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A generalization of the Ando–Krieger theorem

By ROMAN DRNOVŠEK (Ljubljana)

Abstract. Using a recent remarkable result of Turovskii we prove the following extension of the famous Ando–Krieger theorem to multiplicative semigroups of operators. Let E be a Dedekind complete Banach lattice of dimension at least two, and let $\mathcal{S}$ be a multiplicative semigroup of quasinilpotent positive abstract kernel operators on E. Then $\mathcal{S}$ has a non-trivial invariant closed ideal.

1. Introduction

The Ando–Krieger theorem asserts that every quasinilpotent positive kernel operator on an L^p-space ($1 \leq p < \infty$) has a non-trivial invariant closed ideal. This beautiful theorem has been extended by several authors. For instance, the papers [15], [4], [2], and [1] proved it under weaker assumptions on the operator and on the underlying Banach space. On the other hand, the authors of the papers [5] and [9] have considered multiplicative semigroups of quasinilpotent positive kernel operators on the L^2-space and on the space $\mathcal{C}(\mathcal{K})$, respectively. The aim of this note is to extend the results of these two papers to semigroups of operators on a Banach lattice. For the terminology concerning Banach lattices we refer to the standard books [10], [14], [18], [3], and [11].

Let X be a real or complex Banach space of dimension at least two. By an *operator* on X we mean a bounded linear transformation from X

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into itself. We denote by $\mathcal{B}(X)$ the Banach algebra of all operators on X. Let $\mathcal{C}$ be a subset of $\mathcal{B}(X)$, and let $x \in X$. Define

$$\|\mathcal{C}\| := \sup\{\|T\| : T \in \mathcal{C}\} \quad \text{and} \quad \|\mathcal{C}x\| := \sup\{\|Tx\| : T \in \mathcal{C}\}.$$

If $\mathcal{D}$ is another subset of $\mathcal{B}(X)$, we write $\mathcal{C}\mathcal{D} := \{TS : T \in \mathcal{C},\ S \in \mathcal{D}\}$. The powers $\mathcal{C}^n$ are defined inductively by $\mathcal{C}^1 := \mathcal{C}$, $\mathcal{C}^n := \mathcal{C}\mathcal{C}^{n-1}$ for $n \geq 2$. *The Rota–Strang spectral radius* or *the joint spectral radius* $\hat{r}(\mathcal{C})$ of $\mathcal{C}$ is defined by

$$\hat{r}(\mathcal{C}) = \inf_{n \in \mathbb{N}} \|\mathcal{C}^n\|^{1/n}.$$

If $\mathcal{C}$ is bounded, then the infimum is actually the limit (see [13]). In [6] and [8] the following notion has been introduced. *The joint spectral radius of* $\mathcal{C}$ *at a vector* $x \in X$ is defined by the formula

$$\hat{r}(\mathcal{C}, x) = \limsup_{n \to \infty} \|\mathcal{C}^n x\|^{1/n}.$$

Following [16] the collection $\mathcal{C}$ is said to be *finitely quasinilpotent* if $\hat{r}(\mathcal{F}) = 0$ for any finite subset $\mathcal{F}$ of $\mathcal{C}$. Similarly, the collection $\mathcal{C}$ is *finitely quasinilpotent at a vector* $x_0 \in X$ whenever $\hat{r}(\mathcal{F}, x_0) = 0$ for any finite subset $\mathcal{F}$ of $\mathcal{C}$.

The following remarkable result has been shown very recently by Yu. Turovskii [17].

Theorem 1.1. *Every multiplicative semigroup of quasinilpotent compact operators on X is finitely quasinilpotent.*

Yu. Turovskii has obtained this theorem together with the following result that gives the answer to the long standing open question.

Theorem 1.2. *Let $\mathcal{S}$ be a multiplicative semigroup of quasinilpotent compact operators on X. Then $\mathcal{S}$ has a non-trivial invariant closed subspace.*

Let E be a real or complex Banach lattice. Let S and T be positive operators on E. We say that S is *dominated by* T whenever $S \leq T$, that is, the operator $T - S$ is positive. In such a case we shall also say that T *dominates* S.

We now recall the famous result of Aliprantis and Burkinshaw (see e.g. [3]).

Theorem 1.3. *Let E, F, G, and H be real or complex Banach lattices. If each of positive operators $S : E \to F$, $T : F \to G$, $U : G \to H$ is dominated by a compact positive operator, then the operator UTS is also compact.*

From now on, we assume that E is a real or complex Banach lattice of dimension at least two. In [8] we have shown the following result in a slightly more general form.

