Publ. Math. Debrecen
58 / 3 (2001), 467–490

The order theoretic structure of the set of P-sums of a sequence

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Abstract. Let P be a finite set of reals and (λ_n) a sequence of positive reals with $\sum_{n=1}^{\infty} \lambda_n < \infty$. Define the set S to consist of all values $\sum_{n=1}^{\infty} p_n \lambda_n$ with $p_n \in P$. There are several papers of diverse authors on the topological structure of S. Here we start the order theoretic analysis of this topic. In some cases the order theoretic approach leads to a more lucid insight why certain sets play a universal role. In other cases we see that the list of possibilities known up to now is not complete. This decides a question stated by J. E. Nymann and R. A. Sáenz in a recent paper.

1. Introduction

1.1 Motivation

In [N-S1] and [N-S2] the topological structure of sets $S = S(P, \Lambda)$ of the following type has been studied. Let P be a finite set of reals with at least two elements and $\Lambda = (\lambda_n)$, $\lambda_n > 0$, a sequence of positive reals such that the series

$$\sum_{n=1}^{\infty} \lambda_n < \infty$$

converges. The set $S = S(P, \Lambda)$ contains all P-sums (w.r.t. Λ), i.e. all sums of the type

$$\sum_{n=1}^{\infty} p_n \lambda_n, \quad p_n \in P.$$

Mathematics Subject Classification: 11A63, 06A05, 54F05.
Key words and phrases: digital expansion, sets of P-sums, Cantor set, Cantorval, coloured ordering, order topology.

Such sets S are called sets of P-sums. A short list of possibilities for the topological structure of sets of P-sums has been given in [N-S2]. It has been conjectured that this list is complete. In this paper we show that this is not the case. Before we give a new example not contained in the list of [N-S2] we study the order structure of the sets under consideration. To clarify the essential points of the investigations we also consider arbitrary closed sets and then make additional assumptions which are always satisfied by sets of P-sums. This seems to have several advantages:

- One of the main constructions in [N-S2] (and similar in the appendix of [M-O]) produces a homeomorphism which in fact is an order isomorphism. A purely order theoretic formulation makes the idea clearer and gives hints for further investigations.
- The order theoretic investigation is finer: Two closed sets S_1 and S_2 with the same order structure have the same topological structure, but the converse may fail.
- The order structure is more vivid in a psychological sense and hence leads to a rather clear description and understanding of the situation. It shows how many possibilities are left by the necessary conditions for sets of P-sums known up to now.
- The question whether two given different order structures correspond to homeomorphic spaces can be decided in many cases which are interesting for us. (Further investigations on this question are contained in [W].)

Nevertheless there are many problems left open by this paper. But we hope that they get clearer by our approach.

Finally we want to mention that in [M-O] related topological questions have been treated for sums of Cantor sets instead of sets of P-sums.

1.2 Notations

We are going to study certain subsets S, S' of the real line $\mathbb{R}$. From the order theoretic point of view we call them isomorphic, in symbols $S \cong_o S'$, if there is a strictly increasing bijection $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(S) = S'$. If S, S' are homeomorphic, i.e. there exists a continuous bijection $\varphi : S \to S'$ with continuous inverse, we write $S \cong_t S'$. Since the natural topology on $\mathbb{R}$ is its order topology we have the implication

$$S \cong_o S' \quad \Rightarrow \quad S \cong_t S'.$$

Clearly the converse implication is not true: Take $S = [0,1] \cup \{2\}$ and $S' = \{0\} \cup [1,2]$ or (if we would allow decreasing order isomorphisms) $S = [0,1] \cup \{2\} \cup [3,4]$ and $S' = [0,1] \cup [2,3] \cup \{4\}$. A more interesting example will be presented in Section 7.

We note that for closed sets $A, A' \subseteq \mathbb{R}$ we have $A \cong_o A'$ if and only if there is an increasing bijection $\psi : A \to A'$, not necessarily defined on the whole real line. This holds since such a ψ can be extended on the (open) components of the complement of A, for instance linearly, to a bijection $\varphi : \mathbb{R} \to \mathbb{R}$ as required in the definition of $\cong_o$.

For given P we consider the space Y of all sequences $(p_1, p_2, \dots)$, $p_n \in P$, equipped with the compact Tychonoff topology. Then, for $\Lambda = (\lambda_n)$, the set $S = S(P, \Lambda)$ can be considered to be the image of Y under the mapping $\Phi = \Phi_\Lambda$, which maps the sequence $(p_n) \in Y$ to the real number

$$\Phi((p_n)) = \sum_{n=1}^{\infty} p_n \lambda_n.$$

Φ is continuous, hence $S = \Phi(Y)$ is compact. Since the $\lambda_n > 0$ get arbitrarily small and P contains at least two different elements, S contains no isolated points and, therefore, is a perfect set.

In the following intervals or, more precisely, nonvoid convex subsets of linear orderings, play a very important role. As usual, in a linear ordering let (a, b) denote the open interval between a and b, $[a, b]$ the closed one and $(a, b]$ resp. $[a, b)$ the left resp. right half open interval. In the left open case also the value $-\infty$ is allowed for a, in the right open case the value ∞ for b. Sometimes it is convenient to leave open whether the end points are contained. In this case we write $\langle a, b\rangle$ etc.

To each closed $A \subseteq \mathbb{R}$ we will associate a countable linear ordering consisting of pairwise disjoint intervals I, I_n, J etc. (The order relation is inherited from $\mathbb{R}$.) In this context we are interested in successor relations: If $I_1 < I_2$ and there is no J with $I_1 < J < I_2$, we write $I_1 \prec I_2$ or $I_2 \succ I_1$. A chain of length n is a finite set $\{I_0 \prec I_1 \prec \cdots \prec I_n\}$. If $I_n \prec I_{n+1}$ for all integers $n \geq 0$, we call the set a right side infinite chain; if the relation holds for all integers $n < 0$ we call it a left side infinite chain; if we have such a relation for all integers n we call the set a both side infinite chain.

Let $\mathcal{X}_1, \mathcal{X}_2$ be subsets of the linear ordering $\mathcal{O} = (\mathcal{X}, \leq)$. $\mathcal{X}_1$ is called dense in $\mathcal{X}_2$ if for all $J_1 < J_2$ in $\mathcal{X}_2$ there is an $I \in \mathcal{X}_1$ with $J_1 < I < J_2$. $\mathcal{X}_1$ is called dense if it is dense in $\mathcal{X}$. In this case it follows that $\mathcal{X}$ is dense in itself. $\mathcal{X}$ is called bounded if it contains a minimum and a maximum.

1.3 Content of the paper

After the introduction, Section 2 describes how the order theoretic structure of any closed set $A \subseteq \mathbb{R}$ on the real line is reflected by a countable ordering $\mathcal{O}(A)$. The points of $\mathcal{O}(A)$ correspond to non-singleton connected components of either A or $\mathbb{R}\backslash A$ and are therefore painted with two colours.

Section 3 recalls several properties of sets of P-sums already known, in particular the self similarity property from [N-S2]. We prove a further property of such sets and take all these properties as a definition for what we call admissible sets A or admissible orderings $\mathcal{O}$ which are considered in the rest of the paper.

In Section 4 we distinguish four possible types of admissible orderings. Two of them lead to unique order theoretic isomorphism types and hence unique homeomorphism types (we call them Cantor sets resp., according to [M-O], M-Cantorvals). The remaining two cases (R/L-Cantorvals and interval type) are more complicated. They are treated in Sections 5 and 6.

Section 7 asks in which cases different admissible orderings correspond to different homeomorphism types. In the well-understood examples of P-sums known up to now the answer is affirmative. Nevertheless there exist homeomorphic sets $A \cong_t B$ with an interval type A and an R-Cantorval B.

Section 8 presents examples of sets of P-sums corresponding to different types. In particular we give an example showing that the list from [N-S2] is not complete.

The concluding Section 9 is devoted to the discussion of open questions which could lead to further investigations.

2. Closed sets of reals and coloured orderings

Let $(\mathcal{X}, \leq)$ be any linear ordering and $\chi : \mathcal{X} \to C$ a mapping, then we call $\mathcal{O} = (\mathcal{X}, \leq, \chi)$ a C-coloured linear ordering. C is called the set of colours. We write $\mathcal{X}_c = \chi^{-1}(c)$ for each colour $c \in C$. Two C-coloured linear orderings $\mathcal{O}_i = (\mathcal{X}_i, \leq_i, \chi_i)$, $i = 1, 2$, are called isomorphic, in symbols $\mathcal{O}_1 \cong \mathcal{O}_2$, if there is a colour preserving order isomorphism $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$. More explicitly this means that φ is a bijection with $x \leq_1 y$ iff $\varphi(x) \leq_2 \varphi(y)$ and $\chi_2(\varphi(x)) = \chi_1(x)$ for all $x, y \in \mathcal{X}_1$.

