Publ. Math. Debrecen
58 / 3 (2001), 537–547

On p-subnormal subgroups of finite p-soluble groups

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Abstract. In this paper we present some results for p-subnormal subgroups (concept introduced by Kegel in [6]). These statements are analogous to some well-known results for subnormal subgroups, mainly due to Wielandt, and they hold in all p-soluble groups.

Nevertheless, since the set of all p-subnormal subgroups of a group fails to be a lattice new arguments have to be used.

1. Introduction

Throughout this paper, we will denote with p a fixed prime number. All groups considered will be finite.

O. Kegel, in a celebrated paper in 1962, [6], introduced the concept of π-subnormal subgroups, with π a set of prime numbers, as the subgroups whose intersections with all Sylow p-subgroups of the group are Sylow p-subgroups of the subgroup, for all $p \in \pi$. In this paper he stated a famous conjecture that was proved almost 30 years later: a subgroup H is subnormal in a group G if and only if every Sylow subgroup of G meets H in a Sylow subgroup of H, i.e. a subgroup H is subnormal in G if and only if H is p-subnormal in G for all primes p. H. Wielandt, in [9], when the classification of finite simple groups was almost achieved, included this conjecture as one of the important research areas after the classification (cf. [9]). He called it the Kegel's Problem and, since then, this question

Mathematics Subject Classification: Primary 20D20, 20D35; Secondary 20D10, 20E28.
Key words and phrases: p-subnormal subgroup, p-quasinormal subgroup, p-subnormalizer of a subgroup.
The author is supported by Proyecto PB 94-1048 of DGICYT, Ministerio de Educación y Ciencia of Spain.

was called the Kegel–Wielandt conjecture. Finally in [7], P. B. Kleidman published a proof of this conjecture (cf. [7]) which depends upon the classification of finite simple groups. This paper focused the attention on the systematic study of p-subnormality as an embedding property. In this line, the first contribution, after Kegel's paper, came again from Kleidman himself together with R. Guralnik and R. Lyons in [4] (cf. [4]).

All subnormal subgroups of a group form a lattice with the usual join and meet of subgroups. Lattice properties are crucial in the study of subnormal subgroups (cf. [8]) and most of the theorems about subnormality make a strong use of lattice properties in their proofs. However, it is easy to find examples showing that the set of all p-subnormal subgroups of a group is not a lattice. In [2], the class of groups in which all p-subnormal subgroups form a lattice is characterized as the class of p-soluble groups whose p-lenght and p'-lenght are both less or equal to 1. In this class it is not difficult to prove for p-subnormality analogous properties to those of subnormality. The aim of this paper is to present some results for p-subnormal subgroups which are analogous to some others for subnormal subgroups (mainly due to Wielandt) and whose validity holds in all p-soluble groups. This is not an empty exercise in generalization since, by the above remarks, we cannot use lattice properties. So we have to build new proofs.

Definition. A subgroup H of a group G is said to be a *p-subnormal subgroup* of G if $P \cap H$ is a Sylow p-subgroup of H, for any $P \in \mathrm{Syl}_p(G)$.

We collect some basic results from p-subnormality.

Proposition 1. *Let G be a group and N a normal subgroup of G.*

i) *If H is a p-subnormal subgroup of G, then HN/N is a p-subnormal subgroup of G/N.*

ii) *If HN/N is a p-subnormal subgroup of G/N, then HN is a p-subnormal subgroup of G. Moreover if N is a p-group, then H is p-subnormal in G.*

iii) *If H is a p-subnormal subgroup of G and K is a p-subnormal subgroup of H, then K is p-subnormal in G.*

iv) *If H and K are p-subnormal subgroups of G and $HK = KH$, then HK and $H \cap K$ are p-subnormal in G.*

v) *If H is p-subnormal in G, then $O_p(H) \leq O_p(G)$.*

Proof. These are straightforward consequences of the definition. □

An important property of p-subnormal subgroups is the following (cf. Lemma 2.6 of [4], and 2.4, 2.5 of [7]).

Lemma. *Let h be a p-element of a group G and write*

$$\Theta_G(h) = \frac{|\operatorname{Syl}_p(G)|}{|\{P \in \operatorname{Syl}_p(G)\colon h \in P\}|}.$$

Then, if H is a p-subnormal subgroup of G, we have that $\Theta_G(h) = \Theta_H(h)$ for any p-element h of H.

