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Landsberg spaces with common geodesics*

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. The purpose of the present paper is to investigate two problems in Landsberg spaces, which are special Finsler spaces. First we prove that a Landsberg space with a vanishing Douglas tensor is a Berwald space. Further on we will study a special geodesic mapping of a Landsberg space into a *P-Finsler space.

Introduction

Let $F^n(M^n, L)$ be an n-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L(x, y)$, where $y^i = \dot{x}^i$(1, is the fundamental function defined on the manifold $T(M)\backslash\mathcal{O}$ of nonzero tangent vectors. In the following we assume that L is positive and the fundamental metric tensor $g_{ij} = \frac{1}{2}L^2_{\cdot i \cdot j}$ $({}_{\cdot i} = \partial/\partial y^i)$ is positive definite. (Throughout the present paper we shall use the terminology and definitions described in MATSUMOTO's monograph [1](2.)

The system of differential equations for geodesic curves of F^n with respect to the canonical parameter t is given by $\frac{d^2x^i}{dt^2} = -2G^i(x, y)$, where

$$G^i = \frac{1}{4}g^{i\alpha}(y^\beta(\partial L^2_{\cdot\alpha}/\partial x^\beta) - \partial L^2/\partial x^\alpha).$$

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(1 The Roman and the Greek indices run over the range $1, \dots, n$; the Roman indices are free but the Greek indices denote summation.

(2 Numbers in brackets refer to the references at the end of the paper.

The Berwald connection coefficients $G^i_j(x,y)$, $G^i_{jk}(x,y)$ can be derived from the function G^i, namely $G^i_j = G^i_{\cdot j}$; $G^i_{jk} = G^i_{j\cdot k}$. The Berwald covariant derivative with respect to Berwald connection can be written as

$$T^i_{j;k} = \partial T^i_j / \partial x^k - T^i_{j\cdot\alpha} G^\alpha_k + T^\alpha_j G^i_{\alpha k} - T^i_\alpha G^\alpha_{jk}\,. \tag{1}$$

Let us consider two Finsler spaces $F^n(M^n, L)$ and $\tilde{F}^n(M^n, \tilde{L})$ on a common underlying manifold M^n. A diffeomorphism $F^n \to \tilde{F}^n$ is called *geodesic* if it maps an arbitrary geodesic of F^n to a geodesic of $\tilde{F}^n$. In this case the change $L \to \tilde{L}$ of the metrics is called *projective*. As it is well known, the mapping $F^n \to \tilde{F}^n$ is geodesic (that is the change $L \to \tilde{L}$ is projective) if and only if there exists a scalar field $p(x,y)$ satisfying

$$\tilde{G}^i = G^i + py^i\,;\; p \neq 0\,. \tag{2}$$

The projective factor $p(x,y)$ is a positively homogeneous function of degree 1 in y. From (2) we obtain the following equations

$$\tilde{G}^i_j = G^i_j + p\delta^i_j + p_j y^i \qquad\qquad p_j = p_{\cdot j} \tag{3}$$

$$\tilde{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p_{jk}y^i\,, \quad p_{jk} = p_{j\cdot k}. \tag{4}$$

The Weyl curvature tensor and the Douglas tensor are invariant under geodesic mappings (that is under projective changes). It is a well known result that a Finsler space F^n is of scalar curvature if and only if its Weyl tensor vanishes ([2],[4]). Thus there arises an interesting question: which properties are satisfied by Finsler spaces with vanishing Douglas tensor?

Landsberg spaces with vanishing Douglas tensor

Definition 1 ([1]). A Finsler space is called an affinely connected (or Berwald) space if the coefficients G^i_{jk} are functions of the position only, that is the hv-curvature tensor $G^i_{jkl} = G^i_{jk\cdot l}$ is zero.

Definition 2 ([1]). A Finsler space is called a Landsberg space if the condition $y_\alpha G^\alpha_{jkl} = -2P_{jkl} = 0$ holds good, where P_{jkl} is the hv-torsion tensor, and $g_{jk;l} = -2P_{jkl}$.

Theorem 1. *A Landsberg space with vanishing Douglas tensor is a Berwald space if* $n > 2$. (This result can be found in [5] without any justification. We were unable to find any source of its proof, so we feel that it is not completely worthless to present a proof here.)