Given a closed subset $A \subseteq \mathbb{R}$ of the real line we are now going to define the $C_0 = \{f, g\}$-coloured ordering $\mathcal{O}(A) = (\mathcal{X}, \leq, \chi)$ (f, g fixed distinct symbols). The point is that $\mathcal{X} = \mathcal{X}(A)$ will be countable and that $\mathcal{O}(A)$

contains all order theoretic and hence all topological information about A. This gives rise to a satisfactory description of closed subsets of $\mathbb{R}$.

Let $\mathcal{X}_f$ be the set of nonsingleton components of A, $\mathcal{X}_g$ the set of components of the complement $\mathbb{R} \setminus A$, $\mathcal{X} = \mathcal{X}_f \cup \mathcal{X}_g$. Every $I \in \mathcal{X}$ is a set of the form $\langle a, b\rangle$ with $a < b$. $\mathcal{X}$ is ordered in the natural way: $I_1 < I_2$ if and only if $x_1 < x_2$ for some and, since the I_j are convex and pairwise disjoint, hence for all $x_j \in I_j$, $j = 1, 2$. Thus we get indeed a linear ordering $\mathcal{O}(A) = (\mathcal{X}, \leq, \chi)$. We call it the coloured ordering induced by A. For each $I \in \mathcal{X}$ the colour $\chi(I) \in C_0 = \{g, f\}$ is defined to be g if $I \in \mathcal{X}_g$ and f if $I \in \mathcal{X}_f$. The way how $\mathcal{O}(A)$ reflects the structure of A is expressed by the following theorem.

Theorem 1. 1. *Let $\mathcal{O} = (\mathcal{X}, \leq, \chi)$ be a C_0-coloured linear ordering. Then there exists a closed set $A \subseteq \mathbb{R}$ with $\mathcal{O}(A) \cong \mathcal{O}$ if and only if $1 \leq |\mathcal{X}| \leq \aleph_0$ and $\mathcal{X}_g$ is dense in $\mathcal{X}_f$.*

2. *For closed sets $A, B \subseteq \mathbb{R}$ we have $A \cong_o B$ if and only if $\mathcal{O}(A) \cong \mathcal{O}(B)$.*

Proof. 1. *Necessity:* Suppose $\mathcal{O} \cong \mathcal{O}(A)$, w.l.o.g. $\mathcal{O} = \mathcal{O}(A)$. If $|\mathcal{X}| = 0$, then $\mathcal{X}_g = \mathbb{R} \setminus A = \emptyset$, hence $A = \mathbb{R} \in \mathcal{X}_f \subseteq \mathcal{X} = \emptyset$, contradiction. The upper bound $|\mathcal{X}| \leq \aleph_0$ follows, since $\mathcal{X}$ is a collection of pairewise disjoint subsets of the real line, each containing a nonempty open set. Such a collection is at most countable. To prove that $\mathcal{X}_g$ is dense in $\mathcal{X}_f$ suppose $F_1 = \langle a_1, b_1] < F_2 = [a_2, b_2\rangle \in \mathcal{X}_f$. If we had $(b_1, a_2) \subseteq A$, then F_1, F_2 were no components of A, contradiction. Hence the open set $(b_1, a_2) \setminus A \neq \emptyset$ contains a component $G \in \mathcal{X}_g$ with $G \subseteq (b_1, a_2)$, i.e. $F_1 < G < F_2$.

Sufficiency: We first treat the trivial cases with $|\mathcal{X}| \leq 2$. For $\mathcal{X} = \{I\}$ we have to take $A = \mathbb{R}$ if $\chi(I) = f$ and $A = \emptyset$ if $\chi(I) = g$. If $\mathcal{X} = \{I_1 < I_2\}$ distinguish according to the pair $p = (\chi(x_1), \chi(x_2))$. The case $p = (f, f)$ is excluded by the density condition. For $p = (f, g)$ take $A = (-\infty, 0]$, for $p = (g, f)$ take $A = [0, \infty)$, and for $p = (g, g)$ take $A = \{0\}$.

Suppose now $|\mathcal{X}| \geq 3$ and take an enumeration $\mathcal{X} = \{I_1, I_2, \dots\}$. We are going to define open or closed intervals $J_n = \langle a_n, b_n\rangle$ in such a way that for the closed set

$$A = \mathbb{R} \setminus \bigcup_{n: J_n \text{ open}} J_n$$

the mapping $\psi : I_n \mapsto J_n$ is the desired isomorphism. If there is a minimum I_{n_0} in $\mathcal{X}$ let $J_{n_0} = (-\infty, 0\rangle$ with $0 \in J_{n_0}$ iff $\chi(I_{n_0}) = f$. Similarly, if a maximum I_{n_1} exists, let $I_{n_1} = \langle 1, \infty)$ with $1 \in J_{n_1}$ iff $\chi(I_{n_1}) = f$. W.l.o.g. we may assume $n_0 = 1$ and $n_1 \in \{1, 2\}$ whenever defined. It remains to define $J_{n+1} = \langle a_{n+1}, b_{n+1} \rangle$ if $J_1, \dots, J_n$ are defined and where I_{n+1} is neither the maximum nor the minimum in $\mathcal{X}$. By induction hypothesis there is a permutation π of $\{1, \dots, n\}$ with

$$I_{\pi(1)} < I_{\pi(2)} < \cdots < I_{\pi(n)}$$

and

$$a_{\pi(1)} < b_{\pi(1)} \leq a_{\pi(2)} < b_{\pi(2)} \leq \cdots \leq a_{\pi(n)} < b_{\pi(n)}.$$

For the position of I_{n+1} we have to distinguish three possibilities.

- $I_{n+1} < I_{\pi(1)}$: Put $a_{n+1} = b_{n+1} - 1$, where $b_{n+1} = a_{\pi(1)}$ if $I_{n+1} \prec I_{\pi(1)}$ and $b_{n+1} = a_{\pi(1)} - 1$ else.
- $I_{n+1} > I_{\pi(n)}$: Symmetric to the first possibility; put $b_{n+1} = a_{n+1} + 1$, where $a_{n+1} = b_{\pi(n)}$ if $I_{n+1} \succ I_{\pi(n)}$ and $a_{n+1} = b_{\pi(n)} + 1$ else.
- $I_{\pi(k)} < I_{n+1} < I_{\pi(k+1)}$ with $1 \leq k \leq n-1$: If $I_{\pi(k)} \prec I_{n+1}$ put $a_{n+1} = b_{\pi(k)}$, else $a_{n+1} = (b_{\pi(k)} + a_{\pi(k+1)})/2$. If $I_{\pi(k+1)} \succ I_{n+1}$ put $b_{n+1} = a_{\pi(k+1)}$, else $b_{n+1} = (a_{n+1} + a_{\pi(k+1)})/2$.

We put $J_{n+1} = [a_{n+1}, b_{n+1}]$ if $\chi(I_{n+1}) = f$ and $J_{n+1} = (a_{n+1}, b_{n+1})$ if $\chi(I_{n+1}) = g$. Note that, by the density assumption on $\mathcal{X}_g$, the J_n have no end points in common and therefore are pairewise disjoint.

The first statement of the theorem follows, if we can prove the following three assertions for A as defined as above.

a) All J_n with $\chi(I_n) = f$ are components of A.

b) If x is in no J_n, then $\{x\}$ is a component of A.

c) All J_n with $\chi(I_n) = g$ are components of $\mathbb{R} \setminus A$.

ad a) By the definition of A and the disjointness of the J_n we have $J_n \subseteq A$ whenever $\chi(I_n) = f$. Since J_n is connected it is contained in a connected component $B \subseteq A$. It remains to show $B \subseteq J_n$. If $J_n = [a, b\rangle$, $a > -\infty$, we have to show that, for each $\varepsilon > 0$, $(a - \varepsilon, a)$ intersects with at least one $J_{n'}$ with $\chi(J_{n'}) = g$. (The symmetric argument works if $b < \infty$.) Note that the construction guarantees that the union of all J_n is dense in $\mathbb{R}$. Together with the density assumption on $\mathcal{X}_g$ this provides such a $J_{n'}$.

ad b) Similarly as in the proof of a) one sees that, given $\varepsilon > 0$, there are n_1, n_2 such that $\chi(I_{n_1}) = \chi(I_{n_2}) = g$, $J_{n_1} \cap (x - \varepsilon, x) \neq \emptyset$ and $J_{n_2} \cap (x, x + \varepsilon) \neq \emptyset$, proving b).

ad c) Let $J_n = \langle a, b\rangle$ and $\chi(I_n) = g$. Since $J_n \subseteq \mathbb{R} \setminus A$ by definition, it suffices (by symmetry) to derive a contradiction from the assumption $-\infty < a$, $a \in J_{n'}$, $n' \neq n$ and $\chi(I_{n'}) = g$. Indeed, since $J_{n'}$ is open, we get $(a-\varepsilon, a+\varepsilon) \subseteq J_{n'}$ for some $\varepsilon > 0$, thus $\emptyset \neq (a, \min\{a+\varepsilon, b\}) \subseteq J_n \cap J_{n'} = \emptyset$, contradiction.

