Publ. Math. Debrecen
58 / 4 (2001), 549–557

Base N Just Touching Covering Systems

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Abstract. Let $A = \{a_1, a_2, \dots, a_N\}$ be a set of integers, where $a_i \equiv i \pmod{N}$ and $N > 1$. Let $B = A - A$. The set of integers expressible in the form $\sum_{i=0}^{k} b_i N^i$, for some nonnegative k with b_i in B, is denoted Z_B. We show $(a_1 - a_N, a_2 - a_N, \dots, a_N - a_N) = d$ implies that $Z_B = d\mathbb{Z}$.

1. Introduction

The theorem proven in this note is suggested by a conjecture of Kátai [1]. The case for $N = 3$ was proven by the author in [4].

Let $A = \{a_1, a_2, \dots, a_N\}$ be a set of integers, where $a_i \equiv i \pmod{N}$ and $N > 1$. Let $B = A - A$. The set of integers expressible in the form $\sum_{i=0}^{k} b_i N^i$, for some nonnegative k with b_i in B, is denoted Z_B. Integers in Z_B are said to be expressible in B. Let $\mathbb{Z}$ represent the integers and $(c_1, c_2, \dots, c_N)$ denote the greatest common divisor of the c_i. We prove the following theorems:

Theorem 1. *$Z_B = d\mathbb{Z}$ where $d = (a_1 - a_N, a_2 - a_N, \dots, a_N - a_N)$.*

Theorem 2. *Assume $a_N = 0$ and $(a_1, a_2, \dots, a_N) = 1$. Then $Z_B = \mathbb{Z}$.*

Theorem 1 is a consequence of Theorem 2:

Proof of Theorem 1. Since $(a_1 - a_N, a_2 - a_N, \dots, a_N - a_N) = d$, d divides every $a_i - a_j$, and hence every element of Z_B, so $Z_B \subset d\mathbb{Z}$. On the other hand, $A' = \{(a_1 - a_N)/d,\ (a_2 - a_N)/d, \dots, (a_N - a_N)/d\}$ is a set of representatives of the N congruence classes and meets the

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Mathematics Subject Classification: 11A99.
Key words and phrases: number systems, just touching covering systems.

hypotheses of Theorem 2. Therefore each integer m can be written in the form $\sum_{i=0}^{k} c_i N^i$, for some nonnegative k, where c_i is of the form $(a - a_N)/d - (a' - a_N)/d = (a - a')/d$ for some a and a' in A. Then any md is expressible in B, so $Z_B \supset d\mathbb{Z}$. □

The remainder of the paper will prove Theorem 2, and we will assume throughout that $a_N = 0$, though this is used only in the final step. The proof of Theorem 2 is illustrated with the example $N = 4$, with $A = \{13, -6, 15, 0\}$.

When $Z_B = \mathbb{Z}$, the set A is called a Just Touching Covering System (JTCS). See the articles by K. H. Indlekofer, I. Kátai, and P. Racskó, [1]–[3] for the original definition of a JTCS and the demonstration of the equivalence to the current formulation.

2. Proof of Theorem 2

Consider the map F_A on the integers defined by $F_A(n) = (n - a_i)/N$, where a_i is in A and $n \equiv a_i \pmod N$. Let ${F_A}^k$ be the kth iterate of F_A. We have:

Lemma 1. *The integer m is expressible in B if there is some integer j and some k for which ${F_A}^k(j) = {F_A}^k(j+m)$.*

Proof. If $r = {F_A}^k(j)$, then $j = N^k r + \sum_{i=0}^{k-1} c_i N^i$ with c_i in A. Similarly, since $r = {F_A}^k(j+m)$, $j + m = N^k r + \sum_{i=0}^{k-1} c'_i N^i$ with c'_i in A. Then $m = \sum_{i=0}^{k-1} (c'_i - c_i) N^i$ is in Z_B. □