Subnormality is the embedding property which arise when we extend the pro-perty of being normal by transitivity. In our first result we obtain that in the class of p-soluble groups, p-subnormality is the transitive extension of p-quasinormality.

Definition. A subgroup H of a group G is said to be *p-quasinormal* in G if for every Sylow p-subgroup P of G, the product HP is a subgroup of G.

An easy order consideration proves that p-quasinormal subgroups are p-subnormal.

Proposition 2. *Let G be a p-soluble group and H a maximal p-subnormal subgroup of G (i.e. H is p-subnormal in G and if K is a p-subnormal subgroup of G with $H \leq K \leq G$, then either $H = K$ or $K = G$). Then one of the following holds:*

i) *if p does not divide the index $|G : H|$, then $O^{p'}(G) \leq H$,*
ii) *if p divides $|G : H|$, then $|G : H| = p^\alpha$.*

In particular, in both cases, H is a p-quasinormal subgroup of G and H is indeed a maximal subgroup of G.

Proof. Part (i) is clear. To prove (ii) we can assume without loss of generality that H contains no nontrivial normal subgroup of G. Therefore, maximality of H as p-subnormal subgroup of G forces $G = HN$ for every minimal normal subgroup N of G. In this case p divides the order of N. Since G is p-soluble, N is a p-group and the conclusion follows. □

In the alternating group of degree 5, $G = \operatorname{Alt}(5)$, the normalizer H in G of each Sylow 5-subgroup of G is a maximal subgroup of G which is 2-subnormal. Nevertheless the index $|G : H| = 6$. Therefore the above proposition fails in non p-soluble groups.

Corollary. *Let G be a p-soluble group. A subgroup H of G is a p-subnormal subgroup of G if and only if there exists a chain of subgroups*

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that H_i is a p-quasinormal subgroup of H_{i+1} for every $i = 0, \ldots, n-1$.

2. Theorems like Wielandt's

Our first result is a version of the famous "Wielandt's first maximizer lemma" (see [8] Lemma 7.3.1) for p-subnormal subgroups in p-soluble groups.

Theorem 1. *Let G be a p-soluble group and let A be a subgroup of G. Suppose that A is not p-subnormal in G, but A is p-subnormal in H for all proper subgroups H of G containing A. Then*

i) *A is contained in a unique maximal subgroup M of G, and*

ii) *if $g \in G$, then $A^g \leq M$ if and only if $g \in M$.*

Proof. (i) Suppose the claim is not true and let G be a minimal counterexample. We choose a subgroup A of G of minimal order such that A is not p-subnormal in G, A is a p-subnormal subgroup of every proper subgroup of G containing A and A is contained in at least two different maximal subgroups, say M and L, of G.

We take a normal subgroup N of G and consider $C = AN$. Assume that C is a proper subgroup of G. Then A is p-subnormal in C. Moreover, C is not p-subnormal in G. This implies that C/N is not p-subnormal in G/N. On the other hand, it is clear that if K is a proper subgroup of G such that $C \leq K$, then A is p-subnormal in K. Therefore $C/N = AN/N$ is p-subnormal in K/N. By minimality of G, the subgroup C/N is contained in a unique maximal subgroup of G/N. From here we deduce that either $N \nleq L$ or $N \nleq M$. Suppose that $N \nleq M$ then $G = MN$. From the p-subnormality of A in M, we deduce the p-subnormality of AN/N in $G/N = MN/N$, and then C is p-subnormal in G, a contradiction. Therefore $G = AN$ for any normal subgroup of G. In particular, $\mathrm{core}_G(A) = 1$. Moreover, $\mathrm{core}_G(M) = \mathrm{core}_G(L) = 1$ and then G is a primitive group. Since A is not a maximal subgroup of G, the Socle of G is non-abelian.

Suppose that N is a minimal normal subgroup of G. Now N is a p'-group and the index $|M : A|$ is a p'-number. Since A is p-subnormal in M, the subgroup $O^{p'}(A) = O^{p'}(M)$ is normal in M. Analogously we deduce that $O^{p'}(A)$ is a normal subgroup of L. Therefore $O^{p'}(A)$ is normal in $\langle L, M\rangle = G$. This implies that A is p-subnormal in G, a contradiction.