2. It is clear that an order automorphism $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(A) = B$ induces the colour preserving isomorphism $\psi : \mathcal{O}(A) \to \mathcal{O}(B)$ by $\psi(I) = \varphi(I)$. Therefore it remains to construct φ from a given ψ. $\mathcal{X}(A)$ consists of intervals $I = \langle a, b\rangle$. Accordingly let $I' = \langle a', b'\rangle = \psi(I)$. For the definition of φ on the open kernel of I we have to distinguish four cases. If $I_n = \mathbb{R}$, then everything is trivial with the identity map $\varphi(x) = x$. If $-\infty = a < b < \infty$ define $\varphi(b - x) = b' - x$ for $x > 0$. Similarly, if $-\infty < a < b = \infty$, define $\varphi(a+x) = a'+x$ for $x > 0$. If $-\infty < a < b < \infty$ define $\varphi(a + x(b - a)) = a' + x(b' - a')$ for $0 < x < 1$. Now consider an x which is not in the open kernel of any I. Let $\mathcal{X}_0$ be the set of all $I \in \mathcal{X}(A)$ with $\sup I \leq x$ and $\mathcal{X}_1 = \mathcal{X}(A) \setminus \mathcal{X}_0$ the rest. It is clear that

$$x = \sup \bigcup \mathcal{X}_0 = \inf \bigcup \mathcal{X}_1.$$

Thus we have to define

$$\varphi(x) = x' = \sup \bigcup \mathcal{X}_0' = \inf \bigcup \mathcal{X}_1'$$

with $\mathcal{X}_i' = \psi(\mathcal{X}_i)$, $i = 1, 2$. By the construction this φ has all required properties. $\square$

3. Admissible orderings

In the introduction we restated the result, already presented in [N-S1], that sets $S = S(P, \Lambda)$ of P-sums are perfect and compact. Furthermore (cf. [N-S2]) they have the following self similarity property:

Let $a_0 = \max S$, $b_0 = \min S$ and $J = (a, b) \subseteq (b_0, a_0)$ a component of $\mathbb{R} \setminus S$. Then there is an $\varepsilon > 0$ such that

$$(S \cap (a - \varepsilon, a]) - a = (S \cap (a_0 - \varepsilon, a_0]) - a_0$$

and

$$(S \cap [b, b + \varepsilon)) - b = (S \cap [b_0, b_0 + \varepsilon)) - b_0.$$

Let us call a compact perfect set S with this property self similar.

According to this we call a $\{g, f\}$-coloured ordering $\mathcal{O} = (\mathcal{X}, \leq, \chi)$ self similar if the following holds: $G_0 = \min \mathcal{X}$ and $G^0 = \max \mathcal{X}$ exist with $\chi(G_0) = \chi(G^0) = g$ and for any $G_0 < G < G^0$ with $\chi(G) = g$, there exist $I_1, I_1', I_2, I_2' \in \mathcal{X}$ ($G_0 < I_1$, $G < I_1'$, $I_2' < G$, $I_2 < G^0$) such that we have the isomorphisms $[G_0, I_1] \cong [G, I_1']$ and $[I_2, G^0] \cong [I_2', G]$ between intervals in $\mathcal{O}$.

We collect the following simple facts showing how properties of $A \subseteq \mathbb{R}$ are reflected in $\mathcal{O}(A)$.

Theorem 2. *Let $A \subseteq \mathbb{R}$ be a closed set, $\mathcal{O} = (\mathcal{X}, \leq, \chi) = \mathcal{O}(A)$ its induced coloured ordering.*

1. *A is compact if and only if $G_0 = \min \mathcal{X}$ and $G^0 = \max \mathcal{X}$ exist and $\chi(G_0) = \chi(G^0) = g$.*
2. *A is perfect, i.e. has no isolated points, if and only if there are no $G_1 \prec G_2 \in \mathcal{X}$ with $\chi(G_1) = \chi(G_2) = g$.*
3. *If A is self similar then $\mathcal{O}$ is self similar.*

Proof. Clear. $\square$

We want to mention that it is an interesting and not trivial question whether something like a converse of Theorem 2.3 holds. In this paper we leave this question open. A further interesting property of sets of P-sums is the following one.

Theorem 3. *Let $\mathcal{O} = (\mathcal{X}, \leq, \chi) = \mathcal{O}(S)$ with the set $S = S(P, \Lambda)$ of P-sums. If $\mathcal{X}_f \neq \emptyset$, i.e. $\chi(F) = f$ for some $F \in \mathcal{X}$, then $\mathcal{X}_f$ is dense in $\mathcal{X}_g$.*

Proof. Note that, if E_k denotes the finite set of all $\sum_{n=1}^{k} p_{i_n} \lambda_n$, $\Lambda_k = (\lambda_n')$ with $\lambda_n' = \lambda_{n+k}$, and $S_k = S(P, \Lambda_k)$, we have the representation

$$S = E_k + S_k = \bigcup_{s \in E_k} s + S_k.$$

First we show that each S_k contains an open interval. $[a, b] = F \subseteq S$ implies that $S = \bigcup_{s \in E_k} (s + S_k)$ is of second category. By Baire's theorem and translation invariance of Baire category, each S_k has to be of second category. Since S_k is closed, this is possible only if S_k contains an open interval.

To prove the theorem suppose now that $G_1 = (a_1, b_1) < G_2 = (a_2, b_2) \in \mathcal{X}_g$. Since G_1 and G_2 are components of $\mathbb{R} \setminus S$ and S is a perfect set, there is an $x \in S \cap (b_1, a_2)$. Since $\lim_{n\to\infty} d_n = 0$ for $d_n = \max S_n - \min S_n$, we have $b_1 + d_{n_0} < x < a_2 - d_{n_0}$ for some n_0. Since $x \in S = E_{n_0} + S_{n_0}$ we get an $s \in E_{n_0}$ such that $s + S_{n_0} \subseteq (b_1, a_2)$. By the above reasoning, $s + S_{n_0} \subseteq S$ contains an interval I, thus $G_1 < I < G_2$ for some $I \in \mathcal{X}_f$. □

Remark. Note that in the situation of the theorem $\mathcal{X}_f$ need not be dense in $\mathcal{X}$, although $\mathcal{X}_g$ is dense in $\mathcal{X}_f$ by Theorem 1. The most simple example is $\mathcal{X} = \{(-\infty, 0) < [0,1] < (1,\infty)\}$ for $S = S(P,\lambda)$ with $P = \{0,1\}$ and $\lambda_n = 2^{-n}$.

For the rest of our investigations of sets of P-sums we restrict our considerations to what we call admissible orderings $\mathcal{O} = (\mathcal{X}, \leq, \chi)$, defined by the following properties.

1. $\mathcal{O}$ is a nonempty and at most countable $C_0 = \{f, g\}$-coloured ordering.
2. $\mathcal{X}_g$ is dense in $\mathcal{X}_f$.
3. In $\mathcal{O}$ minimum G_0 as well as maximum G^0 exist and $G_0, G^0 \in \mathcal{X}_g$.
4. $\mathcal{O}$ is self similar.
5. $\mathcal{X}_f$ is either dense in $\mathcal{X}_g$ or empty.

Accordingly we call a closed set $A \subseteq \mathbb{R}$ admissible, if $\mathcal{O}(A)$ is admissible. By the above results all sets $S(P, \Lambda)$ of P-sums are admissible.

4. A rough order theoretic classification

Using the notions of R/L/M-Cantorvals from [M-O] we distinguish the following cases for an admissible set S resp. an admissible ordering $\mathcal{O} = (\mathcal{X}, \leq, \chi) = \mathcal{O}(S)$ with $\min \mathcal{X} = G_0$ and $\max \mathcal{X} = G^0$.

- Cantor set: S contains no interval, i.e. $\mathcal{X}_f = \emptyset$. In this case S turns out to be order isomorphic to the classical Cantor set $S(\{0,2\}, (3^{-n}))$ by Theorem 4.
- M-Cantorval: $\mathcal{X}_f \neq \emptyset$ but there is no $F \in \mathcal{X}_f$ with $G_0 \prec F$ or $F \prec G^0$. Note that in this case $\mathcal{X}_f$ and $\mathcal{X}_g$ are dense in $\mathcal{X}$. As Theorem 6 will show, this implies that S is uniquely determined up to $\cong_o$. According Mendes and Oliveira [M-O] we call such sets M-Cantorvals.

- L-(R-)Cantorval: S is called an L-Cantorval if $G_0 \prec F$ for some $F \in \mathcal{X}_f$ but there is no $F \prec G^0$. (Similarly S is called an R-Cantorval if $F \prec G^0$ but not $G_0 \prec F$ for any $F \in \mathcal{X}_f$.) For a detailed analysis cf. Section 5.
- Interval type: $G_0 \prec F_0$ and $F^0 \prec G^0$ for some $F_0, F^0 \in \mathcal{X}_f$. For a detailed analysis cf. Section 6.