Consider a listing of the integers in a row referred to as Line 0. Directly below each integer write its image under F_A. This will be called Line 1. Directly below that, list the images of Line 1 elements under F_A, in Line 2. Continue, to define Line k for any nonnegative integer k. In the illustrative example, we show part of the first three lines:

Line 0:	−2	−1	**0**	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Line 1:	1	−4	**0**	**−3**	**2**	**−3**	1	−2	3	−2	2	−1	4	−1	3	0	5	0	4	1	6	1
Line 2:	−3	−1	**0**	**−4**	**2**	**−4**	**−3**	**1**	**−3**	**1**	**2**	**−4**	**1**	**−4**	**−3**	**0**	**−2**	**0**	1	−3	3	−3

A key observation about the Lines is recorded as:

Lemma 2. *Let S denote any substring of Line k, and t be any integer. Then the substring S' of Line k located $N^k t$ places away from S consists of the same pattern as S, but with t added to each number.*

PROOF. Since $F_A(m + Nt) = F_A(m) + t$, for any m and t, it follows that ${F_A}^k(m + N^k t) = {F_A}^k(m) + t$. □

Therefore the numbers in Line k are completely determined by those in positions 0 through $N^k - 1$. (These are the numbers shown in bold in the example.) The N^k numbers in that basic pattern are repeated as the line is extended, shifted up by 1 each time the pattern is repeated N^k places to the right, and shifted by -1 as it is repeated N^k places to the left. Of course, as Lemma 2 points out, this is true for *any* substring of the line, however long.

According to Lemma 1, m is in Z_B if some Line k contains the same number exactly m places apart. In Line 2 above, we see for example that 8 is expressible since there are two occurrences of -4 eight places apart, showing that ${F_A}^2(3) = {F_A}^2(11)$. Theorem 2 will obviously follow from:

Lemma 3. *Assume the hypotheses of Theorem 2. Given any positive integer m, there is some Line k in which there is a substring of length m consisting of the same number repeated m times.*

In fact, the string of length m consisting of a single repeated number shows that any integer less than m in absolute value is in Z_B, by Lemma 1. Since m is arbitrary, this establishes the theorem. To prove Lemma 3 we need to examine the action of F_A more closely, which we do in the next section.

3. Directed graphs

Consider the directed graph where every integer is connected by an arrow to its image under F_A. It is easy to show (see [1]) that $F_A(m)$ is less than m in absolute value, if m lies outside the interval $I_A = [-|a|/(N-1), |a|/(N-1)]$, where a is the member of A largest in absolute value. If m is in I_A, then so is $F_A(m)$. Therefore, the important part

Figure 1: Directed Graph for $A = \{13, -6, 15, 0\}$.

of the directed graph displays the action of F_A on I_A. In the illustrative example, $I_A = [-5, 5]$, and the graph is displayed in Figure 1.

A number m is said to be periodic for A if ${F_A}^k(m) = m$ for some positive k. Of course, all periodic numbers are contained in I_A. In the example there are four loops of periodic numbers, consisting of $\{0\}$, $\{2\}$, $\{-5\}$, and $\{-1, -4\}$.

In general, define r to be the least common multiple of the lengths of the loops for the graph of a given F_A. In the example $r = 2$. Notice that ${F_A}^r(m) = m$ for any periodic number m. Define the map L_A on $\mathbb{Z}$ as follows: $L_A(m)$ is the periodic number that is the image of m under repeated applications of ${F_A}^r$. Of course, for a periodic number m, $L_A(m) = m$. Define the *Table of* L_A to be a listing of all the integers, in columns headed by their images under L_A. In the example, the column headers and some of the integers in the body of the table are displayed in Table 1:

0	2	-1	-4	-5
$0, 13, -6, 15$	$2, 8, 21, 23$	$-1, -3, 4, -2, 6$	$-4, 1, 3, 5, 7, -8$	$-5, -7$

Table 1.

