theorem called the spectral decomposition for generalized domains of semistable attraction, using the embedding property along with the spectral decomposion for regularly varying sequences in [12].

We begin with a preparatory result.

Lemma 3.1. *Let F be a $d \times d$ real matrix. Factor the minimal polynomial of F into $f_1(x) \cdots f_p(x)$ where all roots of f_i have real part a_i and $a_i < a_j$ for $i < j$. Define $V_i = \mathrm{Ker}(f_i(F))$. Then $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^d$ into F-invariant subspaces, and we may write $F = F_1 \oplus \cdots \oplus F_p$ where $F_i : V_i \to V_i$ and every eigenvalue of F_i has real part equal to a_i. We will call this the* spectral decomposition of $\mathbb{R}^d$ relative to F.

Proof. This is a special case of the primary decomposition theorem of linear algebra. See for example [4]. □

Definition 3.2. Assume that $U \subset \mathbb{R}^d$ is a nontrivial linear subspace and (x_n) is a sequence of unit vectors in $\mathbb{R}^d$. We write $x_n \rightharpoonup U$ as $n \to \infty$ if $\min\{\|x_n - u\| : u \in U \cap S^{d-1}\} \to 0$ as $n \to \infty$.

The next result is the key to the spectral decomposition and its related results. It shows that the growth rate and geometry of regularly varying sequences is very special and closely related to regularly varying sequences of the form (n^F). In fact if one considers a regularly varying sequence of the simple form $B_n = n^F$ for some index F with spectral decomposition $\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_p$ and real spectrum $a_1 < \cdots < a_p$ then we get for $\tilde{L}_i = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_i$ using the fact that each $\tilde{V}_i$ is F-invariant:

If $x \in \tilde{L}_i$ then $B_n x = n^F x \in \tilde{L}_i$ and hence $B_n x / \|B_n x\| \rightharpoonup \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_i$. Furthermore, the norm of B_n restricted to the subspace $\tilde{L}_i$ is controlled by a_i, i.e. $\|B_n|_{\tilde{L}_i}\| \leq C n^{a_i + \delta}$ for every $\delta > 0$, where C is a positive constant depending only on δ but not on n. Then if $x \in \tilde{L}_i$ we have $\|B_n x\| \leq C\|x\| n^{a_i+\delta}$, so $n^{-\rho}\|B_n x\| \to 0$ for all $\rho > a_i$.

Similar results hold for an arbitrary regularly varying sequence as shown in the next theorem. The next two results are stated in [12]. We include them here for sake of completeness.

Theorem 3.3. *Suppose (B_n) is a regularly varying sequence with index F and let $\mathbb{R}^d = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_p$ be the spectral decomposition of $\mathbb{R}^d$ relative to F. Then there exists a nested sequence of subspaces $\tilde{L}_1 \subset \cdots \subset \tilde{L}_p = \mathbb{R}^d$ such that for each $i = 1, \ldots, p$ we have*

(a) $\dim \tilde{L}_i = \dim(\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_i)$;
(b) *if* $x \in \tilde{L}_i$, *then* $B_n x / \|B_n x\| \rightharpoonup \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_i$;
(c) *if* $x \notin \tilde{L}_i$, *then* $B_n x / \|B_n x\| \rightharpoonup \tilde{V}_{i+1} \oplus \cdots \oplus \tilde{V}_p$;
(d) *if* $x \in \tilde{L}_i$, *then* $n^{-\rho}\|B_n x\| \to 0$ *for all* $\rho > a_i$;
(e) *if* $x \notin \tilde{L}_i$, *then* $n^{-\rho}\|B_n x\| \to \infty$ *for all* $\rho < a_{i+1}$.

Proof. See [12], Lemma 2.3. and [16] for a detailed proof. □

The next result is the basic characterization of regular variation for sequences of linear operators. Again it is stated in [12]. It will enable us later to choose norming operators (A_n) in (1.1) which are of a particular simple type. They are block diagonal with respect to the spectral decomposition of the exponent of an operator semistable law and this yields a more detailed description of generalized domains of semistable attraction as well as to many other results as shown later in this paper. Let us agree to write $D_n \sim B_n$ if $D_n B_n^{-1} \to I$.