Now we are going to prove the announced facts that in case of Cantor sets and M-Cantorvals the order theoretic structure is unique, first for Cantor sets. (Theorem 4 is folklore but the new proof, using Cantor's theorem on the uniqueness of countable dense orderings, may be of some interest.)

Theorem 4. *If a nonempty compact perfect set $A \subseteq \mathbb{R}$ contains no interval then $A \cong_o S = S(\{0,2\},(3^{-n}))$, Cantor's middle-third set.*

Proof. Since A and S are nowhere dense, both $\mathcal{X}_f(A)$ and $\mathcal{X}_f(S)$ are empty. On the other hand Theorem 2.2 implies that both $\mathcal{X}_g(A)$ and $\mathcal{X}_g(S)$ are countable dense orderings with minimum and maximum. By a well known theorem of Cantor (cf. for instance [H] p. 100 without minimum and maximum, or the back-and-forth argument in the proof of Theorem 5) any order with this properties is isomorphic to the order of all rationals in $[0,1]$. It follows immediately that $\mathcal{O}(A) \cong \mathcal{O}(S)$. Thus, by Theorem 1, $A \cong_o S$. □

To treat the everywhere dense case, we first prove a purely order theoretic result. It is the coloured analogue of Cantor's theorem used in the previous proof. Theorem 5 may also be considered as a special case of the more general concepts of back-and-forth structures as treated for instance in [H]. But in our simple situation the presentation of these notions would take more space than an explicit proof.

Theorem 5. *Let C be a set of colours and $\mathcal{O} = (\mathcal{X}, \leq, \chi)$, $\mathcal{O}' = (\mathcal{X}', \leq', \chi')$ two countable infinite C-coloured bounded orderings with $\min \mathcal{X} = I_0$, $\max \mathcal{X} = J_0$, $\min \mathcal{X}' = I'_0$ and $\max \mathcal{X}' = J'_0$. If each $\mathcal{X}_c$, resp. $\mathcal{X}'_c$, $c \in C$, is dense in $\mathcal{X}$ resp. in $\mathcal{X}'$ then $\mathcal{O} \cong \mathcal{O}'$ if and only if $\chi(I_0) = \chi'(I'_0)$ and $\chi(J'_0) = \chi'(J'_0)$.*

Proof. It is clear that the condition is necessary. The interesting part is to show that, provided that min and max have the same colours in both orderings, we can construct a colour-preserving order isomorphism

$\varphi : \mathcal{X} \to \mathcal{X}'$. It is clear that we have to define $\varphi(I_0) = I'_0$ and $\varphi(J_0) = J'_0$. Let I_n and I'_n, $n \geq 1$, be enumerations of the members of $\mathcal{X} \setminus \{I_0, J_0\}$ resp. $\mathcal{X}' \setminus \{I'_0, J'_0\}$. For notational convenience (identifying the action $I_n \mapsto I'_{n'}$ with the action $n \mapsto n'$) we consider φ to be a permutation of the positive integers. We define $\varphi = \bigcup_n \varphi_n$, $a_k \mapsto b_k$, with $\varphi_n = \{(a_k, b_k) : k \leq n\}$ inductively. Suppose that φ_{n-1} is a bijection $a_k \mapsto b_k$ for $k < n$ such that $I_{a_k} \mapsto I_{b_k}$ and which preserves colours and the ordering. If n is odd let a_n be the minimal positive integer with $a_n \neq a_k$ for all $k < n$. Since $\mathcal{X}'_c$ is dense in $\mathcal{X}'$, for $c = \chi(I_{a_n})$ there is a $b_n \neq b_k$ for $k < n$ with $\chi'(I'_{b_n}) = c$ and such that φ_n respects the order of the I_{a_k} and I'_{b_k} for $k \leq n$. If n is even change the roles of $\mathcal{O}$ and $\mathcal{O}'$. It is obvious that φ constructed in this way has all required properties. $\square$

Theorem 6. *If S and S' are M-Cantorvals, then $S \cong_o S'$.*

Proof. By Theorem 1 it suffices to show that $\mathcal{O}(S) \cong \mathcal{O}(S')$. We have to check that for M-Cantorvals S and S' the orderings $\mathcal{O} = \mathcal{O}(S)$ and $\mathcal{O}' = \mathcal{O}(S')$ satisfy the assumptions of Theorem 5. But this follows from the fact that each $\mathcal{O}$ associated to an M-Cantorval is a countable infinite $C_0 = \{f, g\}$-coloured bounded ordering with

$$\chi(\min \mathcal{X}) = \chi(\max \mathcal{X}) = g,$$

and that for M-Cantorvals $\mathcal{X}_g$ and $\mathcal{X}_f$ are dense in $\mathcal{X}$. $\square$

An example for an M-Cantorval which is a set $S = S(P, \Lambda)$ of P-sums has already been given in [Gu-N] and will be recalled in Section 8 (example 3).

5. L/R-Cantorvals

It suffices to consider L-Cantorvals. (R-Cantorvals can be treated in a symmetric way.) Thus this section deals with admissible orderings $\mathcal{O}$ with minimum $G_0 \in \mathcal{X}_g$ and neighbour $F_0 \succ G_0$, $F_0 \in \mathcal{X}_f$, but without $F \prec G^0 = \max \mathcal{X}$. By self similarity, each $G \in \mathcal{X}_g \setminus \{G^0\}$ has a successor $F \succ G$. On the other hand $F \prec G$ is impossible. Therefore we can, without ambiguity, consider the pairs (G, F) of neighbours $G \prec F$ to be points with a new colour p (pairs). With the exception of the maximum G^0 the remaining structure contains no points with colour g. Let us, for simplicity, omit G^0. Then all $F \in \mathcal{X}_f$ which are not member in a pair (G, F) cannot have neighbours. Let them be coloured with a (accumulation point). Let us consider the resulting $C^* = \{p, a\}$-coloured structure $\mathcal{O}^*$ with supporting set $\mathcal{X}^*$. If $\mathcal{O}^*$ stems from a set $A \subseteq \mathbb{R}$ via $\mathcal{O}(A)$ we write $\mathcal{O}^* = \mathcal{O}^*(A)$.

Theorem 7. *If $A \subseteq \mathbb{R}$ is an admissible L-Cantorval, then $\mathcal{O}^* = \mathcal{O}^*(A)$ has the following properties:*

1. *$\mathcal{O}^*$ has the minimum $P_0 = (G_0, F_0) \in \mathcal{X}_p^*$ but no maximum.*
2. *$\mathcal{X}_p^*$ is dense in $\mathcal{X}^*$, hence there are no successors in $\mathcal{O}^*$.*
3. *$\mathcal{O}^*$ is self similar at $\mathcal{X}_p$ to the left, i.e. for each $P_1, P_2 \in \mathcal{X}_p^* \setminus \{P_0\}$ there are $P_1' < P_1$ and $P_2' < P_2$ such that the C^*-coloured intervals (P_1', P_1) and (P_2', P_2) are isomorphic.*

Proof. The properties are inherited from $\mathcal{O}$, so we may omit a formal proof. $\square$

Let us call a $C^* = \{p, a\}$-coloured ordering an $*$-admissible ordering if it has the three properties stated in Theorem 7. We also call any closed set $A \subseteq \mathbb{R}$ $*$-admissible if it induces a $*$-admissible ordering $\mathcal{O}^*(A)$.

Theorem 8.

1. *For every $*$-admissible ordering $\mathcal{O}^*$ there is a $*$-admissible $A \subseteq \mathbb{R}$ such that $\mathcal{O}^* = \mathcal{O}^*(A)$.*
2. *If A_1 and A_2 are $*$-admissible, then $\mathcal{O}^*(A_1) \cong \mathcal{O}^*(A_2)$ if and only if $A_1 \cong_o A_2$.*
3. *If A is $*$-admissible with $\mathcal{X}_a^* = \emptyset$, then $A \cong_o A_0$ for a universal closed set $A_0 \subseteq \mathbb{R}$ (unique up to $\cong_o$).*
4. *If A is $*$-admissible with $\mathcal{X}_a^*$ dense in $\mathcal{X}^*$, then $A \cong_o A_1$ for a universal closed set $A_1 \subseteq \mathbb{R}$ (unique up to $\cong_o$).*

Proof. 1. It is clear that to each $\mathcal{O}^*$ there is an $\mathcal{O}$ giving rise to an appropriate L-Cantorval by Theorem 1.

2. The reconstruction of A from $\mathcal{O}^*$ in part 1 is unique up to $\cong_o$.

3. $\mathcal{X}_a^* = \emptyset$ implies that $\mathcal{O}^*$ is in fact a $\{p\}$-coloured (i.e. monochromatic or uncoloured), countable, dense order with minimum and without maximum. Thus $\mathcal{O}^*$ is order isomorphic to the rationals in $[0, 1)$. Therefore, by the second part, all A with $\mathcal{O}^*(A) \cong \mathcal{O}^*$ are pairewise order isomorphic.

