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Quasi contraction nonself mappings on Banach spaces and common fixed point theorems

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Abstract. LJUBOMIR ĆIRIĆ [2] has proved recently fixed point theorems for quasi contraction nonself mappings on Banach spaces. In this paper we consider quasi contraction nonself mappings on Banach spaces and common fixed point theorems for a pair of maps, and offer an etension of Ćirić's result. The main results of K. M. Das and K. V. Naik are also recovered.

Let X be a complete metric space. A map $T : X \mapsto X$ such that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$

$$(0.1)\quad d(Tx,Ty) \leq \lambda \cdot \max\big\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big\}$$

is called *quasicontraction*. Let us remark that ĆIRIĆ [1] introduced and studied quasicontraction as one of the most general contactive type map. The well known ĆIRIĆ's ([1], [3], [5]) result is that quasicontraction f possesses a unique fixed point.

For the convenience of the reader we recall the following recent Ćirić's result.

Theorem (ĆIRIĆ [2, Theorem 2.1]) **1.** *Let X be a Banach space, C a nonempty closed subset of X, and ∂C the boundary of C. Let $T : C \mapsto X$ be a nonself mapping such that for some constant $\lambda \in (0,1)$ and for every*

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$x, y \in C$

$$d(Tx,Ty) \le \lambda \cdot \max\big\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big\}. \tag{1.1}$$

Suppose that

$$T\big(\partial C\big) \subset C. \tag{1.2}$$

Then T has a unique fixed point in C.

Following Ćirić [2], let us remark that *problem to extend the known fixed point theorem for self mappings $T : C \mapsto C$, defined by* (0.1)*, to corresponding nonself mappings $T : C \mapsto X$, $C \neq X$, was open more than 20 years*.

In [2] Ćirić has used new methods and proved a fixed point theorem for the class of nonself mappings defined by (1.1), which satisfy added condition (1.2).

Assume now that X is a normed space. For $x, y \in X$ we shall write

$$\mathrm{seg}[x,y] = \{z \in X : z = (1-t)x + ty,\ 0 \le t \le 1\}.$$

In the proof of the next result we shall use the following observation. Let us remark that if $u \in X$, and $z_0 = (1-t_0)x + t_0 y \in \mathrm{seg}[x,y]$, $0 \le t_0 \le 1$, then

$$\begin{aligned}\|u - z_0\| &= \|(1-t_0)u + t_0 u - (1-t_0)x - t_0 y\| \\ &\le (1-t_0)\|u-x\| + t_0\|u-y\| \le \max\big\{\|u-x\|, \|u-y\|\big\}.\end{aligned}$$

Following Sessa [6] we shall say that $f, g : X \mapsto X$ are weakly commuting if

$$d(fgx, gfx) \le d(fx, gx) \quad \text{for every} \quad x \in X.$$

Clearly weak commutativity of f and g is a generalization of the conventional commutativity of f and g.

In this paper we offer the following extension of Ćirić's result.

Theorem 2. *Let X be a Banach space, C a nonempty closed subset of X, and ∂C the boundary of C. Let $g : C \mapsto X$, $f : X \mapsto X$ and $f : C \mapsto C$. Suppose that $\partial C \neq \emptyset$, f is continuous, and let us assume that f and g satisfy the following conditions:*

(i) *There exists a constant $\lambda \in (0,1)$ such that for every $x, y \in C$*

$$d(gx, gy) \leq \lambda \cdot M(x,y), \tag{2.1}$$

where

$$M(x,y) = \max\big\{d(fx,fy), d(fx,gx), d(fy,gy), d(fx,gy), d(fy,gx)\big\}. \tag{2.2}$$

(ii) *f and g are weakly commutative on C, that is*

$$d(fgx, gfx) \leq d(fx, gx) \quad \text{for every } x \in C. \tag{2.3}$$

$$\text{(iii)} \qquad g(C) \cap C \subset f(C). \tag{2.4}$$

$$\text{(iv)} \qquad g\big(\partial C\big) \subset C. \tag{2.5}$$

$$\text{(v)} \qquad f\big(\partial C\big) \supset \partial C. \tag{2.6}$$

Then f and g have a unique common fixed point in C.