Theorem 3.4. *(B_n) varies regularly with index F if and only if $B_n \sim D_n T$ for some $T \in \mathrm{GL}(\mathbb{R}^d)$ and some (D_n) regularly varying with index F such that: each $\tilde{V}_i$ in the spectral decomposition of $\mathbb{R}^d$ with respect to F is D_n-invariant; and $D_n = D_n^{(1)} \oplus \cdots \oplus D_n^{(p)}$ where each $D_n^{(i)} : \tilde{V}_i \to \tilde{V}_i$ is regularly varying with index F_i.*

Proof. See [12], Theorem 2.4 and [16] for a detailed proof. □

We now apply these results along with the embedding property of the previous section to obtain some powerful results about the behavior of the norming operators in (1.1). We assume that ν is a full (c^E, c) operator semistable law and that $\mu \in \mathrm{GDOSA}(\nu, c)$. As a first corollary we get the following assertion on the growth rate and geometry of the inverse transpose of the norming operators in (1.1), which together with Corollary 3.6 provide crucial information on the growth rate of the norming operators in any radial direction.

Corollary 3.5. *Let $\mu \in \mathrm{GDOSA}(\nu, c)$ for some full (c^E, c) operator semistable law ν with $c > 1$. Let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition of $\mathbb{R}^d$ relative to E. Then for some sequence of norming operators (A_n) satisfying (1.1), there exists a nested sequence of subspaces $\bar{L}_1 \subset \cdots \subset \bar{L}_p = \mathbb{R}^d$ such that for each $i = 1, \dots, p$ we have*

(a) $\dim \bar{L}_i = \dim(V_1^* \oplus \cdots \oplus V_i^*)$;

(b) *if $x \in \bar{L}_i$, then $(A_n^*)^{-1}x / \|(A_n^*)^{-1}x\| \rightharpoonup V_1^* \oplus \cdots \oplus V_i^*$;*

(c) *if $x \notin \bar{L}_i$, then $(A_n^*)^{-1}x / \|(A_n^*)^{-1}x\| \rightharpoonup V_{i+1}^* \oplus \cdots \oplus V_p^*$;*

(d) *if $x \in \bar{L}_i$, then $\lambda^{-n}\|(A_n^*)^{-1}x\| \to 0$ for all $\lambda > c^{a_i}$;*

(e) *if $x \notin \bar{L}_i$, then $\lambda^{-n}\|(A_n^*)^{-1}x\| \to \infty$ for all $\lambda < c^{a_{i+1}}$.*

Here $a_1 < \cdots < a_p$ is the real spectrum of E (see Lemma 3.1).

Proof. Apply Theorem 2.7 to obtain a regularly varying sequence (B_n) with index $(-E)$, where $B_{k_n} = A_n$. Now, taking the inverse transpose we get

$$(B_{[\lambda n]}^*)^{-1}(B_n^*) \to \lambda^{E^*} \quad \text{as } n \to \infty$$

for all $\lambda > 0$, i.e. $\big((B_n^*)^{-1}\big)$ is regularly varying with index E^*. Since $\mathbb{R}^d = V_1^* \oplus \cdots \oplus V_p^*$ is the spectral decomposition relative to E^*, Theorem 3.3 yields a nested sequence $\bar{L}_1 \subset \cdots \subset \bar{L}_p = \mathbb{R}^d$ with $(B_n^*)^{-1}$ satisfying (a)–(e) of that theorem. Note that by (2.3) $B_{k_n} = A_n$ and hence assertions (a)–(c) of the corollary are immediate from the corresponding results of Theorem 3.3. Using the fact that for every $\delta > 0$ and n large enough we have $(c^n)^{1-\delta} \le k_n \le (c^n)^{1+\delta}$, Theorem 3.3(d) implies for $x \in \bar{L}_i$

$$c^{-n\rho(1+\delta)} \|(A_n^*)^{-1}x\| \le k_n^{-\rho} \|(B_{k_n}^*)^{-1}x\| \to 0$$

as $n \to \infty$ for all $\rho > a_i$. Since $\delta > 0$ is arbitrary this implies that $(c^\rho)^{-n}\|(A_n^*)^{-1}x\| \to 0$ for all $\rho > a_i$ which proves (d). The proof of (e) is similar. This concludes the proof. □