4. Generalize the arguments of part 3 in the same way as Theorems 5 and 6 generalize Theorem 4. $\square$

The existence of a set A_0 as in the third part of Theorem 8 will follow from Theorem 9 with the empty ordering $\mathcal{A} = (\emptyset, \leq)$. A more concrete description of such a set is involved in Section 7: The set B in Theorem 12 is of this type. Similarly one can construct a set A_1 as in statement 4 of Theorem 8. We omit a detailed proof.

In the last section of [N-S2] the question has been stated, whether self similarity of sets determines their structure. As we proved in preceding sections this is true for Cantor sets and for M-Cantorvals. Nevertheless, for L/R-Cantorvals and for the interval type the answer is no. This is illustrated by Theorems 9 and 10.

Theorem 9. *Let $\mathcal{A} = (A, \leq)$ be an arbitrary ($\{a\}$-coloured) at most countable ordering. Then there exists a $*$-admissible $\mathcal{O}^* = (\mathcal{X}^*, \leq^*, \chi^*)$ such that $\mathcal{A} \cong \mathcal{X}_a^*$.*

Proof. We first mention that there is an embedding $\varphi : A \to S$, where $S = S(\{0,2\}, (3^{-n}))$ denotes Cantor's middle-third set. This gets clear if one considers the subset $S' \subseteq S$ containing those numbers $x \neq 0$ in S which have a finite representation $x = \sum_{n=1}^{k} \frac{p_n}{3^n}$, $p_n \in \{0,2\}$. S' is a countably infinite set without minimum and maximum and dense in S. Therefore the countable ordered set $A = \{a_1, a_2, \dots\}$ can be embedded into S' via a mapping φ by the obvious inductive construction similar to the proof of Theorem 5. Let $\mathcal{X}_a^* = \varphi(A) \subseteq S' \subseteq S$. We claim that $\mathcal{O}^* = (\mathcal{X}^*, \leq, \chi^*)$ with $\mathcal{X}^* = \mathcal{X}_a^* \cup \mathcal{X}_p^*$, $\mathcal{X}_p^* = [0,1) \cap \mathbb{Q} \setminus S$, has the desired properties. Indeed $\mathcal{X}_a^* \cap \mathcal{X}_p^* = \emptyset$ and $\mathcal{O}^*$ has the minimum $0 \in \mathcal{X}_p^*$ but no maximum. Furthermore, since S is nowhere dense in $\mathbb{R}$, and $\mathbb{Q}$ is dense, we conclude that $\mathcal{X}_p^*$ is dense in $\mathcal{X}^*$ and there are no successors. $\mathcal{X}^*$ is self similar to the left at $x \in \mathcal{X}_p^*$: For every $x \in S'$ there is an $\varepsilon > 0$ with $(x - \varepsilon, x) \cap S = \emptyset$. Thus in this neighbourhood we have a countable dense ordering, isomorphic to the rationals, independently from the special choice of $x \in \mathcal{X}_p^*$. $\square$

Remark. Since $\mathcal{X}_p^*$ is dense, its structure is uniquely determined, namely $\mathcal{X}_p^* \cong \mathbb{Q} \cap [0,1)$. But even if the structure of $\mathcal{X}_a^*$ is given, the structure of $\mathcal{X}^*$ is not uniquely determined. To illustrate this fact consider the following example. Let $\mathcal{X}_p^* = \mathbb{Q} \cap [0,1)$ and $A = \{\frac{\sqrt{2}}{4n} : n = 1, 2, \dots\} \subseteq [0, 0.5]$. Both sets $A_1 = \{\frac{\sqrt{2}}{4}\} \cup (\frac{\sqrt{2}}{4} + A)$ and $A_2 = \{\frac{\sqrt{2}}{8}\} \cup (\frac{\sqrt{2}}{4} + A)$ have the same order structure, namely $A_1 \cong_o A_2 \cong (\omega+1)^{-1}$. (Here α^{-1} denotes the inverse order of the ordinal type α.) Nevertheless, with $\mathcal{X}_a^* = A_i$ we get different $*$-admissible orders depending whether $i = 1$ or $i = 2$. In the first case $\frac{\sqrt{2}}{4} = \min \mathcal{X}_a^*$ is an accumulation point of $\mathcal{X}_a^*$, in the second case $\frac{\sqrt{2}}{8} = \min \mathcal{X}_a^*$ is an isolated point of $\mathcal{X}_a^*$.

6. The interval type

Although the interval type is a bit more complicated, it can be treated in a similar way like L/R-Cantorvals. Therefore it suffices to be somewhat sketchy.

First we note that a finite union $A = I_1 \cup \dots \cup I_n$ of pairwise disjoint closed intervals I_n induces an $\mathcal{O}$ of interval type with $\mathcal{X} = \{G_0 < F_1 < G_1 < \dots < G_{n-1} < F_n < G_n = G^0\}$. Obviously the order type is completely determined by the positive integer n and the classification is trivial. Hence, for the rest of this section, we are mainly interested in the case that $\mathcal{O}$ is an infinite ordering.

In this case we have the following situation. For each $G \in \mathcal{X}_g \setminus \{G_0 = \min \mathcal{X},\ G^0 = \max \mathcal{X}\}$ there are $F, F' \in \mathcal{X}_f$ with $F \prec G \prec F'$. But an identification to triples (F, G, F') with a new colour t is not successful for the following reason. F' might be the predecessor of another $G' \in \mathcal{X}_g$. Thus we have to identify the points of maximal chains. Explicitly the following cases are possible: Finite chain of length $n \geq 1$:

$$F_0 \prec G_1 \prec F_1 \prec G_2 \prec \dots \prec F_n$$

(or $G_0 \prec F_1 \dots \prec F_n$ resp. $F_1 \prec \dots \prec F_n \prec G^0$). We identify the members of a finite chain of length n to one point with the colour s_n. Right side infinite chain:

$$(G_0 \prec) F_0 \prec G_1 \prec F_1 \prec G_2 \prec F_2 \prec \dots$$

We identify the members of a right side infinite chain to one point with the colour s^+. Left side infinite chain:

$$\dots \prec G_{-2} \prec F_{-2} \prec G_{-1} \prec F_{-1} (\prec G^0).$$

We identify the members of a left side infinite chain to one point with the colour s^-. Both side infinite chain:

$$\dots \prec G_{-2} \prec F_{-2} \prec G_{-1} \prec F_{-1} \prec G \prec F_0 \prec G_1 \prec F_1 \prec \dots$$

We identify the members of a both side infinite chain to one point with the colour s. Each $F \in \mathcal{X}_f$ which is not a member of a chain has no neighbours and, therefore, is an accumulation point. Let such points be coloured with the colour a. We get a C^{**}-coloured ordering $\mathcal{O}^{**} = (\mathcal{X}^{**}, \leq^{**}, \chi^{**})$ with the infinite set $C^{**} = \{s_n : n = 1, 2, \dots\} \cup \{s^+, s^-, s, a\}$ of colours. Similarly to L/R-Cantorvals we write $\mathcal{O}^{**} = \mathcal{O}^{**}(A)$ if the ordering stems from a set $A \subseteq \mathbb{R}$ of the interval type via $\mathcal{O}(A)$.

Theorem 10. *If A is an admissible set of the interval type, then $\mathcal{O}^{**} = \mathcal{O}^{**}(A)$ is a C^{**}-coloured ordering with the following properties.*

1. *In $\mathcal{O}^{**}$ there exists a minimum S_0 and a maximum S^0 with $\chi^{**}(S_0) \neq a \neq \chi^{**}(S^0)$.*
2. *If $S_1 \prec S_2$ in $\mathcal{O}^{**}$, then $\chi^{**}(S_1) \in \{s, s^+\}$ or $\chi^{**}(S_2) \in \{s, s^-\}$.*
3. *$\mathcal{X}^{**} \setminus \mathcal{X}^{**}_a$ is dense in $\mathcal{X}_a$.*

Proof. Follows from the construction of $\mathcal{O}^{**}$ and the properties characterizing sets of the interval type. $\square$

Let us call a C^{**}-coloured ordering $\mathcal{O}^{**}$ with the properties stated in the theorem an $**$-admissible ordering, the corresponding A a $**$-admissible set. Similarly as for L/R-Cantorvals we get the following theorem.

Theorem 11.