Proof. Starting with an arbitrary $x_0 \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows. By (2.5) $g(x_0) \in C$. Hence, (2.4) implies that there is $x_1 \in C$ such that $f(x_1) = g(x_0)$. Let us consider $g(x_1)$. If $g(x_1) \in C$, again by (2.4) there is $x_2 \in C$ such that $f(x_2) = g(x_1)$. If $g(x_1) \notin C$, by (2.6) there is $x_2 \in \partial C$ such that $f(x_2) \in \partial C \cap \operatorname{seg}[f(x_1), g(x_1)]$.

Hence, by induction we construct a sequence $\{x_n\}$ of points in C as follows. If $g(x_n) \in C$, than by (2.4) $f(x_{n+1}) = g(x_n)$ for some $x_{n+1} \in C$; if $g(x_n) \notin C$, then by (2.6) pick $x_{n+1} \in \partial C$ such that

$$f(x_{n+1}) \in \partial C \cap \operatorname{seg}\big[f(x_n), g(x_n)\big].$$

We shall prove that $f(x_n)$ and $g(x_n)$ are Cauchy sequences.

First let us prove that

$$f(x_{n+1}) \neq g(x_n) \Rightarrow f(x_n) = g(x_{n-1}). \tag{2.7}$$

Suppose the contrary that $f(x_n) \neq g(x_{n-1})$. Then $x_n \in \partial C$. Now, by (2.4) $g(x_n) \in C$, hence $f(x_{n+1}) = g(x_n)$, a contradiction. Thus we prove (2.7). Now set

$$B(n,k) = \{f(x_j), g(x_j) : n \leq j \leq n+k\} \qquad b(n,k) = \operatorname{diam}(B(n,k))$$
$$B(n) = \{f(x_j), g(x_j) : n \leq j\} \qquad b(n) = \operatorname{diam}(B(n))$$

and note that $b(n,k) \uparrow b(n)$ as $k \to \infty$ and $b(n) \downarrow$ and hence $b = \lim_n b(n) \geq 0$ exists. To see that $f(x_n)$ and $g(x_n)$ are Cauchy sequences we must show that $b = 0$. We claim that

$$b(n,k) \leq \lambda b(n-2, k+2), \quad n,k \geq 2. \tag{2.8}$$

To prove (2.8) we have to consider three cases.

Case 1. $b(n,k) = d(f(x_i), g(x_j))$ with $n \leq i,j \leq n+k$.
If $f(x_i) = g(x_{i-1})$, then

$$b(n,k) = d(g(x_{i-1}), g(x_j)) \leq \lambda M(x_{i-1}, x_j) \leq \lambda b(n-2, k+2).$$

If $f(x_i) \neq g(x_{i-1})$, then $f(x_{i-1}) = g(x_{i-2})$ and so
$f(x_i) \in \operatorname{seg}[f(x_{i-1}), g(x_{i-1})] = \operatorname{seg}[g(x_{i-2}), g(x_{i-1})]$. Thus

$$\begin{aligned} b(n,k) = d(f(x_i), g(x_j)) &\leq \max\Big\{d(g(x_{i-2}), g(x_j)), d(g(x_{i-1}), g(x_j))\Big\} \\ &\leq \lambda \max\Big\{M(x_{i-2}, x_j), M(x_{i-1}, x_j)\Big\} \leq \lambda b(n-2, k+2). \end{aligned}$$

Case 2. $b(n,k) = d(f(x_i), d(x_j))$ with $n \leq i,j \leq n+k$.
If $f(x_j) = g(x_{j-1})$, then Case 2 reduces to Case 1.
If $f(x_j) \neq g(x_{j-1})$, then as in the Case 1 we have $j \geq 2$, $f(x_{j-1}) = g(x_{j-2})$, and

$$f(x_j) \in \partial C \cap \operatorname{seg}\big[g(x_{j-2}), g(x_{j-1})\big].$$

Hence

$$b(n,k) = d\big(f(x_i), f(x_j)\big) \leq \max\Big\{d\big(f(x_i), g(x_{j-2})\big), d\big(f(x_i), g(x_{j-1})\big)\Big\}$$

and Case 2 reduces to Case 1.

Case 3. The remaining case $b(n,k) = d(g(x_i), g(x_j))$ with $n \leq i, j \leq n+k$ is trivial.