In the proof of the corollary above we applied Theorem 3.3 to the regularly varying sequence $\big((B_n^*)^{-1}\big)$ of index E^*, where (B_n) is the embedding sequence of an appropriate sequence of norming operators (A_n). Applying Theorem 3.3 directly to the sequence (B_n), which by Theorem 3.3 is regularly varying with index $(-E)$ we then get the other half of bounds on the growth rate and the geometry of the norming operators (A_n). The following result was also proved in [13] without using the embedding property and Theorem 3.3 but rather proving it directly by applying regular variation techniques. Since the proof in [13] is quite long and technical, the approach given here may give more insight to the problem. It also shows, that the key result to the spectral decomposition for $\mathrm{GDOA}(\nu)$ in [9], Theorem 4.1 and to the spectral decomposition for $\mathrm{GDOSA}(\nu, c)$ in [13], Theorem 2.1 are in fact equivalent. All that is needed for such a result is a regularly varying sequence of linear operators or a sequence of linear operators which can be embedded in a regularly varying sequence.

Corollary 3.6. *Suppose that $\mu \in \mathrm{GDOSA}(\nu, c)$ where ν is a full (c^E, c) operator semistable law for some $c > 1$. Let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition relative to E. Then for some sequence of norming operators (A_n) in (1.1) there exists a nested sequence of subspaces $\mathbb{R}^d = L_1 \supset \cdots \supset L_p$ such that for each $i = 1, \ldots, p$*

(a) *$\dim L_i = \dim(V_i \oplus \cdots \oplus V_p)$;*
(b) *if $x \in L_i$, then $A_n x/\|A_n x\| \rightharpoonup V_i \oplus \cdots \oplus V_p$;*
(c) *if $x \notin L_i$, then $A_n x/\|A_n x\| \rightharpoonup V_1 \oplus \cdots \oplus V_{i-1}$;*
(d) *if $x \in L_i$, then $\lambda^n \|A_n x\| \to 0$ for all $\lambda < c^{a_i}$;*
(e) *if $x \notin L_i$, then $\lambda^n \|A_n x\| \to \infty$ for all $\lambda > c^{a_{i-1}}$.*

Here $a_1 < \cdots < a_p$ is the real spectrum of E (see Lemma 3.1).

PROOF. Apply Theorem 2.7 to obtain a regularly varying sequence (B_n) with index $(-E)$ where $B_{k_n} = A_n$. Then $(-E)$ has real spectrum $-a_p < \cdots < -a_1$ with spectral decomposition $\mathbb{R}^d = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_p$ where $\tilde{V}_j = V_{p-j+1}$ and $V_1, \ldots, V_p$ is the spectral decomposition of $\mathbb{R}^d$ relative to E. Now apply Theorem 3.3 to obtain a nested sequence $\tilde{L}_1 \subset \cdots \subset \tilde{L}_p = \mathbb{R}^d$ with (a)–(e) of that theorem. Then if we define $L_j = \tilde{L}_{p-j+1}$ we get for each $j = 1, \ldots, p$ that

(a) $\dim L_j = \dim \tilde{L}_{p-j+1} = \dim(\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_{p-j+1}) = \dim(V_j \oplus \cdots \oplus V_p)$;

(b) if $x \in L_j = \tilde{L}_{p-j+1}$ then $B_n x / \|B_n x\| \rightharpoonup \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_{p-j+1} = V_j \oplus \cdots \oplus V_p$;

(c) if $x \notin L_j$ then $B_n x / \|B_n x\| \rightharpoonup \tilde{V}_{(p-j+1)+1} \oplus \cdots \oplus \tilde{V}_p = V_1 \oplus \cdots \oplus V_{j-1}$;

(d) if $x \in L_j$ then $n^{-\rho} \|B_n x\| \to 0$ for all $\rho > -a_j$ which is equivalent to $n^{\rho} \|B_n x\| \to 0$ for all $\rho < a_j$;

(e) if $x \notin L_j$ then $n^{-\rho} \|B_n x\| \to \infty$ for all $\rho < -a_{j-1}$ which is equivalent to $n^{\rho} \|B_n x\| \to \infty$ for all $\rho > a_{j-1}$.

