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Modified version of Jensen equation and orthogonal additivity

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Abstract. Functional equation of Jensen type arrising in the theory of Orlicz spaces is studied. The 1-dimensional case turned out to be exceptional and has been treated separately. The shape of solutions of this equation is expressed in terms of multiplicative functions. After necessary modifications the case of higher dimensions is considered. Close connection between the resulting equation and certain kind of orthogonal additivity is visualized.

1. Introduction

We start with a functional inequality that occurs in some considerations concerning Orlicz spaces. For example P. Kolwicz and R. Płuciennik [2] deal with the inequality

$$f\left(\frac{x+y}{2}\right) \le \gamma[f(x)+f(y)]$$

postulated for some $\gamma \in \left(0, \frac{1}{2}\right)$ and all $x, y \in \mathbb{R}^+ := (0, \infty)$ such that $\frac{x}{y} \le a$ for some $a \in \mathbb{R}^+$. This inequality was an inspiration for author's paper [6] where a stronger version was considered. Namely, we were looking for functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\bigwedge_{a\in\mathbb{R}^+} \bigvee_{\gamma(a)} \bigwedge_{x,y\in\mathbb{R}^+, x\le ay} f\left(\frac{x+y}{2}\right) \le \gamma(a)[f(x)+f(y)].$$

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It turned out that the equality case

$$\bigwedge_{a\in\mathbb{R}^+}\bigvee_{\gamma(a)}\bigwedge_{x\in\mathbb{R}^+} f\left(\frac{x+ax}{2}\right)=\gamma(a)[f(x)+f(ax)]$$

plays a distinguished role.

In the present paper we deal with that kind of equations admitting the unknown functions defined on more abstract spaces. To this aim we have to replace the condition $\frac{x}{y}\le a$ by some other condition. Some way to do this is found by S. Chen and H. Hudzik in [1]. In that paper the functional inequality in question is assumed to be satisfied for all x, y fulfilling the inequality

$$|x-y|\ge\frac{1-\sigma}{2}|x+y|.$$

Thus we are going to solve the equation

$$f\left(\frac{x+y}{2}\right)=\gamma\left(\frac{|x-y|}{|x+y|}\right)[f(x)+f(y)] \tag{1}$$

for all $x,y\in\mathbb{R}$, $x\neq -y$. More generally, we shall deal with the same equation but with the modulus sign replaced by the norm in some real linear space. As an unexpected result we obtain the close connection between our equation and that of orthogonal additivity. Since there are no nontrivial orthogonalities in $\mathbb{R}$, the case of dimension equal to 1 will be considered separately.

It is easy to check that function defined by the following formula

$$f(x):=\begin{cases}0 & x\neq 0\\ c & x=0,\end{cases}$$

where c is a fixed real number, is a solution of equation (1). Since this solution is trivial, we make a general assumption that f is not of this form (f is nontrivial).

2. One-dimensional case

Theorem 1. *If $f:\mathbb{R}\to\mathbb{R}$ is a solution of equation* (1) *where $\gamma:[0,\infty)\to\mathbb{R}$ is a given injection, then either $f(x)=\beta\operatorname{sgn}(x)\phi(|x|)$, $x\in$*

$\mathbb{R}\setminus\{0\}$, or $f(x)=\beta\phi(|x|)$, $x\in\mathbb{R}\setminus\{0\}$, for some real number $\beta\neq 0$ where $\phi:\mathbb{R}^+\to\mathbb{R}$ is a function satisfying the equation

$$\phi(uv)=\phi(u)\phi(v),\quad u,v\in\mathbb{R}^+.$$

Proof. Take $x,y\in\mathbb{R}^+$. If $\frac{x}{y}=b$, then $\frac{|x-y|}{|x+y|}=\frac{|1-b|}{|1+b|}$. Thus

$$f\left(\frac{x+y}{2}\right)=\gamma\left(\frac{|1-b|}{1+b}\right)[f(x)+f(y)]=\gamma_1(b)[f(x)+f(y)]$$

where function $\gamma_1:\mathbb{R}^+\to\mathbb{R}$ is defined as follows:

$$\gamma_1(b):=\gamma\left(\frac{|1-b|}{1+b}\right),\quad b\in\mathbb{R}^+.$$

From [6] we derive the existence of a function $\alpha:\mathbb{R}^+\to\mathbb{R}$ such that