Now let $k \to \infty$ in (2.8) to obtain $b(n) \leq \lambda b(n-2)$ and let $n \to \infty$ to obtain $b \leq \lambda b$, that is $b = 0$. It follows that both $\{f(x_n)\}$ and $\{g(x_n\}$ are Cauchy sequences. Since $f(x_n) \in C$ and C is a closed subset of a Banach space X we conclude that $\lim f(x_n) = y \in C$. Since

$$d\big(f(x_n), g(x_n)\big) \leq b_n \to 0, \quad n \to \infty,$$

we have $\lim g(x_n) = y$. Hence,

$$\lim g(x_n) = \lim f(x_n) = y \in C.$$

By continuity of f

$$\lim f(g(x_n)) = \lim f(f(x_n)) = f(y) \in C.$$

By (2.3), we have

$$\begin{aligned} d\big(gf(x_n), f(y)\big) &\leq d\big(gf(x_n), fg(x_n)\big) + d\big(fg(x_n), f(y)\big) \\ &\leq d\big(f(x_n), g(x_n)\big) + d\big(fg(x_n), f(y)\big) \to 0, \quad n \to \infty. \end{aligned} \tag{2.9}$$

Hence

$$\lim (gf)(x_n) = f(y). \tag{2.10}$$

Now, by (2.9) and (2.10)

$$M(fx_n, y) \to d(fy, gy) \quad n \to \infty,$$

and

$$d(fy, gy) \leq \lambda \cdot d(fy, gy).$$

Hence, as $\lambda < 1$,

$$f(y) = g(y). \tag{2.11}$$

We shall prove that $g(y)$ is a common fixed point for f and g. By (2.11) and (2.3) it follows that

$$fg(y) = gf(y) = gg(y). \tag{2.12}$$

By (2.1), (2.11) and (2.12) we have

$$d\big(gg(y), g(y)\big) \leq \lambda \cdot M(gy, y) = \lambda \cdot d\big(gg(y), g(y)\big),$$

and hence $gg(y) = g(y)$. From (2.12), we conclude that $g(y)$ is also a fixed point of f. The uniqueness of the common fixed point is immediate from (2.1). □

Let us remark that in Theorem 2, setting $f = I_X$, the identity map on X, we get Theorem 1.

If f is not continuous, but f^m is continuous for any fixed integer m, we can prove the next result.

Theorem 3. *Let X be a Banach space, C a nonempty closed subset of X, and ∂C the boundary of C. Let $g : C \mapsto X$, $f : X \mapsto X$ and $f : C \mapsto C$. Suppose that f^m, m any fixed positive integer, is continuous, and let us assume that f and g satisfy* (2.1), (2.4), (2.5), (2.6) *and f and g are commutative on C, that is*

$$(fg)x = (gf)x \quad \text{for each } x \in C. \tag{3.1}$$

Then f and g have a unique common fixed point in C.

Proof. Let $\{x_n\}$, $g(x_n)$ and $f(x_n)$ be the sequences as in the proof of Theorem 2. Hence,

$$\lim g(x_n) = \lim f(x_n) = y \in C.$$

By (2.2), for each $n \geq 1$

$$\begin{aligned} d\big(f^m g(x_n), gf^{m-1}(y)\big) &= d\big(gf^m(x_n), gf^{m-1}(y)\big) \\ &\leq \lambda \cdot M\big(f^m(x_n), f^{m-1}(y)\big) \\ &= \lambda \cdot \max\Big\{d\big(f^m f(x_n), f^m(y)\big), d\big(f^m f(x_n), f^m g(x_n)\big), \\ &\quad d\big(f^m(y), gf^{m-1}(y)\big), d\big(f^m f(x_n), gf^{m-1}(y)\big), d\big(f^m(y), f^m g(x_n)\big)\Big\}. \end{aligned}$$

Now, by continuity of f^m

$$d\big(f^m(y), gf^{m-1}(y)\big) \leq \lambda \cdot d\big(f^m(y), gf^{m-1}(y)\big).$$

Whence, $f^m(y) = gf^{m-1}(y)$, since $\lambda < 1$. Now $f^m(y)$ is a common fixed point for f and g (see (2.11) and (2.12)). The uniqueness of the common fixed point follows immediate from (2.1). $\square$

Let us remark that in Theorem 3, setting $f = I_X$, the identity map on X, we get Theorem 1.

The next result is connected with [2, Theorem 2.2] and Theorem 2.

Theorem 4. *Let X be a Banach space, C a nonempty compact subset of X, and ∂C the boundary of C. Let $g : C \mapsto X$, $f : X \mapsto X$ and $f : C \mapsto C$. Suppose that g and f are continuous, and let us assume that f and g satisfy* (2.3), (2.4), (2.5), (2.6) *and for all $x, y \in C$, $x \neq y$*

$$d(gx, gy) < M(x, y), \tag{4.1}$$

where

$$M(x, y) = \max\big\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\big\}. \tag{4.2}$$

Then f and g have a unique common fixed point in C.