Now the proof of Corollary 3.6 is almost identical to the proof of Corollary 3.5 and therefore omitted. □

Now let ν be a full (c^E, c) operator semistable law and let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition with respect to E. The idempotent operators $\pi_i : \mathbb{R}^d \to \mathbb{R}^d$ with $\mathrm{Im}(\pi_i) = V_i$ satisfy $\pi_1 + \cdots + \pi_p = I$ and $\pi_i \pi_j = 0$ if $i \neq j$. Now define $\nu_i = \pi_i(\nu)$, then ν_i is a probability measure on $\mathbb{R}^d$ which is supported on the subspace V_i. We call $(\nu_1, \ldots, \nu_p)$ the *spectral decomposition* of ν. The restriction of ν_i to the E-invariant subspace V_i will be denoted by $\bar{\nu}_i$. Then it follows that $\bar{\nu}_i$ is a full (c^{E_i}, c) operator semistable law on V_i, where $E = E_1 \oplus \cdots \oplus E_p$ (see Lemma 3.1). The real spectrum of E_i consists of one single element a_i. We will say that $\bar{\nu}_i$ is *spectrally simple*. If $a_i = \frac{1}{2}$, then $\bar{\nu}_i$ is normal and otherwise ($a_i > \frac{1}{2}$) $\bar{\nu}_i$ is a full nonnormal operator semistable law of an especially simple type.

Suppose that $\mu \in \mathrm{GDOSA}(\nu, c)$ and (1.1) holds. In the presence of a large degree of symmetry in the limit, the norming operators in (1.1) may exhibit a wild behavior. For example suppose that μ is a mean zero finite second moment spherically symmetric probability distribution on $\mathbb{R}^d$. By the central limit theorem, (1.1) holds with $A_n = k_n^{-1/2} I$, for any sequence (k_n) with $k_{n+1}/k_n \to c$ and ν centered normal law. But (1.1) still holds for $A_n = k_n^{-1/2} U_n$ for any sequence (U_n) of orthogonal transformations. Alternatively we can replace A_n by any sequence of operators $A'_n \sim A_n$.

Let $\nu = (\nu_1, \dots, \nu_p)$ be the spectral decomposition of the limit in (1.1). As before we will denote by $\bar{\mu}_i$ the restriction of $\mu_i = \pi_i(\mu)$ to V_i. If π_i and A_n commute in general (i.e. if $V_1, \dots, V_p$ are A_n-invariant subspaces for all n), then $\bar{\mu}_i \in \mathrm{GDOSA}(\bar{\nu}_i, c)$ for all $i = 1, \dots, p$. This reduces the analysis of $\mu \in \mathrm{GDOSA}(\nu, c)$ to the case of a spectrally simple limit.

In general it is too much to expect that the norming sequence (A_n) in (1.1) is as well-behaved as in the preceding paragraph. For example, suppose $T \in \mathrm{GL}(\mathbb{R}^d)$ and let $\mu_0 = T\mu$. Then $\mu_0 \in \mathrm{GDOSA}(\nu, c)$ and in fact

$$A'_n \mu_0^{k_n} * \delta(b_n) \Rightarrow \nu$$

with $A'_n = A_n T^{-1}$. We cannot decompose the sequence (A'_n) as we did before. All we can say is that there is another direct sum decomposition $\mathbb{R}^d = W_1 \oplus \cdots \oplus W_p$ such that $A'_n(W_i) = V_i$ for all $i = 1, \dots, p$ (take $W_i = T(V_i)$). The next theorem says that this kind of a decomposition is always possible. It follows that for any $\mu_0 \in \mathrm{GDOSA}(\nu, c)$, there exists a $T \in \mathrm{GL}(\mathbb{R}^d)$ such that $\mu = T\mu_0$ decomposes into $(\mu_1, \dots, \mu_p)$ where $\bar{\mu}_i \in \mathrm{GDOSA}(\bar{\nu}_i, c)$.

We say that a $\mu \in \mathrm{GDOSA}(\nu, c)$ is *spectrally compatible* with ν, if there is a sequence of norming operators (A_n) such that (1.1) holds and $V_1, \dots, V_p$ are A_n-invariant subspaces for all n.

Theorem 3.7. *For any $\mu_0 \in \mathrm{GDOSA}(\nu, c)$ there exists $T \in \mathrm{GL}(\mathbb{R}^d)$ such that $\mu = T\mu_0$ is spectrally compatible with ν. Equivalently, μ_0 is spectrally compatible with $T^{-1}\nu$.*

Even though this theorem was already proved in [13], we will show here that it in fact follows from the more general spectral decomposition result for regularly varying sequences of linear operators, Theorem 3.4.