The significance of the table is this: Consider any finite length substring in some Line k. Then there is a substring in some lower Line k', with $k' = k + ir$ for some i, where each of the numbers in the original substring has been replaced by the number at the head of its column in the Table of L_A. We will examine the implications of this, in the example.

4. Proof of Lemma 3 in the example

Consider the substring of Line 2 of length 22 in the example above. (The same technique will work for any finite substring, though the Line numbers may differ.) Table 1 implies that a later line (Line 6 actually, using the directed graph) will contain the revised substring, where each number is replaced by its column header in Table 1:

$$\begin{array}{llllllllll} \text{Line 2:} & -3 & -1 & 0 & -4 & 2 & -4 & -3 & 1 & \ldots \\ \text{Line 6:} & -1 & -1 & 0 & -4 & 2 & -4 & -1 & -4 & \ldots \end{array}$$

This latter string will involve only the column headers $\{0, 2, -1, -4, -5\}$. By Lemma 2, a substring of any line will be repeated further along in that same line, but with each integer shifted by t, where t is arbitrary. In this example, $3(4^6)$ places to the right in Line 6, we have the same sequence shifted up by 3:

$$\begin{array}{ll} \text{Line 6:} & -1\ -1\ 0\ -4\ 2\ -4\ -1\ -4\ \ldots\ 2\ 2\ \ 3\ -1\ \ 5\ -1\ 2\ -1\ \ldots \\ \text{Line 10:} & \qquad\qquad\qquad\qquad\qquad\quad\ \ldots\ 2\ 2\ -4\ -1\ -4\ -1\ 2\ -1\ \ldots \end{array}$$

That is, there is a string of the same length, involving the numbers $\{3, 5, 2, -1, -2\}$. Then referring back to Table 1, we see that some lower line (Line 10 for example, by the directed graph) has the same sequence replaced by *its* column headers. But the set of such headers is only $\{-4, 2, -1\}$: 3 and 5 are both in the column headed by -4, while -2 and -1 are both in the column headed by -1. This is the key observation: a string of any finite length involving any integers implies the existence of a string of the same length involving members of $\{0, 2, -1, -4, -5\}$. This in turn implies the existence of a string of the same length involving only members of $\{-4, 2, -1\}$. The table above can be consolidated into a new table as follows:

-4	2	-1
$0, 13, -6, 15, 2, 8, 21, 23$	$4, -3, -1, -2, 6$	$1, 3, -4, 5, 7, -8, -5, -7$

Table 2.

Here, the numbers formerly in the columns headed by 0 and 2 in Table 1 are now in the column headed by $-4 = L_A(0+3) = L_A(2+3)$. Similarly, the numbers originally headed by -4 or -5, will now be headed by $-1 = L_A(-4+3) = L_A(-5+3)$, and numbers formerly headed by -1 will now be headed by $L_A(-1+3) = 2$.

The new table is to be interpreted exactly as the original table was: Any finite string in a particular line implies the existence of the same string in some other line, but with each number replaced by its column header.

It is important to note that numbers which were in the same column in the original table are also together in the consolidated table.

The reason we were able to reduce the number of columns is because the original column headers, when shifted by 3, gave results which were contained in only three of the original columns. This suggests the reason for shifting by 3 and an appropriate shift for the next step. If a and b are column headers, find two numbers in a column which differ by $a - b$. If these two numbers are $t + a$ and $t + b$, then shifting a and b by t will lead to a table with fewer columns. In Table 2, we have the column headers $\{-4, 2, -1\}$. Since -4 and 2 have a difference of 6, find numbers in a column of Table 2 which are 6 apart, for example, -5 and 1. That suggests a shift of -1. Then a string involving $\{-4, 2, -1\}$ implies one involving $\{-5, 1, -2\}$ due to shifting by -1. That implies, by Table 2, a string involving only $\{-1, 2\}$. Tracing through the implications, we can again combine columns to produce:

-1	2
$0, -13, -6, 15, 2, 8, 21, 23, 4, -3, -1, -2, 6$	$1, 3, -4, 5, 7, -8, -5, -7$

Table 3.