Theorem 1.4. *Let $\mathcal{C}$ be a collection of positive operators on E that is finitely quasinilpotent at a non-zero positive vector. If $\mathcal{C}$ contains an operator that dominates a non-zero positive compact operator, then $\mathcal{C}$ has a non-trivial invariant closed ideal.*

The proof of the following lemma can be found in [7]. Recall that a subset $\mathcal{I}$ of a multiplicative semigroup $\mathcal{S}$ is said to be a *semigroup ideal* if ST and TS belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$.

Lemma 1.5. *Let $\mathcal{S}$ be a multiplicative semigroup of positive operators on E. If a non-zero semigroup ideal of $\mathcal{S}$ has a non-trivial invariant closed ideal, then the semigroup $\mathcal{S}$ has a non-trivial invariant closed ideal as well.*

For the rest of this section let us assume that E is Dedekind complete. Denote by E^* the norm dual of E. A *rank-one operator* on E is any operator of the form $u \otimes f$, where $u \in E$ and $f \in E^*$. A finite sum of such operators is known as a *finite rank operators*. Let $(E^* \otimes E)^{dd}$ be the band in the Banach lattice of all regular operators on E that is generated by all finite rank operators. As in [1] we call the operators in this band *abstract kernel operators*. It should be noted that in the usual definition of abstract kernel operators the norm dual E^* is replaced by the subband E_n^* in E^* of all order continuous functionals (see e.g. [18, Chap. 13]).

If E is a Banach function space with respect to σ-finite positive measure μ, then $(E^* \otimes E)^{dd}$ contains all kernel operators on E (see [18, Chap. 13]). Recall that an operator T on a Banach function space E is said to be a *kernel operator* if there exists a $\mu \times \mu$-measurable function $k_T(s,t)$ such that

(1) $(Tf)(s) = \int k_T(s,t) f(t)\, d\mu(t)$ for almost all s,

(2) $\int |k_T(\,\cdot\,,t) f(t)|\, d\mu(t) \in E$ for each $f \in E$.

2. Results

We first apply Theorems 1.1 and 1.4 to obtain the following extension of the famous result of de Pagter [12].

Theorem 2.1. *Let $\mathcal{S}$ be a multiplicative semigroup of quasinilpotent positive operators on a Banach lattice E. If $\mathcal{S}$ contains an operator that dominates a non-zero positive compact operator on E, then $\mathcal{S}$ has a non-trivial invariant closed ideal.*

Proof. Let K be a non-zero positive compact operator on E that is dominated by some $S \in \mathcal{S}$. The semigroup generated by $\mathcal{S}$ and K is a non-zero semigroup of quasinilpotent positive operators on E. So, there is no loss of generality in assuming that $K \in \mathcal{S}$. Let $\mathcal{I}$ be the semigroup ideal of $\mathcal{S}$ generated by the operator K. By Theorem 1.1, the semigroup $\mathcal{I}$ is finitely quasinilpotent. It follows that $\mathcal{I}$ has a non-trivial invariant closed ideal by Theorem 1.4. Now Lemma 1.5 implies that $\mathcal{S}$ has a non-trivial invariant closed ideal. This completes the proof of the theorem. □

Given a collection $\mathcal{C}$ of positive operators on E, let $N(\mathcal{C})$ denote the intersection of all null ideals of operators from $\mathcal{C}$, that is,

$$N(\mathcal{C}) = \bigcap_{T \in \mathcal{C}} \{x \in E : T|x| = 0\}.$$

A slight modification of Lemma 6.2 of [1] gives the following generalization of it.

Lemma 2.2. *Let $\mathcal{C}$ be a collection of positive abstract kernel operators on a Dedekind complete Banach lattice E. If $N(\mathcal{C}) = \{0\}$, then for each non-zero $x_0 \in E^+$ there exist operators S, T, and $U \in \mathcal{C}$ and a positive compact operator K on E such that $K \le UTS$ and $Kx_0 > 0$.*

Proof. Let $x_0 \in E^+$ be a non-zero vector. Since $N(\mathcal{C}) = \{0\}$, there exists $S \in \mathcal{C}$ such that $Sx_0 > 0$. Since S is an abstract kernel operator, there exists a net $\{S_\alpha\}$ such that $0 \le S_\alpha \uparrow S$ and each S_α is dominated by a positive finite rank operator. From $S_\alpha x_0 \uparrow Sx_0$ and $Sx_0 > 0$ it follows that there exists some index α_0 such that $S_\alpha x_0 > 0$ for all $\alpha \ge \alpha_0$. Since $N(\mathcal{C}) = \{0\}$, there exists $T \in \mathcal{C}$ such that $T(S_{\alpha_0} x_0) > 0$. Furthermore, there exists a net $\{T_\beta\}$ such that $0 \le T_\beta \uparrow T$ and each T_β is dominated by a positive finite rank operator. Similarly, from $T_\beta(S_{\alpha_0} x_0) \uparrow T(S_{\alpha_0} x_0)$