PROOF. Let $\mu_0 \in \mathrm{GDOSA}(\nu, c)$, where ν is a full (c^E, c) operator semistable law on $\mathbb{R}^d$. Apply Theorem 2.7 to obtain a sequence (B_n) of linear operators regularly varying with index $(-E)$, such that $A_n = B_{k_n}$ is a suitable sequence of norming operators for (1.1), i.e. $B_{k_n} \mu_0^{k_n} * \delta(b_n) \Rightarrow \nu$. By Theorem 3.4 there exists a $T \in \mathrm{GL}(\mathbb{R}^d)$ such that $B_n \sim D_n T$ and every V_i in the spectral decomposition with respect to E is D_n-invariant for all n. Then $\mu = T\mu_0$ is spectrally compatible with ν. On the other hand $T^{-1}\nu$ is a $(c^{T^{-1}ET}, c)$ operator semistable law and $\mathbb{R}^d = W_1 \oplus \cdots \oplus W_p$, where $W_i = T^{-1}(V_i)$ is the spectral decomposition of $\mathbb{R}^d$ with respect to $T^{-1}ET$. It follows that $(T^{-1} D_{k_n} T)\mu_0^{k_n} * \delta(T^{-1} b_n) \Rightarrow T^{-1}\nu$ as well as $T^{-1} D_n T(W_i) = W_i$, proving that μ_0 is spectrally compatible with $T^{-1}\nu$.
□

4. The behavior of the partial sum

In [11] and [14] we described the behavior of the partial sum S_n along certain subsequences $(k_n)_n$ of natural numbers. It turned out that if $k_{n+1}/k_n \to c \geq 1$ then the affine normalized sequence $(S_{k_n})_n$ converges in distribution to an operator semistable limit. We gave necessary and sufficient conditions on the distribution μ of one summand of the i.i.d. sum S_n for that to happen.

In this section we will derive complementary results on the asymptotic behavior of the whole partial sum $(S_n)_n$ if μ is in the generalized domain of semistable attraction of an operator semistable law. The main tools we use in the proofs are the embedding property of the sequence of norming operators in (1.1) and the spectral decomposition for regularly varying sequences of linear operators proved in Section 3.

In the process we will consider the behavior of the vector valued sum $S_n = X_1 + \cdots + X_n$ as well as the properties of the projection of S_n along any radial direction. Our results are formulated in terms of *stochastic compactness.*

Definition 4.1. Let $(Y_n)_n$ be a sequence of random vectors on $\mathbb{R}^d$. We say that $(Y_n)_n$ is stochastically compact if $(\mathcal{L}(Y_n))_n$ is weakly relatively compact and all limit points are full (resp. nondegenerate if $d = 1$). Here $\mathcal{L}(Y)$ denotes the distribution of Y.

Suppose that μ is in the generalized domain of semistable attraction of some full (c^E, c) operator semistable law ν. Then (1.1) holds for some linear operators A_n and shifts a_n. If ν is a normal law then we already know (see [14]) that $\mathrm{GDOSA}(\nu, c) = \mathrm{GDOA}(\nu)$, where $\mathrm{GDOA}(\nu)$ denotes the generalized domain of attraction of ν, that is the set of all μ such that (1.1) holds for $k_n = n$. Hence there exist linear operators B_n and nonrandom vectors b_n with $B_n S_n - b_n \Rightarrow \nu$ proving the stochastic compactness of $(B_n S_n - b_n)_n$. Therefore, in view of Theorem 2.1 in [14], it is enough to consider the case of a limit ν which is not a normal law.

We first consider the vector valued sum S_n.

Theorem 4.2. *Suppose X is a random vector with distribution μ in the* GDOSA *of some full (c^E, c) operator semistable law ν, where $c > 1$. Then there exists a sequence (B_n) regularly varying with index $(-E)$ and a sequence (b_n) of nonrandom vectors such that*

$$\big(B_n(X_1 + \cdots + X_n) - b_n\big)_{n\geq 1}$$

is stochastically compact with limit set contained in

$$\left\{\lambda^{-E}\nu^{\lambda} : \lambda \in [1,c]\right\}.$$

PROOF. Apply Theorem 2.7 to obtain a sequence (B_n) regularly varying with index $(-E)$ such that $A_n = B_{k_n}$ is a suitable norming sequence for μ in (1.1). Note that by the construction of B_n in the proof of Theorem 2.7 we have $B_n = \lambda_n^{-E} A_{p_n}$, where we write $n = \lambda_n k_{p_n}$ and $k_{p_n} \leq n < k_{p_n+1}$. Then $k_{n+1}/k_n \to c$ implies that (λ_n) is relatively compact with limit set contained in $[1,c]$. Let $b_n = \lambda_n \lambda_n^{-E} a_{p_n}$. Using Lévy's continuity theorem we get

$$\left(A_n\mu^{k_n} * \delta(-a_n)\right)\widehat{\ }(x) \to \hat{\nu}(x) \tag{4.1}$$

uniformly on compact subsets of $\mathbb{R}^d$, where $\hat{\rho}$ denotes the Fourier transform of a measure ρ.