Proof. The Douglas tensor is given by

$$D^h_{ijk} = G^h_{ijk} - (y^h G_{ij\cdot k} + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij})/(n+1) \tag{5}$$

where $G_{ij} = G^{\alpha}_{ij\alpha}$. If we assume that $D^h_{ijk} = 0$, and $P_{ijk} = 0$, then contracting (5) by $h^l_h = (\delta^l_h - l^l l_h)$, (where $l^l = y^l/L$ and $l_h = \partial L/\partial y^h$) we get

$$G^l_{ijk} = \frac{1}{n+1}(h^l_i G_{jk} + h^l_j G_{ik} + h^l_k G_{ij}) \tag{6}$$

(We used here the fact that in any Finsler space F^n $(n > 2)$ condition $D^h_{ijk} = 0$ is equivalent to $h^h_\alpha D^\alpha_{ijk} = 0$ [6].) We consider the identities in the Landsberg space

$$G_{ihjk} + G_{hijk} = 2C_{hik;j} \tag{7}$$

$$G_{ihjk} - G_{hijk} = 0, \tag{8}$$

where $G_{ihjk} = g_{h\alpha} G^\alpha_{ijk}$ and $2C_{hik} = g_{hi.k}$. Substitute from (6) into (8) we get

$$G_{ik} = \frac{1}{n-1} G h_{ik}\,; \qquad G = G_{\alpha\beta} g^{\alpha\beta}.$$

(6) can be rewritten in the form

$$G_{ihjk} = \frac{G}{n^2-1}(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}), \quad G_{jk} = Gh_{jk}/(n-1). \tag{9}$$

From one of the Bianchi identities follows that in Landsberg spaces

$$S_{ijkh;l} = 0$$

holds good ([1, (17.17)]), where S_{ijkl} denotes the Cartan's third curvature tensor

$$S_{ijkh} = C_{ih\alpha} C^\alpha_{jk} - C_{ik\alpha} C^\alpha_{jh}\,; \quad C^i_{jk} = C_{\alpha jk} g^{\alpha i}\,. \tag{10}$$

Differentiating (10) covariantly from (7)–(8) and (9) we obtain

$$G(h_{ih}C_{jkl} + h_{jk}C_{ihl} - h_{ik}C_{jhl} - h_{jh}C_{ikl}) = 0. \tag{11}$$

Transvecting (11) after the substitution $h = l$ and to $j = k$, we have

$$(n-2)GC_i = 0\,; \quad C_i = C_i{}^\alpha{}_\alpha,$$

i.e. the Landsberg space is a Berwald or a Riemannian space. □

Problem. Determine all the Finsler spaces which have common geodesic with some Riemannian space, that is determine all the Finsler spaces which admit geodesic mapping onto a Riemannian space.

From Theorem 1 and from Szabó's [3] result, by which any Berwald connection is Riemannian metrizable one, follows an answer to the problem above in the case of Landsberg spaces:

Corollary. *In the set of Landsberg spaces only Berwald spaces have common geodesics with some Riemannian spaces.*

On a special geodesic mapping

A. Moór [7] investigated pairs of Finsler spaces F_n and $\tilde{F}_n$ in which the h(hv)-torsion tensors coincide, that is

$$\tilde{C}_{ijk} = C_{ijk}. \tag{12}$$

We will give an example for this kind of spaces. For this we will need the following lemma:

Lemma 1. *Let there be given two Finsler fundamental functions by $L(x,y)$ and $\hat{L}(x,y)$ respectively. Then $\tilde{L}(x,y) := \sqrt{L^2(x,y) + \hat{L}^2(x,y)}$ is also a Finsler fundamental function.*

Proof. One has to prove only that $\tilde{L}(x,y)$ is a convex function in y, i.e.

$$\tilde{L}(x, y+\bar{y}) \le \tilde{L}(x,y) + \tilde{L}(x,\bar{y}). \tag{13}$$

We know that

$$L(x, y+\bar{y}) \le L(x,y) + L(x,\bar{y}) \tag{14}$$

and

$$\hat{L}(x, y+\bar{y}) \le \hat{L}(x,y) + \hat{L}(x,\bar{y}). \tag{15}$$

It is enough to prove that

$$L^2(x, y+\bar{y}) + \hat{L}^2(x, y+\bar{y}) \le L^2(x,y) + \hat{L}^2(x,y) + L^2(x,\bar{y}) + \hat{L}^2(x,\bar{y}) + 2\tilde{L}(x,y)\tilde{L}(x,\bar{y}).$$