(ii) This is an immediate consequence of (i). If $A^g \leq M$, then $A \leq M^{g^{-1}}$ and by (i), we deduce that $g \in N_G(M)$. If M were a normal subgroup of G, then A would be p-subnormal in G, a contradiction. Hence $M = N_G(M)$ and $g \in M$. □

Example. The p-solubility restriction in the above result is necessary. Let $G = \mathrm{Alt}(6)$ be the alternating group of degree 6 and consider in G the following subgroups:

$$A = \langle t_1 = (123),\ t_2 = (12)(34),\ t_3 = (12)(45)\rangle,$$

and

$$B = \langle s_1 = (124)(653),\ s_2 = (16)(25),\ s_3 = (14)(53)\rangle.$$

It is clear that A is the stabilizer of the point 6 and then $A \cong \mathrm{Alt}(5)$. We see also that $t_1^3 = 1 = s_1^3$, $t_2^2 = s_2^2 = 2 = t_3^2 = s_3^2$, $(t_1t_2)^3 = (t_2t_3)^3 = 1 = (s_1s_2)^3 = (s_2s_3)^3$, $(t_3t_1)^2 = (t_1t_3)^2 = 1 = (s_3s_1)^2 = (s_1s_3)^2$. This implies that the mapping θ defined by $t_i^\theta = s_i$, for $i = 1, 2, 3$, can be extended to an isomorphism from A to B and then $B \cong \mathrm{Alt}(5)$.

Now $a = s_2^{s_1} = (25)(43)$ and $b = s_3s = (14523)$. Then $b^a = b^{-1}$. Thisimplies that the subgroup $H = \langle b, a\rangle \cong \mathrm{Dih}(5)$ and $H \leq A \cap B$. In fact $P = \langle b\rangle$ is a Sylow 5-subgroup of A and B and $H = N_A(P) = N_B(P)$. At least, there are two different maximal subgroups of G containing H.

In fact if K is a proper subgroup of G such that $H \leq K$, then either $H = K$ of $K \cong \mathrm{Alt}(5)$ and then $H = N_K(P)$. Therefore H is 2-subnormal in every proper subgroup of G containing H.

On the other hand, for any involution $x \in H$ we have $\Theta_H(x) = 5$. G possesses 45 involutions all conjugate. Each Sylow 2-subgroup of G is isomorphic to $\mathrm{Dih}(4)$ and contains 5 involutions. Then $\Theta_G(x) = \frac{45}{5} = 9$. Then H is not 2-subnormal in G

After the above theorem it is not difficult to deduce a criterion for p-subnor-mality in p-soluble groups analogous to Wielandt's (see [8] Theorem 7.3.3).

Theorem 2. *Let G be a p-soluble group and a subgroup H of G. The following statements are pairwise equivalent:*

i) *H is a p-subnormal subgroup of G;*

ii) *for every $g \in G$, H is a p-subnormal subgroup of $\langle H, g\rangle$;*

iii) *for every $g \in G$, H is a p-subnormal subgroup of $\langle H, H^g\rangle$.*

Proof. The only non-obvious implication is (iii) $\Longrightarrow$ (i). To prove it, we assume that there exists a counterexample G of minimal order: there exists a proper subgroup H of G which is not p-subnormal in G despite H is a p-subnormal subgroup of $\langle H, H^g\rangle$ for all $g \in G$.

By minimality of G, we deduce that the subgroup H is p-subnormal in every proper subgroup of G containing H and no proper p-subnormal subgroup of G contains H. We apply the Theorem 1a and we deduce that there exists a unique maximal subgroup M of G such that $H \leq M$. Take an element $g \in G \setminus M$ and consider the subgroup $K = \langle H, H^g \rangle$. Clearly K is a proper subgroup of G and then K is contained in some maximal subgroup of G. This maximal subgroup contains also H. The uniqueness of M forces $H^g \leq M$. But by Theorem 1b, we have $g \in M$. This is the final contradiction. The minimal counterexample does not exist and the claim is true. □

Notice that in the last theorem, taking all the primes that divide the order of the group G, we obtain a proof of the criterion of subnormality, due to Wielandt, for subgroups of a soluble group.

Theorem 3. *Let G be a p-soluble group and $H, K \leq G$ such that $G = HK$. Let A be a p-subnormal subgroup of H and of K. Then A is a p-subnormal subgroup of G.*

Proof. Let G be a counterexample of minimal order to the theorem. Consider a triple of subgroups H, K, A such that $G = HK$, A is p-subnormal in both H and K but A is not p-subnormal in G and $|H| + |K|$ is of maximal order with these requirements.