and $T(S_{\alpha_0}x_0) > 0$ it follows that there exists some index β_0 such that $T_\beta(S_{\alpha_0}x_0) > 0$ for all $\beta \geq \beta_0$. Repeating this once more, we obtain an operator $U \in \mathcal{C}$ such that $U(T_{\beta_0}S_{\alpha_0}x_0) > 0$, and a net $\{U_\gamma\}$ such that $0 \leq U_\gamma \uparrow U$ and each U_γ is dominated by a positive finite rank operator. Also, there exists some index γ_0 such that $U_{\gamma_0}T_{\beta_0}S_{\alpha_0}x_0 > 0$.

If we let $K := U_{\gamma_0}T_{\beta_0}S_{\alpha_0}$, then K is a non-zero positive operator satisfying $K \leq UTS$ and $Kx_0 > 0$. Since it is also compact by Theorem 1.3, the proof is finished. □

We are now able to prove the following extension of the Ando–Krieger theorem.

Theorem 2.3. *Let $\mathcal{S}$ be a multiplicative semigroup of quasinilpotent positive abstract kernel operators on a Dedekind complete Banach lattice E. Then $\mathcal{S}$ has a non-trivial invariant closed ideal.*

Proof. We may assume that $\mathcal{S}$ is non-zero. If $N(\mathcal{S}) \neq \{0\}$, then $N(\mathcal{S})$ is a non-trivial closed ideal that is also invariant under any member of $\mathcal{S}$. So, assume that $N(\mathcal{S}) = \{0\}$. Then Lemma 2.2 implies that there exist operators S, T, and $U \in \mathcal{S}$ and a non-zero positive compact operator K on E such that $K \leq UTS \in \mathcal{S}$. Now, an application of Theorem 2.1 completes the proof. □

In the case of Banach function spaces we obtain the following

Corollary 2.4. *Let $\mathcal{S}$ be a multiplicative semigroup of quasinilpotent positive kernel operators on a Banach function space of dimension greater than 1. Then $\mathcal{S}$ has a non-trivial invariant closed ideal.*

Observe that the last corollary is an extension of Theorem 3.4 of [5], where kernel operators on a L^2-space with some additional assumptions were considered.

We conclude this note by giving another application of Theorem 2.1. To prove an extension of a very recent result of Jahandideh [9, Theorem 3.4], we recall some relevant definitions and remarks from [9].

Let X be a locally compact Hausdorff–Lindelöf space. We consider the Banach lattice $C_0(X)$ of all real continuous functions on X that vanish at infinity. Let μ be a positive σ-finite Borel measure on X. By the Lindelöf property we may assume that every non-empty open subset of X has positive measure; see [5, Section 3] for the proof of this. An operator T

on $C_0(X)$ is said to be *integral operator* on $C_0(X)$ *by way of* μ if the exists $\mu \times \mu$-measurable real function k_T on $X \times X$ such that

$$(Tf)(x) = \int_X k_T(x,y) f(y)\, d\mu(y)$$

for each $f \in C_0(X)$. The following result is the promised extension of [9, Theorem 3.4].

Theorem 2.5. *Let $\mathcal{S}$ be a multiplicative semigroup of quasinilpotent positive operators on $C_0(X)$. Assume that $\mathcal{S}$ contains an integral operator S by the way of μ such that its kernel $k_S : X \times X \to [0,\infty)$ is lower semicontinuous at some point $(x_0, y_0) \in X \times X$ with $k_S(x_0, y_0) > 0$. Then $\mathcal{S}$ has a non-trivial invariant closed ideal.*

Proof. Denote $k_S(x_0, y_0) = 2c$. Lower semicontinuity implies that there exist open sets U and V of X such that $k_S(x,y) \geq c$ for all $x \in U$ and $y \in V$. By Urysohn's lemma, there exists a continuous function $u : X \to [0,1]$ such that $u(x_0) = 1$ and the support of u lies in U. Similarly, there is a continuous function $v : X \to [0,1]$ such that $v(y_0) = 1$ and the support of v lies in V. Define the function $k_A : X \times X \to [0,c]$ by $k_A(x,y) = c\, u(x)\, v(y)$. The corresponding integral operator A on $C_0(X)$ by the way of μ (with the kernel k_A) is a non-zero positive compact operator that is dominated by S. An application of Theorem 2.1 now completes the proof. □

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ROMAN DRNOVŠEK
FACULTY OF MATHEMATICS AND PHYSICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19
SI–1000 LJUBLJANA
SLOVENIA

E-mail: roman.drnovsek@fmf.uni-lj.si

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