$$f(bx)=\alpha(b)f(x) \tag{2}$$

for all $b,x\in\mathbb{R}^+$. In the above mentioned paper function γ_1 was assumed to be increasing. This assumption was a consequence of some applications made in that paper. But, a careful inspection of the proof presented in [6], shows that what we really need is

$$\Gamma_3(b):=\gamma_1\left(\frac{b}{2b-1}\right)\neq\gamma_1\left(\frac{1}{b}\right)=:\Gamma_1(b)$$

for all $b>1$ (we preserve the notation from [6]). Since γ is assumed to be injective, if we had $\Gamma_1(b)=\Gamma_3(b)$ for some $b>1$, then

$$\frac{\left|1-\frac{1}{b}\right|}{1+\frac{1}{b}}=\frac{\left|1-\frac{b}{2b-1}\right|}{1+\frac{b}{2b-1}}, \tag{3}$$

which implies that $b=1$, a contradiction. Thus $\Gamma_1(b)\neq\Gamma_3(b)$ for all $b\in(1,\infty)$. Using (2), we get

$$\alpha(bc)f(x)=f(bcx)=\alpha(b)f(cx)=\alpha(b)\alpha(c)f(x)$$

for all $b,c,x\in\mathbb{R}^+$. Since $f\neq 0$, we have $\alpha(bc)=\alpha(b)\alpha(c)$ for all $b,c\in\mathbb{R}^+$. Furthermore $f(x)=f(x\cdot 1)=\alpha(x)f(1)$, and consequently, writing $\beta_1:=f(1)$ and $\phi(x):=\alpha(x)$ we get $f(x)=\beta_1\phi(x)$ for all $x\in\mathbb{R}^+$.

Let us now take $x, y \in \mathbb{R}^- := (-\infty, 0)$. In the same manner one can show that $f(x) = \beta_2\psi(-x)$ for all $x \in \mathbb{R}^-$, some constant β_2 and some nonzero function $\psi : \mathbb{R}^+ \to \mathbb{R}$ satisfying the equation

$$\psi(uv) = \psi(u)\psi(v), \quad u, v \in \mathbb{R}^+.$$

Our task now is to show that $\phi = \psi$. Fix arbitrary $x > 0$. Then

$$\begin{aligned} f\left(\frac{x}{2}\right) &= f\left(\frac{2x - x}{2}\right) = \gamma\left(\frac{2+1}{2-1}\right)[f(2x) + f(-x)] \\ &= \gamma(3)[\beta_1\phi(2)\phi(x) + \beta_2\psi(x)]. \end{aligned}$$

One the other hand, we have

$$f\left(\frac{x}{2}\right) = \beta_1\phi\left(\frac{x}{2}\right) = \beta_1\phi\left(\frac{1}{2}\right)\phi(x).$$

One can easily show that $\beta_1\phi\left(\frac{1}{2}\right)\phi(x) \neq 0$ for every $x > 0$. Thus

$$\frac{\phi(x)}{\beta_1\phi(2)\phi(x) + \beta_2\psi(x)} = \frac{\gamma(3)}{\beta_1\phi\left(\frac{1}{2}\right)} = \text{const}\,.$$

That means that also

$$\frac{\psi(x)}{\phi(x)} = c$$

for all x and some fixed $c \in \mathbb{R}$. Since $\psi(1) = \phi(1) = 1$, we get $c = 1$ and $\phi = \psi$.

The next step of the proof is to show that $|\beta_1| = |\beta_2|$. To this end let us write

$$\begin{aligned} \beta_2 = f(-1) &= f\left(\frac{-3+1}{2}\right) \\ &= \gamma\left(\frac{|-3-1|}{|-3+1|}\right)[f(-3) + f(1)] = \gamma(2)[\beta_2\phi(3) + \beta_1]. \end{aligned}$$

Similarly

$$\beta_1 = f(1) = f\left(\frac{3-1}{2}\right) = \gamma(2)[\beta_1\phi(3) + \beta_2].$$

Eliminating $\gamma(2)$ from above equations we conclude that

$$\frac{\beta_1}{\beta_1\phi(3)+\beta_2}=\frac{\beta_2}{\beta_1+\beta_2\phi(3)}.$$

Consequently

$$\beta_1^2+\beta_1\beta_2\phi(3)=\beta_1\beta_2\phi(3)+\beta_2^2,$$

whence $\beta_1^2=\beta_2^2$. □

The following remark will show that under some assumptions the above theorem gives a characterization of solutions of equation (1).