1. *For every $**$-admissible $\mathcal{O}^{**}$ there is a $**$-admissible set A with $\mathcal{O}^{**} \cong \mathcal{O}^{**}(A)$.*
2. *If A_1 and A_2 are $**$-admissible sets, then $\mathcal{O}^{**}(A_1) \cong \mathcal{O}^{**}(A_2)$ if and only if $A_1 \cong_o A_2$.*
3. *Let $\mathcal{A}$ be an arbitrary ($\{a\}$-coloured) at most countable ordering. Then there exists a $**$-admissible $\mathcal{O}^{**} = (\mathcal{X}^{**}, \leq^{**}, \chi^{**})$ such that $\mathcal{A} \cong \mathcal{X}^{**}_a$.*

Proof. Similar as for the corresponding properties of L/R-Cantorvals, cf. Theorems 8.1, 8.2 and 9. $\square$

The Remark at the end of Section 5 holds mutatis mutandis in the interval type case. Note that statements 3 and 4 of Theorem 8 have no simple analogues for interval type sets. This is due to the fact that C^{**} contains more than two colours.

7. Order theoretic and topological structure

Motivated from the fact that for closed sets $A, B \subseteq \mathbb{R}$ a topological isomorphism $A \cong_t B$ in general does not imply an order isomorphism $A \cong_o B$ one might ask which admissible orderings can be distinguished from the topological point of view. (For certain types of compact sets of reals, [W] treats the question how many different order theoretic isomorphism types lead to the same homeomorphism type.) For notational convenience we restrict our attention to compact sets A, B. We start with the observation that every homeomorphism $\varphi : A \to B$ maps components of connectedness on components of connectedness. Singleton components $\{x\}$ of A (we call such an x a single point of A) are mapped on singleton components $\{\varphi(x)\}$ of B, interval components $I = [a, b]$, $a < b$ are mapped on proper intervals $\varphi(I)$. Among all points $x \in I$ the end points a and b can be characterized topologically by the property that $I \setminus \{x\}$ is connected if and only if $x \in \{a, b\}$. Thus φ maps end points of an interval of A to end points of an interval of B. For the end point a the following two situations can be distinguished. If $G \prec I = [a, b]$ for some $G \in \mathcal{X}_g(A)$ then $a \in A$ has a neighbourhood base of connected subsets $[a, a + \varepsilon) \subseteq A$. We call I an isolated interval (from the left). Similarly we call I isolated from the right if $I \prec G$ for some $G \in \mathcal{X}_g(A)$. If $G_1 \prec I \prec G_2$ we call I both-sided isolated. If no G with $G \prec I = [a, b]$ exists, then a has no connected neighbourhood in A and we call I accumulated from the left (similarly from the right or both-sided). Thus both-sided isolated intervals are preserved by homeomorphisms as well as one-sided isolated (hence one-sided accumulated) intervals and both-sided accumulated intervals. Therefore we can distinguish the following topological cases for an admissible set A:

- A contains no interval. By Theorem 4, A is a Cantor set and uniquely determined from the order theoretic point of view.
- A contains intervals which are all both-sided accumulated. Then A is an M-Cantorval. By Theorem 6, A is determined uniquely from the order theoretic point of view.
- A contains intervals which are not both-sided accumulated. This includes the L/R-Cantorval and the interval type case. Theorems 9 and 11.3 show that very different situations are possible.

The question arises whether $A \cong_t B$ is possible with an interval type A and an L-Cantorval B. Since in B we have $G \prec F$ but never $F \prec G$ ($F \in \mathcal{X}_f, G \in \mathcal{X}_g$), there are no both-sided isolated intervals of B. Hence $A \cong_t B$ yields that in A only chains of the type $F_1 \prec G \prec F_2$ are possible and of course must occur, but no longer chains. It may be surprising at first glance that such sets A and B indeed might be homeomorphic.

Theorem 12. *Consider an interval type $A \subseteq \mathbb{R}$ such that $\mathcal{O}^{**} = \mathcal{O}^{**}(A)$ is a dense order with all points of colour s_1. Similarly let $B \subseteq \mathbb{R}$ denote an admissible L-Cantorval such that $\mathcal{O}^* = \mathcal{O}^*(B)$ is a dense order with all points of colour p. Then $A \cong_t B$. Such sets A and B exist.*

Proof. From Theorems 1.1, 8 and 11.1 it gets clear that sets A and B with the given order theoretic structure exist. The proof of $A \cong_t B$ proceeds in five steps.

First step, parametrization for A resp. $\mathcal{O}^{**}$ and B resp. $\mathcal{O}^*$: Note that the order structures of $\mathcal{O}^{**}$ and $\mathcal{O}^*$ are uniquely determined by the assumption. Let D denote the set of all dyadic numbers $d \in [0,1]$, i.e. $d = \frac{k}{2^n}$ with integers $n \geq 0$ and $0 \leq k \leq 2^n$. Let $\mathcal{X}(A)$ consist of $F_d = [\alpha_d, \beta_d] \in \mathcal{X}_f(A)$ ($d \in D \setminus \{0\}$), $G_d = (\beta_d, \gamma_d) \in \mathcal{X}_g(A)$ ($d \in D$) and $F'_d = [\gamma_d, \delta_d] \in \mathcal{X}_f(A)$ ($d \in D \setminus \{1\}$) such that $\delta_{d_1} < \alpha_{d_2}$ if and only if $d_1 < d_2$. Since D is a countable dense ordering, A is as assumed in the theorem if and only if $\mathcal{O}(A)$ has this structure. Similarly B can be described by $I_d = (\rho_d, \sigma_d) \in \mathcal{X}_g(B)$ ($d \in D$) and $J_d = [\sigma_d, \tau_d] \in \mathcal{X}_f(B)$ ($d \in D \setminus \{1\}$) with $\tau_{d_1} < \rho_{d_2}$ if and only if $d_1 < d_2$.

Second step, further notations: Let 2^n denote the set of all 0-1-sequences $a = (a_1, a_2, \dots, a_n)$, $a_i \in 2 = \{0,1\}$, of length $n \geq 0$, $2^{<\omega}$ the union of all 2^n, 2^ω the set of all infinite 0-1-sequences $\bar{a} = (a_1, a_2, \dots)$. To any finite sequence $a \in 2^n$ we associate the numbers $q_a = \sum_{i=1}^n \frac{a_i}{2^i}$, $q'_a = q_a + \frac{1}{2^n}$ and the sets

$$A_a = \overline{\bigcup \{F_q \cup F'_q : q_a < q < q'_a\} \cup F'_{q_a} \cup F_{q'_a}}$$

and

$$B_a = \overline{\bigcup \{J_q : q_a \leq q < q'_a\}}.$$

The sets A_a and B_a are clopen in A resp. in B. If we are given an infinite sequence $\bar{a} = (a_1, a_2, \dots) \in 2^\omega$, we write $\bar{a}_n = (a_1, \dots, a_n)$ for its initial segments, furthermore

$$q_{\bar{a}} = \lim_{n \to \infty} q_{\bar{a}_n} = \sum_{n=1}^\infty \frac{a_n}{2^n}$$

and

$$A_{\bar{a}} = \bigcap_n A_{\bar{a}_n}, \quad B_{\bar{a}} = \bigcap_n B_{\bar{a}_n}.$$

Let 2^ω_0 denote the set of all 0-1-sequences where the a_n are eventually 0, 2^ω_1 the set of those where the a_n are eventually 1 and $2^\omega_* = 2^\omega \setminus 2^\omega_0 \setminus 2^\omega_1$ the rest of 2^ω. Note that with $q = q_{\bar a}$ we have $\bar a \in 2^\omega_0$ iff $A_{\bar a} = F'_q$ iff $B_{\bar a} = J_q$; $\bar a \in 2^\omega_1$ iff $A_{\bar a} = F_q$ (in this case $B_{\bar a} = \{y_{q_{\bar a}}\}$ is a single point of B); $\bar a \in 2^\omega_*$ iff $A_{\bar a} = \{x_{q_{\bar a}}\}$ is a single point of A. Furthermore we mention that $\lim_{n\to\infty} x_n = x$, $x_n \in A_{\bar a_n}$ (resp. $B_{\bar a_n}$) implies $x \in A_{\bar a}$ (resp. $B_{\bar a}$).

Third step, construction of a bijection $\bar\psi : 2^\omega \to 2^\omega$:
For $\bar a = (a_1, a_2, \dots) \in 2^\omega$ we define $\bar\psi(\bar a) = \bar b = (b_1, b_2, \dots)$ in the following way. Given $\bar a$, there is a unique finite or infinite sequence $0 = n_0 < n_1 < n_2 < \dots$, $n_{k+1} \geq n_k + 2$, of maximal length such that $(a_{n_{k-1}+1}, \dots, a_{n_k})$ either is of the form $(0, 0, \dots, 0, 1)$ or $(1, 1, \dots, 1, 0)$. In the first case we put $(b_{n_{k-1}+1}, \dots, b_{n_k}) = (0, 0, \dots, 0, 1)$, in the second case $= (1, 0, 0, \dots, 0, 0, 1)$. If the sequence of the n_k is finite, i.e. $(a_{n_k+1}, a_{n_k+2}, \dots)=(0, 0, \dots)$ or $= (1, 1, \dots)$, then let $(b_{n_k+1}, b_{n_k+2}, \dots)= (0, 0, \dots)$ in the first case and $= (1, 0, 0, \dots)$ in the second one:

$$\begin{aligned}
(0,0,0,\dots,0,0,1) &\mapsto (0,0,0,\dots,0,0,1)\\
(1,1,1,\dots,1,1,0) &\mapsto (1,0,0,\dots,0,0,1)\\
(0,0,0,\dots,0,\dots) &\mapsto (0,0,0,\dots,0,\dots)\\
(1,1,1,\dots,1,\dots) &\mapsto (1,0,0,\dots,0,\dots)
\end{aligned}$$

In the remaining steps of the proof we shall frequently use the following properties of $\bar\psi$.