If M is a maximal subgroup of G such that H is a proper subgroup of M, then $M = H(K \cap M)$, by Dedekind's law. Since A is p-subnormal subgroup in H and in $K \cap M$, the subgroup A is p-subnormal in M by minimality of G. Now $G = MK$ and $|M| + |K| > |H| + |K|$. Maximality of $|H| + |K|$ forces A to be p-subnormal in G, a contradiction. Therefore H is a maximal subgroup of G.

Suppose that N is a proper normal subgroup of G. The triple HN/N, KN/N, AN/N satisfy the hypotheses of the theorem in the group G/N. By minimality of G the subgroup AN/N is p-subnormal in G/N. Then AN is a p-subnormal subgroup of G.

Suppose now that N is a minimal normal subgroup of G and $N \leq H$. Since $AN \leq H$, we have that A is a p-subnormal in AN and then in G, a contradiction. Therefore $\mathrm{core}_G(H) = 1$.

Analogously it can be deduced that K is a core-free maximal subgroup of G.

So, G is a soluble primitive group and both H and K complement the unique minimal normal subgroup N of G. Hence H and K are conjugate in G. But in this case $G \neq HK$ by Ore's Theorem (cf. [3] Th. A.16.2). This is the final contradiction. □

In the above example, notice that $G = AB$ and $H = A \cap B$. Therefore H is 2-subnormal in A and B but H is not 2-subnormal in $AB = G$, i.e. the solubility hypothesis is necessary in the Theorem 3.

3. On p-subnormalizers

The classical Wielandt's criteria of subnormality allow the introduction of the subnormalizer of a subgroup in a group (cf. [8], 7.7). Here, the criteria of p-subnormality in Theorem 2 allow us to introduce a similar concept.

Definition. Given a group G and $H \leq G$ we consider the subset

$$S_G^p(H) = \{g \in G \colon H \text{ is } p\text{-subnormal in } \langle H, g\rangle\}.$$

This subset will be called the *p-subnormalizer of H in G*.

Remarks. Let G be a group and $H \leq G$.

1) If G is p-soluble, it is clear, by Theorem 2, that a subgroup H of G is p-subnormal in G if and only if $S_G^p(H) = G$.

2) If $H \leq K \leq G$, then $S_K^p(H) = S_G^p(H) \cap K$.

3) $S_G^p(H) = \bigcup\{K \leq G \colon H \text{ is } p\text{-subnormal in } K\}$. In particular, if $S_G^p(H)$ is a subgroup, then $S_G^p(H)$ is the largest subgroup of G in which H is p-subnormal.

4) If T is a p-subnormal subgroup of H, then $S_G^p(H) \subseteq S_G^p(T)$.

5) If H is a p-subgroup of G, then $S_G^p(H) = S_G(H)$, the subnormalizer of H in G.

6) If N is a normal subgroup of G, then $S_G^p(H)N/N \subseteq S_{G/N}^p(HN/N)$. If moreover $N \leq H$, then $S_G^p(H)/N = S_{G/N}^p(H/N)$.

7) $S_G^p(H) \subseteq S_G^p(O^{p'}(H))$. Moreover, if $S_G^p(O^{p'}(H))$ is a p-soluble subgroup, then $S_G^p(H) = S_G^p(O^{p'}(H))$.

Proof. If $S^p_G(O^{p'}(H))$ is a p-soluble subgroup of G, then $O^{p'}(H)$ is p-subnormal in $S^p_G(O^{p'}(H))$ and $H \leq S^p_G(O^{p'}(H))$. Hence H is p-subnormal in $S^p_G(O^{p'}(H))$. □

8) If G is a p-soluble group, then $S^p_G(H)$ is a subgroup of G if and only if for every pair of subgoups A, B of G such that H is p-subnormal in both of them, we have that H is p-subnormal in $\langle A, B\rangle$.

Proof. If $S^p_G(H)$ is a subgroup of G, it is easy to see, by Remark 3, that the claim is true. To prove the sufficiency, we suppose that $x_1, x_2 \in S^p_G(H)$. Then, for $i = 1, 2$, H is p-subnormal in $\langle H, x_i\rangle = A_i$. Hence H is p-subnormal in $\langle A_1, A_2\rangle$. Since $x_1x_2 \in \langle A_1, A_2\rangle$ we conclude that $x_1x_2 \in S^p_G(H)$; i.e. $S^p_G(H)$ is a subgroup of G. □

9) If H is a subgroup of G and p does not divides to $|G : H|$, then $S^p_G(H) = N_G(O^{p'}(H))$.