Proof. Suppose that f and g do not have a unique common fixed point in C. Then, as $d(fx, gx) > 0$ and $d(fy, gy) > 0$, for all $x, y \in C$, we have $0 < M(x, y)$. Let $Q : C \times C \mapsto [0, 1)$ be the mapping defined by

$$Q(x, y) = \frac{d(gx, gy)}{M(x, y)}, \quad x, y \in C.$$

Clearly, Q is a continuous function and $Q(x, y) < 1$, $x, y \in C$. Now, as in the proof of [2, Theorem 2.2], there exists $x_0, y_0 \in C$ such that $\sup\{Q(x, y) : x, y \in C\} = Q(x_0, y_0) < 1$. Hence g is a quasicontraction, and by Theorem 2 f and g have a common unique fixed point in C. This is in contradiction with our assumption that f and g have not a common unique fixed point. The uniqueness follows from (4.1). $\square$

Again, in Theorem 4, setting $f = I_X$, the identity map on X, we get [2, Theorem 2.2].

By the proof of Theorem 2 we can recover some results of K. M. Das and K. V. Naik [3] and Jungck [4].

Theorem 5 (K. M. Das and K. V. Naik [3, Theorem 2.1])**.** *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy*

$$g(X) \subset f(X) \tag{5.1}$$

and there exists a constant $\lambda \in (0,1)$ such that for every $x, y \in X$

$$d(gx,gy) \leq \lambda \cdot M(x,y), \tag{5.2}$$

where

$$\begin{aligned} M(x,y) = \max\big\{&d(fx,fy), d(fx,gx),\\ &d(fy,gy), d(fx,gy), d(fy,gx)\big\}. \end{aligned} \tag{5.3}$$

Then f and g have a unique fixed point.

Proof. We follow the proof of Theorem 2. Let us remark that the condition (5.1) implies that starting with an arbitrary $x_0 \in X$, we construct a sequence $\{x_n\}$ of points in X such that $f(x_{n+1}) = g(x_n)$, $n = 0, 1, 2, \ldots$. The rest of the proof follows by the proof of Theorem 2. □

Now, by the proof of Theorem 2 we can recover the main result of K. M. Das and K. V. Naik [3].

Theorem 6 (K. M. Das and K. V. Naik [3, Theorem 3.1])**.** *Let X be a complete metric space. Let f^2 be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy*

$$gf(X) \subset f^2(X) \tag{6.1}$$

and $f(g(x)) = g(f(x))$ whenever both sides are defined. Further, let there exists a constant $\lambda \in (0,1)$ such that for every $x, y \in f(X)$

$$d(gx,gy) \leq \lambda \cdot M(x,y), \tag{6.2}$$

where

$$\begin{aligned} M(x,y) = \max\big\{&d(fx,fy), d(fx,gx),\\ &d(fy,gy), d(fx,gy), d(fy,gx)\big\}. \end{aligned} \tag{6.3}$$

Then f and g have a unique common fixed point.

PROOF. Again, we follow the proof of Theorem 2. By (6.1) starting with an arbitrary $x_0 \in f(X)$, we construct a sequence $\{x_n\}$ of points in $f(X)$ such that $f(x_{n+1}) = g(x_n) = y_n$, $n = 0, 1, 2, \dots$. Now $f(y_{n+1}) = f(g(x_n)) = g(f(x_n)) = g(y_{n-1}) = z_n$, $n = 1, 2, \dots$. By (2.11) $\{z_n\}$ is a Cauchy sequence in X and hence convergent to some $z \in X$. Further as in the proof of [3, Theorem 3.1] or as in the proof of Theorem 2, $m = 2$, we conclude that $f^2 z = gfz$, and gfz is a unique common fixed of f and g. □

Let us remark that from Theorem 2 and the proof of Theorem 5, we get

Theorem 7. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that weakly commutes with f. Further let f and g satisfy* (5.1) *and* (2.1). *Then f and g have a unique common fixed point.*

Now as a corollary we get the following result of G. JUNGCK [4].

Corollary 8 (JUNGCK [4]). *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy* (5.1) *and there exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in X$*

$$d(gx, gy) \le \lambda \cdot d(fx, fy).$$

Then f and g have a unique common fixed point.

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