- $(b_1, \dots, b_n)$ depends only on $(a_1, \dots, a_n)$, not on a_k for $k > n$. Thus we have mappings $\psi_n : 2^n \to 2^n$ with $\psi_n(\bar a_n) = \bar\psi(\bar a)_n$. Whenever convenient we write $\psi = \bigcup_n \psi_n : 2^{<\omega} \to 2^{<\omega}$.
- ψ, ψ_n, $\bar\psi$ are bijections. It is easy to check that the following mapping of blocks induces the inverse of $\bar\psi$ in a similar way as ψ was defined:

$$\begin{aligned}
(0,0,0,\dots,0,0,1) &\mapsto (0,0,0,\dots,0,0,1)\\
(1,0,0,\dots,0,0,1) &\mapsto (1,1,1,\dots,1,1,0)\\
(0,0,0,\dots,0,\dots) &\mapsto (0,0,0,\dots,0,\dots)\\
(1,0,0,\dots,0,\dots) &\mapsto (1,1,1,\dots,1,\dots)
\end{aligned}$$

Here, for given $\bar{b}=(b_1, b_2, \dots)$, the considered blocks $(b_{n_{k-1}+1}, \dots, b_{n_k})$ are defined by taking n_{k+1} as the minimal $l \geq n_k + 2$ with $b_l = 1$ (if such an l exists).

- $\bar{\psi}(2^\omega_0 \cup 2^\omega_1) = 2^\omega_0$.
- $\bar{\psi}(2^\omega_*) = 2^\omega_1 \cup 2^\omega_*$.

Fourth step, construction of φ via $\bar{\psi}$: The mapping φ is defined in such a way that $\varphi(A_{\bar{a}}) = B_{\bar{\psi}(\bar{a})}$. If $\bar{a} \in 2^\omega_*$, then $A_{\bar{a}} = \{x\}$ is a singleton component of A, and $B_{\bar{\psi}(\bar{a})} = \{y\} \in 2^\omega_1 \cup 2^\omega_*$ is a singleton component of B, hence $\varphi(x) = y$ is unique by this property. If $\bar{a} \in 2^\omega_0 \cup 2^\omega_1$ then both $A_{\bar{a}} = [x, y]$ and $B_{\bar{\psi}(\bar{a})} = [v, w]$ are proper intervals. Hence we define $\varphi|_{[x,y]}$ to be any homeomorphism between these intervals such that the following condition is fulfilled: If $[x, y] = F_q$ for some q then let φ be decreasing, if $[x, y] = F'_q$ for some q let φ be increasing. This takes care of the fact that the F'_q and the J_q are accumulated from the right side and the F_q are accumulated from the left side.

Fifth step, φ has the desired properties: From the properties listed above it is clear that $\bar{\psi}$ is a bijection on the set of 0-1-sequences. This implies that $\varphi : A \to B$ is also a bijection. We are going to check the continuity of φ at each point $x \in A$. Since A is compact, this will imply that φ is a homeomorphism. We distinguish three different cases for x:

1. Let $x \in F \setminus \overline{A \setminus F}$ for some $F \in \mathcal{X}_f(A)$, i.e. x is an inner point of some interval with respect to the subspace topology on $A \subseteq \mathbb{R}$. Then φ, whose restriction to intervals is a homeomorphism, is continuous in x.
2. If $x \in A \setminus \bigcup \mathcal{X}_f$, i.e. $\{x\} = A_{q_{\bar{a}}}$ is a singleton component of A and $\bar{a} \in 2^\omega_*$, consider any sequence (x_k) in A converging to x. For given $\varepsilon > 0$ we can find an n such that $B_{\bar{\psi}(\bar{a})_n} \subseteq (y-\varepsilon, y+\varepsilon)$, where $y = \varphi(x)$ is the single point of B forming the component $B_{\bar{\psi}(\bar{a})}$. (Here we used the fact mentioned at the end of the second step.) For sufficiently large k we have $x_k \in A_{\bar{a}_n}$, therefore $\varphi(x_k) \in B_{\bar{\psi}(\bar{a})_n}$, yielding $|\varphi(x_k) - \varphi(x)| < \varepsilon$, i.e. $\varphi(x_k)$ converges to $\varphi(x)$.
3. If $x \in F \cap \overline{A \setminus F}$ for some $F \in \mathcal{X}_f(A)$, i.e. x is an accumulation end point of an interval, then let (x_k) be an arbitrary sequence in A converging to x. We can split (x_k) into two subsequences, one in F and one outside of F. Thus it suffices to show that $\lim_{k\to\infty} x_k = x$ implies $\lim_{k\to\infty} \varphi(x_k) = \varphi(x)$ for each of those types. The first type can be treated as the first case for x (inner point of an interval) and the second type as the second case (x single point). $\square$

8. Several old and one new example of sets of P-sums

In this section we reconsider four examples of sets of P-sums in our context and give one further example showing that the list from [N-S2] of possible structures of sets $S = S(P, \Lambda)$ of P-sums is not complete.

1. Finite union of intervals. This subordinates to our so-called interval type. In [N-S1], Theorem 2.4, two conditions on P and Λ are given; one is sufficient and one is necessary for S to be a finite union of intervals. An interesting example has been presented in [N-S1]: Taking $P = \{0, 1, 2, 7, 8, 9\}$ and $\lambda_n = 3^{-n}$ one gets $S = [0, \frac{13}{6}] \cup [\frac{7}{3}, \frac{9}{2}]$.
2. Cantor set. The classical example is $P = \{0, 2\}$ and $\lambda_n = 3^{-n}$. Theorem 2.4. in [N-S1] contains a sufficient condition on P and Λ which implies that S is homoeomorphic to the Cantor set. In our paper Theorem 4 is the order theoretic version of this result, stating that every nowhere dense perfect compact subset of $\mathbb{R}$ has the same order theoretic structure.
3. M-Cantorval. The reader may check that the example of an M-Cantorval presented in Section 2 in [Gu-N] and reconsidered in [N-S1] and [N-S2] can be rewritten as $S = S(P, \Lambda)$ with $P = \{0, 2, 3, 5\}$ and $\lambda_n = 4^{-n}$. Theorem 1 in [Gu-N] says that if $P = \{0, 1\}$ then S is either a finite union of intervals or a Cantor set or homeomorphic to this example. In our paper, Theorem 6 says that all M-Cantorvals have the same order theoretic structure.
4. L-Cantorval. In the last section of [N-S2] the authors give a short discussion of the example $P = \{0, 1, 2, 9\}$, $\lambda_n = 3^{-n}$, and mention that $[0, \frac{9}{8}]$ is an interval of S, that S has infinitely many gaps in every neighbourhood of $\frac{9}{2} = \max S$ and S is not homeomorphic to one of the previous types. Thus we have indeed an example of an L-Cantorval. In [N-S2] the conjecture has been stated that now the list of possible topological structures of such sets is complete. The next example shows that this is not the case.
5. Interval type with infinitely many intervals. If one takes $\lambda_n = 4^{-n}$ and $P = \{-12, -11, -10, -9, 0, 9, 10, 11, 12\}$, one gets a set S of the interval type. In fact $S = -S$ consists of exactly one single point 0 plus the disjoint union of one increasing and one decreasing sequence of intervals, both converging to $0 \in S$. This follows from the theorem proved below. Since S contains infinitely many both-sided isolated intervals, it cannot be homeomorphic to any of the previous examples.

Theorem 13. *Let $P = P_0 \cup \{0\}$ with $P_0 = \{-12, -11, -10, -9, 9, 10, 11, 12\}$, $\Lambda = (\lambda_n)$, $\lambda_n = 4^{-n}$, $S_0 = S(P_0, \Lambda)$ and $S = S(P, \Lambda)$. Then the following statements hold:*

1. $S_0 = [-4, -\frac{5}{4}] \cup [\frac{5}{4}, 4]$.
2. $S = \{0\} \cup \bigcup_{n=0}^{\infty} \frac{1}{4^n} S_0$.
3. $\mathcal{O}(S) \cong \{G_0^- \prec F_0^- \prec G_1^- \prec F_1^- \prec \cdots \prec F_1^+ \prec G_1^+ \prec F_0^+ \prec G_0^+\}$, *i.e. S is $**$-admissible with $\mathcal{O}^{**}(S) \cong \{I_1 \prec I_2\}$, $\chi^{**}(I_1) = s^+$, $\chi^{**}(I_2) = s^-$.*

Proof. 1. Let $S' = [-4, -\frac{5}{4}] \cup [\frac{5}{4}, 4]$.