If $\lambda_n \to \lambda$ along a subsequence then by (4.1)

$$\begin{aligned}
\left(B_n\mu^n * \delta(-b_n)\right)\widehat{\ }(x) &= \hat{\mu}(B_n^* x)^n e^{-i\langle b_n, x\rangle} \\
&= \hat{\mu}\left(A_{p_n}^*(\lambda_n^{-E^*}x)\right)^{\lambda_n k_{p_n}} e^{-i\langle a_{p_n}, \lambda_n^{-E^*}x\rangle \lambda_n} \\
&= \left(\hat{\mu}\left(A_{p_n}^*(\lambda_n^{-E^*}x)\right)^{k_{p_n}} e^{-i\langle a_{p_n}, \lambda_n^{-E^*}x\rangle}\right)^{\lambda_n} \\
&= \left(\left(A_{p_n}\mu^{k_{p_n}} * \delta(-a_{p_n})\right)\widehat{\ }\left(\lambda_n^{-E^*}x\right)\right)^{\lambda_n} \\
&\to \left(\hat{\nu}\left(\lambda^{-E^*}x\right)\right)^{\lambda} = \left(\lambda^{-E}\nu^{\lambda}\right)\widehat{\ }(x)
\end{aligned}$$

along that subsequence, where $\lambda^{-E}\nu^{\lambda}$ is full since ν is full. Since every subsequence has a further subsequence with this property the result is now immediate. □

Remark 4.3. (a) It is easy to see that every law $\nu_{(\lambda)} = \lambda^{-E}\nu^{\lambda}$ in the limit set is also (c^E, c) operator semistable. The Lévy measure of $\nu_{(\lambda)}$ is given by $\phi_{(\lambda)} = \lambda(\lambda^{-E}\phi)$, where ϕ is the Lévy measure of ν.

(b) If ν is not only operator semistable but actually operator stable, it follows from a characterization of operator stable laws in terms of the Lévy measure that $\phi_{(\lambda)} = \phi$ for all $1 \leq \lambda \leq c$ and hence in this case the limit

set in Theorem 4.2 is, if we adjust the centering constants b_n if necessary, just the one point set $\{\nu\}$. This gives a new proof of Theorem 2.1 in [14].

(c) It follows from Theorem 4.2 that for any $\|\theta\| = 1$ the sequence $(\langle B_n S_n, \theta\rangle - \langle b_n, \theta\rangle)_n$ is stochastically compact with limit set $\{\langle Y_\lambda, \theta\rangle : \lambda \in [1, c]\}$, where $\mathcal{L}(Y_\lambda) = \lambda^{-E}\nu^\lambda$.

(d) It follows from the proof of Theorem 4.2 that if the centering constants a_n in (1.1) can be chosen to be zero, then $b_n = 0$ for all n.

In addition to Theorem 4.2 we now consider the behavior of the projection of S_n along any radial direction θ. Note that $\langle S_n, \theta\rangle = \langle X_1, \theta\rangle + \cdots + \langle X_n, \theta\rangle$ is a sum of i.i.d. random variables. We will normalize $\langle S_n, \theta\rangle$ by scalars r_n which is different from considering $\langle B_n S_n, \theta\rangle$, where we first normalize by a linear operator and then project. See Remark 4.3(c).