Remark 1. Let $\phi:\mathbb{R}^+\to\mathbb{R}$ be a function satisfying equation

$$\phi(xy)=\phi(x)\phi(y),\quad x,y\in\mathbb{R}^+.$$

Extend ϕ to $[0,\infty)$ by putting $\phi(0):=0$. Then the function $f:\mathbb{R}\to\mathbb{R}$ given by the formula

$$f(x)=\beta\phi(|x|),\qquad x\in\mathbb{R}, \tag{4}$$

satisfies equation (1); if moreover ϕ is an injection, then the function

$$f(x)=\beta\,\mathrm{sgn}(x)\phi(|x|),\qquad x\in\mathbb{R}, \tag{5}$$

satisfies equation (1).

Proof. We shall prove that any function f defined by formula (5) satisfies equation (1) with the function

$$\gamma^-(a):=\begin{cases}\dfrac{1}{\phi(1+a)+\phi(|1-a|)} & \text{if } a\le 1\\[2ex] \dfrac{1}{\phi(1+a)-\phi(|1-a|)} & \text{if } a>1,\end{cases}$$

whereas if f satisfies (4) then f is a solution of equation (1) with function

$$\gamma^+(a):=\frac{1}{\phi(1+a)+\phi(|1-a|)},\qquad a\in\mathbb{R}.$$

Clearly, without loss of generality we may assume that $\beta=1$. To verify our claim one needs to consider the following cases:

a) $f(x) = \phi(|x|)$, $x \in \mathbb{R}$, and

1. $x \geq y \geq 0$, $x + y \neq 0$
2. $x \leq y \leq 0$, $x + y \neq 0$
3. $y \leq 0 \leq x$, $|x| > |y|$
4. $y \leq 0 \leq x$, $|x| < |y|$

b) $f(x) = \mathrm{sgn}(x)\phi(|x|)$, $x \in \mathbb{R}$, and conditions 1–4 hold true.

We shall present the detailed proof in cases a)1 and b)3. The proof in the remaining cases is quite similar.

So, let $f(x) := \phi(|x|)$, $x \in \mathbb{R}$, and let $x \geq y \geq 0$. Then

$$f\left(\frac{x+y}{2}\right) = \frac{\phi(x+y)}{\phi(2)}.$$

Further

$$\gamma\left(\frac{x-y}{x+y}\right)[f(x)+f(y)] = \frac{\phi(x)+\phi(y)}{\phi\left(1+\frac{x-y}{x+y}\right)+\phi\left(1-\frac{x-y}{x+y}\right)}$$
$$= \frac{(\phi(x)+\phi(y))\phi(x+y)}{\phi(2)\phi(x)+\phi(2)\phi(y)} = \frac{\phi(x+y)}{\phi(2)} = f\left(\frac{x+y}{2}\right).$$

Now, consider case b)3. So, let $f(x) = \mathrm{sgn}\, x\phi(|x|)$, $x \geq 0 \geq y$ and $|x| > |y|$. For such x, y the expression $\frac{|x-y|}{|x+y|}$ is bigger than 1 and therefore we shall apply the second formula from the definition of function γ^-. Now write $0 \leq y' := -y$ to get

$$f\left(\frac{x+y}{2}\right) = \phi\left(\frac{x+y}{2}\right) = \phi\left(\frac{x-y'}{2}\right)$$

and

$$\gamma\left(\frac{|x-y|}{|x+y|}\right)[f(x)+f(y)] = \frac{\phi(x)-\phi(y')}{\phi\left(1+\frac{x+y'}{x-y'}\right)-\phi\left(|1-\frac{x+y'}{x-y'}|\right)}. \tag{6}$$

As we have already seen $\frac{|x-y|}{|x+y|} > 1$. We thus get $\left|1 - \frac{x+y'}{x-y'}\right| = \frac{x+y'}{x-y'} - 1 = \frac{2y'}{x-y'}$. Now, by equation (6)we have

$$\gamma\left(\frac{|x-y|}{|x+y|}\right) = \frac{1}{\phi\left(\frac{x-y'+x+y'}{x-y'}\right) - \phi\left(\frac{x+y'-x+y'}{x-y'}\right)} = \frac{\phi(x-y')}{\phi(2)(\phi(x)-\phi(y'))}.$$

Hence $\gamma\left(\frac{|x-y|}{|x+y|}\right)[f(x)+f(y)] = f\left(\frac{x+y}{2}\right)$. □

3. Multi-dimensional case

In the remaining part of the paper we assume that X is a real linear space with $\dim X \geq 2$.