Proof. Obviously $S^p_G(H) \subseteq N_G(O^{p'}(H))$. Now, we suppose that H is p-subnormal in $\langle H, g\rangle = K$. Since p does not divides to $|G : H|$ we can deduce that $O^{p'}(H) = O^{p'}(K)$. Then K is a subgroup of $N_G(O^{p'}(H))$ and therefore $N_G(O^{p'}(H) \subseteq S^p_G(H)$. □

Now we turn our attention to the study of the class of p-soluble groups in which the p-subnormalizers of all their subgroups are subgroups too. We denote this class of groups by $\mathfrak{X}$. Casolo obtained important results about the class of finite groups, which he call sn-groups, in which the subnormalizers of all subgroups are subgroups too (cf. [1]). This results will be helpful in study of the class $\mathfrak{X}$.

Casolo (see [1] Lemma 1.2) shows that if H is a subnormal subgroup of G and $Q \in \mathrm{Syl}_p(H)$, then $G = H\langle S_G(Q)\rangle$. In the case of p-subnormal subgroups of a p-soluble group we obtain the following result.

Proposition 5. *If H is a p-subnormal subgroup of a p-soluble group G and $Q \in \mathrm{Syl}_p(H)$, then $G = H\langle S^p_G(Q)\rangle$.*

Proof. First we suppose that H is a maximal p-subnormal subgroup of G and we apply Proposition 2. If p does not divide the index $|G : H|$, then $\mathrm{Syl}_p(H) = \mathrm{Syl}_p(G)$. An easy Frattini argument proves that $G = HN_G(Q)$. If $p^\alpha = |G : H|$, then $G = HP$ for any Sylow p-subgroup P of G. Moreover, $Q = P \cap H \in \mathrm{Syl}_p(H)$ and Q is a p-subnormal subgroup of P, i.e. $P \subseteq \langle S^p_G(Q)\rangle$. Hence the result follows clearly in both cases.

In the general case, let K be a maximal p-subnormal subgroup of G such that $H \leq K$. We can assume that $H < K < G$ and by induction $K = H\langle S_K^p(Q)\rangle$ with $Q \in \mathrm{Syl}_p(H)$. Now, we take a Sylow p-subgroup P of K such that $Q \leq P$. Since Q is p-subnormal in P, we have $S_G^p(P) \subseteq \langle S_G^p(Q)\rangle$. Hence, $G = K\langle S_G^p(P)\rangle = H\langle S_K^p(Q)\rangle\langle S_G^p(Q)\rangle = H\langle S_G^p(Q)\rangle$. □

Proposition 5 suggests that if G is a p-soluble group in $\mathfrak{X}$ and H is a subgroup of G, then $S_G^p(H) = HS_G^p(Q)$, where $Q \in \mathrm{Syl}_p(H)$. Nevertheless this is not true in general. Some others conditions are needed as we see next.

Proposition 6. *If G is a p-soluble group and $H \leq G$ is a subgroup of G then:*

i) $S_G^p(H) \subseteq H\langle S_G(Q)\rangle$, *for any* $Q \in \mathrm{Syl}_p(H)$;

ii) *if G is a sn-group and $HS_G(Q)$ is a subgroup of G, for some $Q \in \mathrm{Syl}_p(H)$, then $S_G^p(H) = HS_G(Q)$*

Proof. (i) Let K be a subgroup of G such that H is p-subnormal in K. By Proposition 5, for any $Q \in \mathrm{Syl}_p(H)$, we have $K = H\langle S_K(Q)\rangle$. Notice that $\langle S_K(Q)\rangle \leq \langle S_G(Q)\rangle \cap K$. This implies that $K = H\langle S_K(Q)\rangle \leq H(\langle S_G(Q)\rangle \cap K) = H\langle S_G(Q)\rangle \cap K$. Hence $K \leq H\langle S_G(Q)\rangle$. Finally, we obtain $S_G^p(H) \subseteq H\langle S_G(Q)\rangle$, by Remark 3.