From (14) and (15) we get

$$L^2(x, y+\bar{y}) \le L^2(x,y) + L^2(x,\bar{y}) + 2L(x,y)L(x,\bar{y}) \tag{16}$$

and

$$\hat{L}^2(x, y+\bar{y}) \le \hat{L}^2(x,y) + L^2(x,\bar{y}) + 2\hat{L}(x,y)\hat{L}(x,\bar{y}). \tag{17}$$

This shows that we must prove that

$$2L(x,y)L(x,\bar{y}) + 2\hat{L}(x,y)\hat{L}(x,\bar{y}) \le \tilde{L}(x,y)\tilde{L}(x,\bar{y})$$

which is equivalent with the following inequality

$$L^2(x,y)L^2(x,\bar{y}) + 2L(x,y)L(x,\bar{y})\hat{L}(x,y)\hat{L}(x,\bar{y}) + \hat{L}^2(x,y)\hat{L}^2(x,\bar{y}) \le (L^2(x,y) + \hat{L}^2(x,y))(L^2(x,\bar{y}) + \hat{L}^2(x,\bar{y})).$$

From this we obtain

$$0 \le (L(x,y)\hat{L}(x,\bar{y}) - L(x,\bar{y})\hat{L}(x,y))^2. \tag{18}$$

From this it follows (13), for every function we were using is positive. □

Using the above lemma it can be easily seen that any sum of Finsler fundamental tensors is a Finsler fundamental tensor.

Now if $\hat{L}$ is a Riemannian fundamental function then we get that

$$\tilde{g}_{ij}(x,y) = g_{ij}(x,y) + \hat{g}_{ij}(x) \tag{19}$$

is a Finsler fundamental tensor which satisfies (12).

A. Moór studied geodesic mappings between Finsler spaces related by equality (19) in the special case when

$$\hat{g}_{ij} = 1/2(s_i r_j + r_i s_j),$$

and he gave a sufficient and necessary condition for F_n and $\tilde{F}_n$ have common geodesics. Note that it is easy to show that fundamental metric tensors $\tilde{g}_{ij}(x,y)$, $g_{ij}(x,y)$, and $\hat{g}_{ij}(x)$ cannot have the same set of geodesics at the same time.

Definition 3 ([8]). If a Finsler space satisfies the condition $P_{ijk} - \lambda C_{ijk} = 0$ the space is called a *P-Finsler space. Scalar function $\lambda(x,y)$ is given by $P_\alpha C^\alpha / C_\alpha C^\alpha$, where $P_\alpha = P^\beta_{\alpha\beta}$.

Lemma 2. *If we assume that there exists a geodesic mapping between F^n and $\tilde{F}^n$ (for which condition (12) is satisfied), then we get*

$$\tilde{P}_{ijk} = P_{ijk} - pC_{ijk}. \tag{20}$$

Proof. We obtain (20) by taking the covariant derivative of (12), using (3), (4) and then contracting by y. □

Thus we have the following

Theorem 2. *Let there be given two Finsler spaces $\tilde{F}^n$ and F^n which are related by condition (12). If $\tilde{F}^n$ is a Landsberg space and it can be geodesically mapped onto Finsler space F^n, then F^n must be a *P-space, and the corresponding geodesic mapping is not trivial (i.e. $p \neq 0$).*

Theorem 3. *Let be given two Landsberg spaces with the condition (12). If these spaces have common geodesics, and the corresponding geodesic mapping is not trivial (i.e. $p \neq 0$) then they are Riemannian spaces.*

Added in proof. In a personal letter professor M. Matsumoto confirmed that Theorem 1 has not yet been proved anywhere. He proposed the following Lemma and Corollary in order to make the picture more complete:

Lemma (Using the notations of [1]). *A Finsler space F^n is a Landsberg space if and only if* $G_i{}^h{}_{jk} = C_i{}^h{}_{k|j}$.

PROOF. If F^n is Landsberg then we have $F_i{}^h{}_{jk} = C_i{}^h{}_{k|j}$ from [1, (18.2)] and $G_i{}^h{}_{jk} = F_i{}^h{}_{jk}$ from [1, (18.16)]. Conversely, $G_i{}^h{}_{jk} = C_i{}^h{}_{k|j}$ implies $C_i{}^h{}_{k|0} = G_i{}^h{}_{0k} = 0$. □

Corollary. *For a Landsberg space we have*

(1) $G_{hijk} = C_{hik|j} = G_{ihjk}$,

(2) $G_h{}^i{}_{jk}$ *is indicatory* ([1, p. 219]) *in all indices,*

(3) $G_{hj} = C_{h|j}$.

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