$S' \subseteq S_0$: First we prove that if $x = \sum_{n=1}^{\infty} \frac{a_n}{4^n} \in S_0$ with $a_n \in \{-12, 9\}$ then $[x, x+1] \subseteq S_0$. To see this take $y \in [x, x+1]$, i.e. $y = x + z$ with $0 \leq z \leq 1$. Such a z has a representation $z = \sum_{n=1}^{\infty} \frac{b_n}{4^n}$ with $b_n \in \{0, 1, 2, 3\}$. Thus $y = \sum_{n=1}^{\infty} \frac{c_n}{4^n}$ with $c_n = a_n + b_n \in \{-12, 9\} + \{0, 1, 2, 3\} = P_0$ and $y \in S_0$. As special values for x we consider

$$\begin{aligned}
x_1 &= -4 = \textstyle\sum_{n=1}^{\infty} \frac{-12}{4^n}, \\
x_2 &= -4 + \tfrac{21}{64} = \tfrac{-12}{4} + \tfrac{-12}{4^2} + \tfrac{9}{4^3} + \textstyle\sum_{n=4}^{\infty} \frac{-12}{4^n}, \\
x_3 &= -3 + \tfrac{20}{64} = \tfrac{-12}{4} + \tfrac{9}{4^2} + \textstyle\sum_{n=3}^{\infty} \frac{-12}{4^n}, \text{ and} \\
x_4 &= -3 + \tfrac{3}{4} = \tfrac{-12}{4} + \textstyle\sum_{n=2}^{\infty} \frac{9}{4^n}.
\end{aligned}$$

Since $x_i \leq x_{i+1}$ and $x_{i+1} \leq x_i + 1$ for $i = 1, 2, 3$, we get $[-4, -\frac{5}{4}] = [x_1, x_4 + 1] \subseteq S_0$. Since $P_0 = -P_0$ we have $S_0 = -S_0$ and get $[\frac{5}{4}, 4] \subseteq S_0$ and therefore $S' \subseteq S$.

$S_0 \subseteq S'$: Pick $x \in S_0$, i.e. $x = \sum_{n=1}^{\infty} \frac{a_n}{4^n}$ with $a_n \in P_0$. $x \geq \sum_{n=1}^{\infty} \frac{-12}{4^n} = -4$ and similarly $x \leq 4$. If $a_1 < 0$ we get $x \leq \frac{-9}{4} + \sum_{n=2}^{\infty} \frac{12}{4^n} = -\frac{5}{4}$. Similarly $a_1 > 0$ implies $x \geq \frac{5}{4}$. This shows $x \in S'$ in each case, hence $S_0 \subseteq S'$.

2. Now we have to show $S = S''$ with $S'' = \{0\} \cup \bigcup_{n=0}^{\infty} \frac{1}{4^n} S_0$.

$S'' \subseteq S$: Pick $x \in S''$. If $x = 0$ then $x = \sum_{n=1}^{\infty} \frac{0}{4^n} \in S$. If $x \neq 0$ then we can find an integer $n_0 \geq 0$ and an $x_0 \in S_0$ such that $x = \frac{x_0}{4^{n_0}}$. By the first statement there is a representation $x_0 = \sum_{n=1}^{\infty} \frac{a_n}{4^n}$ with $a_n \in P_0$. Thus

$$x = \frac{1}{4^{n_0}} \sum_{n=1}^{\infty} \frac{a_n}{4^n} = \sum_{n=1}^{\infty} \frac{a'_n}{4^n}$$

with $a'_1 = a'_2 = \cdots = a'_{n_0} = 0 \in P$ and $a'_n = a_{n_0+n} \in P_0 \subseteq P$ for $n > 0$. Thus $x \in S$ and $S'' \subseteq S$.

$S \subseteq S''$: If $x \in S$ then $x = \sum_{n=1}^{\infty} \frac{a_n}{4^n}$ with $a_n \in P$. If $a_n = 0$ for all n then $x = 0 \in S''$. Otherwise there is a minimal k with $x = \sum_{n=k+1}^{\infty} \frac{a_n}{4^n}$ and $a_{k+1} \neq 0$. Similar to the first part of the proof we get $-\frac{1}{4^{k-1}} \leq x \leq \frac{1}{4^{k-1}}$, furthermore $x \leq \frac{1}{4^k}(-\frac{5}{4})$ if $a_{k+1} < 0$ and $x \geq \frac{1}{4^k}\frac{5}{4}$ if $a_{k+1} > 0$. This shows $x \in \frac{1}{4^k} S_0 \subseteq S''$.

3. From the second statement it gets immediately obvious that with $\frac{1}{4^n} S_0 = F_n^- \cup F_n^+$, $F_n^- = -F_n^+$, $F_n^+ = \frac{1}{4^n}[\frac{5}{4}, 4]$, $G_n^- = -G_n^+$, $G_0^+ = (4, \infty)$ and $G_n^+ = \frac{1}{4^n}(4,5)$ for $n \geq 1$ we get the pairwise disjoint intervals of S and $\mathbb{R} \setminus S$ which are ordered as stated in the theorem. $\square$

9. Open problems

In this final section we mention several open problems which could be a motivation for further research on our topic. We make some observations which might be considered as indications that there are many possibilities left open by the examples treated up to now.

First we note that, in all examples treated explicitly up to now, P only contains integers and $\lambda = \lambda^{-n}$ with an integer λ. We do not have strong arguments to believe that this restriction does not affect the generality w.r.t. the topological or order theoretic structure. We must not forget that in this special case the set S has the strong self similarity property $S_n = \lambda^{-n} S$ (S_n as in the proof of Theorem 3) which in general does not hold if the λ_n do not form a geometric progression.

An almost trivial illustration of this aspect is the following observation: If $S = S(P, (\lambda_n))$ induces the coloured ordering $\mathcal{O} = \mathcal{O}(S)$ then we (essentially) can produce the order structure $\mathcal{O} \times |P| = \mathcal{O} + \cdots + \mathcal{O}$ by replacing (λ_n) by λ'_n with $\lambda'_{n+1} = \lambda_n$ for $n \geq 1$ and λ'_1 sufficiently large.

Iterating this process we can, starting with our new example from Theorem 13, generate an infinite series of non homeomorphic examples, namely, for each integer $k \geq 0$ an example with exactly 9^k singleton components.

To be less trivial consider the L-Cantorval $S = S(P, \Lambda)$ with $P = \{0, 1, 2, 9\}$ and $\lambda_n = 3^{-n}$ (Section 8, example 4). The results of Section 5 show that L-Cantorvals can have various structures, maybe quite complicated. This depends on the question whether there are both-sided accumulated intervals in S and how they are embedded. It seems to be not trivial to decide this question. In the concrete example number theoretic arguments may be expected. But this is due to the special example. Therefore from our perspective it cannot be excluded that, for example, replacing $9 \in P$ by an irrational/transcendental number $9 + \varepsilon$ would affect the fractal structure of S in a way which cannot be predicted easily.

On the other hand we can imagine the following situation. Suppose that under certain assumptions on P and $\Lambda = (\lambda_n)$ (for instance $\lambda_n = \lambda^{-1}$ with λ and all $p \in P$ rational or, stronger, integer) one could show that S has no both-sided accumulated intervals. Then Theorem 8.3 would yield that in such cases the order theoretic and hence the topological structure of S is uniquely determined. By Theorem 8.4 similar considerations apply for the case that one could show that the both-sided accumulated intervals of S are dense in $\mathcal{O}(S)$.

Concerning the interval type we mention that all investigated examples of $**$-admissible sets of P-sums only contain both-sided isolated intervals, finitely or infinitely many. Hence they are all not homeomorphic to L/R-Cantorvals ($*$-admissible sets). Is it possible that, nevertheless, the phenomenon of Theorem 12 occurs for sets A, B of P-sums?

We also recall the open question mentioned after Theorem 2: Does self similarity of $\mathcal{O}$ imply the existence of a self similar A with $\mathcal{O}(A) \cong \mathcal{O}$?

The problem of earlier papers, namely to give a complete list of the topological structures of P-sets, now can be modified by asking the following two questions: 1.) Given P and Λ, is $S(P, \Lambda)$ a Cantor set, an M-Cantorval, an R/L-Cantorval or of interval type? 2.) Which $*$- and $**$-admissible orderings correspond to sets of P-sums and which of them correspond to homeomorphic sets? It seems that an answer to this question and looking for satisfactory classification theorems describing the topological or order theoretic structure of $S(P, \Lambda)$ for any given P and Λ requires much deeper investigations than ours.

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(Received July 26, 1999; revised February 15, 2000)