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed space and let $f : X \to \mathbb{R}$ be a solution of the equation*

$$f\left(\frac{x+y}{2}\right) = \gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)[f(x)+f(y)], \tag{7}$$

for all $x, y \in X$, $x \neq -y$, with some function $\gamma : [0,\infty) \to \mathbb{R}$.

Then:

a) *if X is an inner product space, then*

$$f(x) = a(\|x\|^2) + b(x) + c, \quad x \in X, \tag{8}$$

for some additive functions $a : [0,\infty) \to \mathbb{R}$, $b : X \to \mathbb{R}$ and some constant $c \in \mathbb{R}$;

b) *if the norm $\|\cdot\|$ does not come from an inner product and* $\dim X \geq 3$, *then*

$$f(x) = b(x) + c, \quad x \in X$$

for some additive function $b : X \to \mathbb{R}$ and some constant $c \in \mathbb{R}$.

Proof. Assume that $x, y \in X$ and

$$\|x-y\| = \|x+y\|. \tag{9}$$

Then

$$f\left(\frac{x+y}{2}\right) = \gamma(1)[f(x)+f(y)] \tag{10}$$

and in particular,

$$f\left(\frac{x}{2}\right) = \gamma(1)[f(x) + f(0)] \tag{11}$$

for all $x \in X \setminus \{0\}$. Combining equations (10) and (11) we conclude that

$$\gamma(1)[f(x+y) + f(0)] = \gamma(1)[f(x) + f(y)]$$

for all x, y satisfying (9). Let us note that $\gamma(1) \neq 0$. Supposing the opposite, we see that $f\left(\frac{x}{2}\right) = 0$ for all $x \in X \setminus \{0\}$ and consequently $f(x) = 0$ for all $x \neq 0$ which contradicts the assumption that f is nontrivial. Since $\gamma(1) \neq 0$, we have

$$f(x+y) + f(0) = f(x) + f(y) \tag{12}$$

for x, y satisfying (9). Define a function by the formula $g(x) := f(x) - f(0)$, $x \in X$. This function satisfies the following conditional equation

$$\|x - y\| = \|x + y\| \Rightarrow g(x+y) = g(x) + g(y), \quad y \in X.$$

Relation

$$xRy \Longleftrightarrow \|x - y\| = \|x + y\|$$

is called *James orthogonality*. In inner product spaces this orthogonality coincides with the usual orthogonality defined by inner product. Therefore it is enough to apply suitable result for orthogonally additive functions in inner product spaces (see [3] page 43, Corollary 10).

Now, let X be normed space which is not an inner product space. Since $\dim X \geq 2$, it is known that there is no nonzero even James-orthogonally additive function (see [5] page 374, Theorem 1.1). Observe that the odd and even parts of f are James orthogonally additive as well. Consequently we may assume that g is odd. In the paper [4] (page 270) Gy. Szabó proved that if $\dim X \geq 3$, then odd James-orthogonally additive function must be unconditionally additive. This finishes the proof. □

From the proof presented above we can derive the following two remarks.

Remark 2. Let X be a normed space such that the norm $\|\cdot\|$ does not come from an inner product. Then there is no nonzero even function satisfying (7).

Remark 3. Let X be a normed space such that the norm $\|\cdot\|$ does not come from an inner product. Each solution of equation (7) vanishing at zero has to be odd. In the case where $\dim X = 2$ the shape of solutions of equation (7) remains unknown (similarly as the form of James-orthogonally additive functions in such a case).

Theorem 2. *Let X be an inner product space. Then $f : X \to \mathbb{R}$ satisfies equation (7) if and only if either $\gamma(\alpha) = \frac{1}{2}$ for $\alpha \geq 0$ and*

$$f(x) = b(x) + c, \qquad x \in X,$$

for some additive function $b : X \to \mathbb{R}$ and some real constant c, or $\gamma(\alpha) = \frac{1}{2(1+\alpha^2)}$ for $\alpha \geq 0$ and

$$f(x) = k\|x\|^2, \qquad x \in X,$$

for some constant $k \in \mathbb{R}$.