Theorem 4.4. *Suppose X is a random vector with distribution μ in the* GDOSA *of some full (c^E, c) operator semistable law ν where $c > 1$. Then for all unit vectors $\theta \in \mathbb{R}^d$ there exists a sequence (r_n) of positive real numbers tending to zero and a sequence (s_n) of shifts such that*

$$\left(r_n \sum_{i=1}^{n} \langle X_i, \theta\rangle - s_n\right)_{n\geq 1} \tag{4.2}$$

is stochastically compact. Moreover, if $\mathbb{R}^d = V_1^* \oplus \cdots \oplus V_p^*$ is the spectral decomposition of $\mathbb{R}^d$ with respect to E^* and if $\bar{L}_1 \subset \cdots \subset \bar{L}_p = \mathbb{R}^d$ is the nested sequence of subspaces constructed in Corollary 3.5 we have: If $\theta \in \bar{L}_i \setminus \bar{L}_{i-1}$ then the limit set of the sequence in (4.2) *is contained in the set*

$$\left\{\langle Y_\lambda, \theta_0\rangle : \lambda \in [1, c],\ \theta_0 \in V_i^*,\ \|\theta_0\| = 1\right\}$$

where $\mathcal{L}(Y_\lambda) = \lambda^{-E}\nu^\lambda$.

Proof. Apply Theorem 4.2 to obtain a regularly varying sequence (B_n) with index $(-E)$ and shifts (b_n) such that $\big(B_n(X_1 + \cdots + X_n) - b_n\big)_n$ is stochastically compact with limit set $\{\lambda^{-E}\nu^\lambda : \lambda \in [1, c]\}$.

Fix any $\|\theta\| = 1$ and let

$$r_n = \left\|\left(B_n^*\right)^{-1}\theta\right\|^{-1}. \tag{4.3}$$

Write $(B_n^*)^{-1}\theta = r_n^{-1}\theta_n$ where $\|\theta_n\| = 1$. Then an application of Theorem 3.3 to the regularly varying sequence $\big((B_n^*)^{-1}\big)$ with index E^* gives a nested sequence of subspaces $\bar{L}_1 \subset \cdots \subset \bar{L}_p = \mathbb{R}^d$. Note that these subspaces are the same as those constructed in the proof of Corollary 3.5. Then by Theorem 3.3 we infer that $\theta \in \bar{L}_i \setminus \bar{L}_{i-1}$ implies $\theta_n \to V_i^*$. Since

$$\langle X_i, \theta\rangle = \langle B_n^{-1} B_n X_i, \theta\rangle = \langle B_n X_i, (B_n^*)^{-1}\theta\rangle = r_n^{-1}\langle B_n X_i, \theta_n\rangle,$$

we get if we let $s_n = \langle b_n, \theta_n\rangle$ that

$$r_n \sum_{i=1}^n \langle X_i, \theta\rangle - s_n = \sum_{i=1}^n \langle B_n X_i, \theta_n\rangle - \langle b_n, \theta_n\rangle = \Big\langle B_n \sum_{i=1}^n X_i - b_n, \theta_n \Big\rangle.$$

Now every subsequence (n') has a further subsequence $(n'') \subset (n')$ such that $\theta_n \to \theta_0 \in V_i^*$ along (n''). Hence $h_n(x) = \langle x, \theta_n\rangle \to h_0(x) = \langle x, \theta_0\rangle$ along the same subsequence (n''). Using the fact that every full operator semistable law has a Lebesque density by [6], Theorem 2.2, we get using [2], Theorem 5.5 that

$$r_n \sum_{i=1}^n \langle X_i, \theta\rangle - s_n = h_n\Big(B_n \sum_{i=1}^n X_i - b_n\Big) \longrightarrow h_0(Y_\lambda) = \langle Y_\lambda, \theta_0\rangle$$

along a subsequence $(n''') \subset (n'')$ such that $B_n \sum_{i=1}^n X_i - b_n \Rightarrow Y_\lambda$ along (n'''). Here, as above, $\mathcal{L}(Y_\lambda) = \lambda^{-E}\nu^\lambda$ for some $\lambda \in [1, c]$. Hence we have shown that every subsequence (n') has a further subsequence (n''') along which we get convergence. This concludes the proof. □

Remark 4.5. (a) Let $\theta \in \bar{L}_i \setminus \bar{L}_{i-1}$ for some $i = 1, \ldots, p$. It follows from Theorem 3.3 that for every $\delta > 0$ there exists a $n_0 \geq 1$ such that

$$n^{-a_i - \delta} \leq r_n \leq n^{-a_i + \delta}$$

whenever $n \geq n_0$, showing sharp bounds on the rate at which r_n tends to zero. This rate will vary along the direction θ.

(b) We can choose $s_n = 0$ if the centering constants a_n in (1.1) were zero. See also Remark 4.3(d).

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