(ii) We denote by K the subgroup $HS_G(Q)$. We prove that $K \subseteq S_G^p(H)$. By an easy order consideration we have, $|K|_p = |S_G(Q)|_p$, i.e. $\mathrm{Syl}_p(S_G(Q)) \subseteq \mathrm{Syl}_p(K)$. Choose $P \in \mathrm{Syl}_p(K)$. It is clear that $P^k \in S_G(Q)$, for some $k \in K$. Moreover if $k = hx$, with $h \in H$ and $x \in S_G(Q)$, we deduce that $P^h \leq S_G(Q)$. Since G is a sn-group, by [1] Proposition 1.4, we have that $S_G(Q) = N_G(R)$, where R is the intersection of all Sylow p-subgroups of G containing Q. In particular, $Q \leq R \leq O_p(S_G(Q)) \leq P^h$ and then $Q^{h^{-1}} = P \cap H$. Then $P \cap H \in \mathrm{Syl}_p(H)$, i.e. H is p-subnormal in K and $K \subseteq S_G^p(H)$. Therefore, we conclude that $K = S_G^p(H)$. □

In the next example we consider a soluble group G in $\mathfrak{X}$, $p = 3$, and we see that there exists a subgroup H of G, such that $S_G^3(H) \neq \langle HS_G(Q)\rangle$.

Example. Let C be the cyclic group of order 6. Denote $C = \langle a, b\rangle$ with $a^3 = 1$ and $b^2 = 1$. The group C acts on a two dimensional $GF(7)$-vector space V in such a way that there exists a basis $\{w_1, w_2\}$ of G in which the action of a has matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ and the action of b is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the semidirect product $G = [V]C$.

Since $|G| = 7^2.3.2$, each proper subgroup of G is either a $3'$-group (and therefore is 3-subnormal in G, i.e. $S^3_G(H) = G$) or its index is prime to 3. Hence, by Remark 9, we deduce that the 3-subnormalizer of each subgroup is a subgroup of G.

Consider in G the subgroup $H = [\langle w_1\rangle]\langle a\rangle$. The subgroup H is isomorphic to the Frobenius group of order 21. For $Q = \langle a\rangle$, a Sylow 3-subgroup of H, we have $S_G(Q) = N_G(Q) = C$. Notice that the subset $HC = HS_G(Q)$ is not a subgroup of G. On the other hand, if $K < G$ and H is 3-submormal in K then, using the above argument, $|G : K| = 7$. Therefore $K \leq N_G(H) \leq N_G(\langle w_1\rangle) = V\langle a\rangle$, a contradiction. Hence $S^3_G(H) = H$.

Consequently $S^3_G(H)$ is a subgroup of G, G is a soluble group in $\mathfrak{X}$ and $HS_G(Q)$ is not a subgroup of G; moreover $H = S^3_G(H) \neq \langle HS_G(Q)\rangle = G$

The class $\mathfrak{X}$, as the class of the sn-groups, verifies the following properties:

Proposition 7. *The class $\mathfrak{X}$ verifies the following properties:*

i) *if $G \in \mathfrak{X}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{X}$; in other words, $\mathfrak{X}$ is an homomorph;*

ii) *if $G \in \mathfrak{X}$ and H is a subgroup of G, then $H \in \mathfrak{X}$; this is to say that $\mathfrak{X}$ is a S-closed homomorph;*

iii) *if N is a normal p-subgroup of a group G such that $G/N \in \mathfrak{X}$, then $G \in \mathfrak{X}$; in the notacion of* [3], *this is $\mathfrak{S}_p\mathfrak{X} = \mathfrak{X}$.*

Proof. (i) Let G be a group in $\mathfrak{X}$ and N a normal subgroup of G. For any subgroup H/N of G/N we have, by Remark 6, that $S^p_{G/N}(H/N) = S^p_G(H)/N$. Therefore $S^p_{G/N}(H/N)$ is a subgroup of G/N. Hence $G/N \in \mathfrak{X}$.

(ii) Let G be a group in $\mathfrak{X}$ and H a subgroup of G. If A is a subgroup of H, then $S^p_H(A) = S^p_G(A) \cap H$. In particular $S^p_H(A)$ is a subgroup of H. Hence $H \in \mathfrak{X}$.

(iii) Let G be a group and N a normal p-subgroup of G such that $G/N \in \mathfrak{X}$. If H is a subgroup of G, then H is p-subnormal in HN. Suppose that H is a p-subnormal subgroup of two subgroups A and B of G. Since the p-subnormalizers in G/N are subgroups, we have that HN/N is a p-subnormal subgroup of $\langle A, B\rangle N/N$. Then HN is p-subnormal in $\langle A, B\rangle N$. Hence H is p-subnormal in $\langle A, B\rangle$; this implies that $S^p_G(H)$ is a subgroup of G. Therefore $G \in \mathfrak{X}$. □

Acknowledgement. The author would like to thank Professor L. M. Ezquerro for his very helpful comments and suggestions while this article was being written.

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(Received October 4, 1999; file received January 27, 2000)