Proof. **Part I.**

In the first part of the proof we are going to show that either

$$\gamma(1) = \frac{1}{4} \quad \text{or} \quad a = 0.$$

Applying formula (8) from Lemma 1 to equation (7) we infer that

$$\frac{1}{4}a\left(\|x+y\|^2\right) + \frac{1}{2}b(x+y) + c = \gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)\left[a\left(\|x\|^2 + \|y\|^2\right) + b(x+y) + 2c\right]$$

for all $x, y \in X$, $x \neq -y$. Taking $y = 0$ we obtain

$$\frac{1}{4}a\left(\|x\|^2\right) + \frac{1}{2}b(x) + c = \gamma(1)\left[a\left(\|x\|^2 + \|y\|^2\right) + b(x+y) + 2c\right], \qquad x \in X \setminus \{0\},$$

and, consequently,

$$\left(\gamma(1)-\frac{1}{4}\right)a\left(\|x\|^2\right)+\left(\gamma(1)-\frac{1}{2}\right)b(x)+c(2\gamma(1)-1)=0 \tag{13}$$

for all $x \in X \setminus \{0\}$. Replacing x by $2x$ in equation (13) we obtain

$$4\left(\gamma(1)-\frac{1}{4}\right)a\left(\|x\|^2\right)+2\left(\gamma(1)-\frac{1}{2}\right)b(x)+c(2\gamma(1)-1)=0, \quad x \neq 0. \tag{14}$$

Now we shall subtract (13) from (14) to get

$$3\left(\gamma(1)-\frac{1}{4}\right)a\left(\|x\|^2\right)+\left(\gamma(1)-\frac{1}{2}\right)b(x)=0 \quad \text{for all } x \in X. \tag{15}$$

Again taking $2x$ instead of x we can rewrite (15) as

$$12\left(\gamma(1)-\frac{1}{4}\right)a\left(\|x\|^2\right)+2\left(\gamma(1)-\frac{1}{2}\right)b(x)=0 \quad \text{for all } x \in X. \tag{16}$$

Now we multiply both sides of the equation (15) by 2 to get

$$6\left(\gamma(1)-\frac{1}{4}\right)a\left(\|x\|^2\right)+2\left(\gamma(1)-\frac{1}{2}\right)b(x)=0 \quad \text{for all } x \in X. \tag{17}$$

Subtracting (17) from (16) and dividing the result by 6 yields

$$\left(\gamma(1)-\frac{1}{4}\right)a\left(\|x\|^2\right)=0, \quad \text{for all } x \in X. \tag{18}$$

At this moment first aim of the proof has been achieved.

Part II.

Lemma 1 provides the form of f by the formula (8). Using the result of the first part – namely equation (18), it is easy to prove that in this form either $a=0$ or $b=0$, $c=0$.

Let us distinguish two cases:

1° $\gamma(1)=\frac{1}{4}$; in this case by (15) we have $\left(\gamma(1)-\frac{1}{2}\right)b(x)\equiv 0$. Consequently $b=0$ and

$$f(x)=a\left(\|x\|^2\right)+c, \quad x \in X \setminus \{0\}.$$

But by (14), $c\left(\frac{1}{2}-1\right)=0$ and hence

$$f(x)=a\left(\|x\|^2\right), \quad x\in X. \tag{19}$$

2° $\gamma(1)\neq\frac{1}{4}$; then by (18) we get $a=0$ and, consequently $f(x)=b(x)+c$ for all $x\in X$. It is obvious that in this case $\gamma(\alpha)=\frac{1}{2}$, $\alpha\in[0,\infty)$.

Part III.

It remains to prove that in the first case $f(x)=k\|x\|^2$ for all $x\in X$. We already have

$$f(x)=a\left(\|x\|^2\right) \quad \text{for all } x\in X,$$

and some additive function a. That means that in this part of the proof we want to obtain regularity of a. It will be done by proving the boundedness below on some set.

Applying the above form of f into (7) we get

$$\frac{1}{4}a\left(\|x+y\|^2\right)=\gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)a\left(\|x\|^2+\|y\|^2\right) \tag{20}$$

for all $x,y\in X$, $x\neq -y$, which may equivalently be written as

$$\frac{1}{2}a\left(\|x+y\|^2\right)=\gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)a\left(\|x+y\|^2+\|x-y\|^2\right).$$