For the final step, the column headers are 3 apart, as are the numbers -8 and -5, which belong to the "2" column. By shifting -7, a sequence involving $\{-1, 2\}$ will involve instead the numbers $\{-8, -5\}$, which according to Table 3 implies a sequence involving only the number 2. Thus we have consolidated all integers into a single column headed by a 2. Remembering the interpretation of the tables, this implies that any finite string, however long, implies the existence of one of equal length involving the single digit 2. This establishes Lemma 3 in the case of the example.

5. Proof of Lemma 3 in general

Begin with the Table of L_A as described above. If two numbers in a column have the same difference as two column headers, columns may be consolidated as described in the last section. If the Table of L_A has as column headers $c_1, c_2, \ldots, c_n$ (all the periodic numbers), and some column contains both c_i+t and c_j+t, for some integer t, then the consolidated table can be described as follows: If a number m was in the column headed by c_i in the original table, in the new table m will be in the column headed by the header of c_i+t in the original table. The new headers are some proper subset of the original headers. The new table has the same significance as the previous: Any finite length substring in any line will imply the existence of the same string in some other line with the original numbers replaced by the column headers.

Repeat this consolidation process as many times as possible. In fact, according to Lemma 4 below, the end result will be a single column headed by a single periodic number. According to the meaning of the table, this implies the existence of strings of arbitrary length consisting of a single repeated digit, proving Lemma 3, and Theorem 2. It remains only to establish:

Lemma 4. *In a table of more than one column, created by the process described above, with headers $c_1, \ldots, c_k$, there exists a column containing two numbers with difference equal to some $c_i - c_j$.*

Proof of Lemma 4. Assume the c_i are in increasing order. Replace the header c_i by h_i, where $h_i = c_i - c_1$. Then the h_i are nonnegative and in increasing order, with $h_1 = 0$. For simplicity write the largest header h_k as d. (The h_i are no longer necessarily periodic numbers, but that will not matter to the proof. Essentially, we are replacing column headers by their new values after a shift.) Since $h_i - h_j = c_i - c_j$, we must show some column contains two numbers differing by some $h_i - h_j$. *Assume this is not the case*. This assumption will severely restrict the ways in which integers can be positioned in the body of the table.

The integers are partitioned into k columns headed by $h_1, h_2, \ldots, h_k$, where $h_1 = 0$ and $h_k = d$. Consider the numbers 0 through $d-1$ in the body of the table. Call this set of numbers Block 0, or B_0 for short. Likewise, B_1 refers to the numbers d through $2d-1$, and in general B_i is the set $B_0 + id$, where i is any integer. Given the column locations of the numbers in B_0, the locations of the numbers in B_1 are completely determined: Since the numbers in the set $S = \{d-h_1, d-h_2, \ldots, d-h_{k-1}, d-h_k\}$ are located in k different columns by the assumption, the location of $d = d-h_1$

is determined by the location of the other $k-1$ numbers in S, all of which are in B_0. Similarly $d+1$ has its location determined by the locations of $\{d+1-h_2, d+1-h_3, \ldots, d+1-h_k\}$, all of which are in $B_0 \cup \{d\}$. Then continuing, any m in B_1 has its location determined by the locations of elements in B_0 or of elements smaller than m in B_1, which were themselves previously located using the knowledge about B_0. In similar fashion, we can show the location of numbers in B_0 determines the location of numbers in B_{-1}. By continuing the argument, we see the location of the numbers of B_0 (or in fact of any one B_i) determines the location of *all* the integers in the table.