Setting $x+y=s, x-y=t$ we can rewrite the above equation as

$$\frac{1}{2}a\left(\|s\|^2\right)=\gamma\left(\frac{\|t\|}{\|s\|}\right)a\left(\|s\|^2+\|t\|^2\right)$$

for all $s,t\in X$, $s\neq 0$. Now, putting $t=\alpha s$, $\alpha\in[0,\infty)$, we obtain

$$\frac{1}{2}a\left(\|s\|^2\right)=\gamma(\alpha)a\left(\|s\|^2+\|\alpha s\|^2\right)=\gamma(\alpha)a\left(\left(1+\alpha^2\right)\|s\|^2\right),\ s\in X\setminus\{0\}.$$

Thus, for any $s\in X$ with $\|s\|=1$ we obtain

$$\frac{1}{2}a(1)=\gamma(\alpha)a\left(1+\alpha^2\right), \quad \alpha\in[0,\infty). \tag{21}$$

Multiply (21) by $a\left(1+\left(\frac{\|x-y\|}{\|x+y\|}\right)^2\right)$ to get

$$\frac{1}{4}a\left(\|x+y\|^2\right)a\left(1+\frac{\|x-y\|^2}{\|x+y\|^2}\right)$$
$$=\gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)a\left(1+\left(\frac{\|x-y\|}{\|x+y\|}\right)^2\right)\cdot a\left(\|x\|^2+\|y\|^2\right).$$

Now, by (21), $\frac{1}{2}a(1)=\gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)a\left(1+\left(\frac{\|x-y\|}{\|x+y\|}\right)^2\right)$ whence

$$\frac{1}{2}a\left(\|x+y\|^2\right)a\left(1+\frac{\|x-y\|^2}{\|x+y\|^2}\right)=a(1)a\left(\|x\|^2+\|y\|^2\right)$$

for all $x,y\in X$, $x\neq -y$.

Put $q:=\|x+y\|^2$ and $r:=\|x-y\|^2$. Then

$$\|x\|^2+\|y\|^2=\frac{q+r}{2}.$$

Since for each pair of positive numbers q, r we can find $x,y\in X$ such that $q=\|x+y\|^2$, $r=\|x-y\|^2$, we get the following

$$\frac{1}{2}a(q)a\left(1+\frac{r}{q}\right)=a(1)a\left(\frac{q+r}{2}\right),\quad q,r>0,$$

and, in consequence,

$$a(q)a\left(1+\frac{r}{q}\right)=a(1)a(q+r),\qquad q,r>0.$$

Further, writing rq instead of r yields

$$a(q)a(1+r)=a(1)a(q+qr)=a(1)a(q(1+r)),\quad q,r>0.$$

Take in the above equation $\lambda:=1+r$ and $q=\lambda$ to get

$$0\leq a(\lambda)^2=a(1)a(\lambda^2).$$

for all $\lambda\in[1,\infty)$. Hence a is a an additive function bounded below on $[1,\infty)$ which implies that

$$a(\lambda)=k\lambda$$

for some $k \in \mathbb{R}$ and all $\lambda \in [0, \infty)$. This jointly with (19) shows that $f(x) = k\|x\|^2$, $x \in X$.

Part IV.

In the last part of the proof we want to obtain the form of function γ. We also have to prove that the function

$$f(x) = k\|x\|^2$$

satisfies our equation.

Observe that the formula

$$\gamma(\alpha) = \frac{1}{2(1+\alpha^2)}$$

yields an immediate consequence of (21).

To finish the proof it remains to show that $f(x) = \|x\|^2$ satisfies (7). In fact, to have

$$\frac{\|x+y\|^2}{4} = \frac{\|x\|^2 + \|y\|^2}{2\left(1 + \frac{\|x-y\|^2}{\|x+y\|^2}\right)}, \quad x, y \in X,$$

i.e.

$$\frac{\|x+y\|^2}{2} = \frac{\|x\|^2 + \|y\|^2}{\frac{\|x+y\|^2 + \|x-y\|^2}{\|x+y\|^2}}, \quad x, y \in X,$$

we must check that

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad x, y \in X.$$

Since X is an inner product space, this equality is obviously satisfied. □

Corollary 1. *Let X be a normed space. Then $f(x) = \|x\|^2$, $x \in X$, satisfies equation* (7) *if and only if X is an inner product space.*

Proof. From the above theorem we infer that if X is an inner product space, then f is a solution of equation (7) with function $\gamma(\alpha) = \frac{1}{2(1+\alpha^2)}$, $\alpha \in [0, \infty)$. On the other hand if X is not an inner product space, then since f is even, Remark 2 finishes the proof. □

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