Given B_i, let P_i denote the string of locations (column numbers) of the members of B_i in their natural order. This produces a string of length d using the symbols $\{1, 2, \ldots, k\}$. Since only a finite number of such strings exist, we must have $P_i = P_{i+t}$ for some i and t. That is, the locations of the numbers in block B_i are the same as for block B_{i+t}. (More accurately, the number id $+m$ in B_i is located in the same column as $(i+t)d+m$ in B_{i+t}, where m is between 0 and $d-1$.) By the argument above, since each block will determine its adjacent block, this means that $P_{i+1} = P_{i+t+1}$. In fact, since each block will determine all other blocks, $P_{i+j} = P_{i+t+j}$ for any j. In short, $P_j = P_{j+t}$ for any j. This means that any number m is in the same column as $m + td$. Thus, numbers which are congruent modulo td are in the same column of the table. This fact will lead to the desired contradicton.

With r as defined earlier in the definition of L_A, choose positive j and s such that $(N^r)^{(j+is)} \equiv (N^r)^j$ modulo (td) for any positive integer i. This is possible since the pattern of (mod td) congruence classes of the powers of N^r must begin to repeat at some stage.

Since $(a_1, a_2, \ldots, a_N) = 1$, there are integers k_i' such that $\sum k_i' a_i = d$. This implies there is a number $\sum k_i a_i$ congruent to d (mod (td)) with each k_i nonnegative. (We have $d + (\sum a_i)(tdm) = \sum (tdm + k_i')a_i$. For sufficiently large m, letting $k_i = tdm + k_i'$, we have each k_i nonnegative and $\sum k_i a_i \equiv d$ modulo (td) as desired.)

For the contradiction, let $C = [N^{r(j+s)}a_1 + N^{r(j+2s)}a_1 + \cdots + N^{r(j+k_1 s)}a_1] + [N^{r(j+k_1 s+s)}a_2 + N^{r(j+k_1 s+2s)}a_2 + \cdots + N^{r(j+k_1 s+k_2 s)}a_2] + \cdots + [\, N^{r(j+k_1 s+k_2 s+\cdots+k_{N-1}s+s)}a_N + \cdots + N^{r(j+k_1 s+k_2 s+\cdots+k_N s)}a_N]$. Here there are k_i terms involving a_i, and the powers of N^r increase from term to term by s, but are all congruent to N^{rj} modulo (td).

It is easy to see that C is congruent to N^{rj} $(\sum k_i a_i)$ modulo (td), and hence to $N^{rj}d$ modulo (td). Then C is in the same column as N^{rj}d as noted above. Since $a_N = 0$, we have ${F_A}^{rj}(N^{rj}d) = d$, and hence $L_A(d) = L_A(N^{rj}d)$. Then d and $N^{rj}d$ are in the same column in the Table of L_A, and hence in all derived tables as well. Then C is in the same column as d.

On the other hand, repeated application of ${F_A}^r$ to C eventually results in 0. (For each i, F_A $(a_i) = 0$, and hence ${F_A}^r(a_i) = 0$, since $a_N = 0$. Moreover, ${F_A}^r(mN^r) = m$ for any m, again because $a_N = 0$.) Since $a_N = 0$, 0 is periodic, and therefore $L_A(C) = 0$. Then C is in the same column as 0, which is a contradiction: As noted above, 0 and d are in separate columns since their difference is $d - 0 = h_k - h_1 = c_k - c_1$. $\square$

References

[1] I. Kátai, Generalized Number Systems and Fractal Geometry, Monograph, *Janus Pannonius University*, *Pécs, Hungary*, 1995.

[2] K. H. Indlekofer, I. Kátai and P. Racskó, Some remarks on generalized number systems, *Acta Sci. Math.* **57** (1993), 543–553.

[3] K. H. Indlekofer, I. Kátai and P. Racskó, Number systems and fractal geometry, Probability Theory and its Applications, *Kluwer*, 1992, 319–334.

[4] G. Michalek, Base Three Just Touching Covering Systems, *Publicationes Mathematicae Debrecen* **51**/3–4 (1997), 241–263.

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(Received October 22, 1999; file received February 8, 2000)