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# More about some functional equations of multiplicative symmetry

By N. BRILLOUET-BELLUOT (Nantes)

**Abstract.** In the present paper, we consider the following functional equations of multiplicative symmetry:

$$f(x.f(y)) = f(y.f(x)) \qquad (x, y \in K) \tag{1}$$

$$f(x.f(y)) = f(x).f(y) \qquad (x, y \in K) \tag{2}$$

where $K = \mathbb{R}$ or $\mathbb{C}$. The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of these equations have been already determined and some partial results concerning their continuous solutions $f : \mathbb{C} \to \mathbb{C}$ have been obtained.

We show here that we may obtain the continuous solutions of these equations with similar arguments in the real and in the complex case. Moreover, in the case $K = \mathbb{C}$, we determine the nonconstant continuous solutions of (1) and (2) under the hypothesis that $f(\mathbb{C}) \setminus \{0\}$ is connected, which leads to a generalization of a previous result of J. Dhombres.

## 1. Introduction

Let $G$ be a set endowed with a binary operation $*$. Following J. G. Dhombres (cf. [3]), we say that a function $f : G \to G$ is *a multiplicative symmetric function on* $(G, *)$ if it satisfies the functional equation: $f(x * f(y)) = f(y * f(x))$ for all $x$, $y$ in $G$. In particular, if $f$ satisfies a functional equation of the following type: $f(x * f(y)) = F(x, y)$ $(x, y \in G)$ where $F : G \times G \to G$ is a symmetric function (possibly depending on $f$), then $f$ is a multiplicative symmetric function on $(G, *)$.

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In [4], J. G. Dhombres considered the multiplicative symmetric functions on the group with zero $(\mathbb{R},.)$. He obtained the continuous solutions $f:\mathbb{R}\to\mathbb{R}$ of the following functional equations:

$$f(x.f(y)) = f(y.f(x)) \qquad (x,y\in\mathbb{R}) \tag{1}$$

$$f(x.f(y)) = f(x).f(y) \qquad (x,y\in\mathbb{R}). \tag{2}$$

In [3], he gave partial results concerning the continuous solutions $f:\mathbb{C}\to\mathbb{C}$ of these functional equations under the hypothesis $f^{-1}(0)=\{0\}$ if $f$ is not constant. We have completed his results in [2] under the same hypothesis.

In the present paper, we put under similar arguments the two cases. Also, concerning the complex case, we consider here a more general class of solutions of (1) and (2).

*In the whole paper, $K$ is either $\mathbb{R}$ or $\mathbb{C}$ and $U$ will denote* $\{z\in\mathbb{C};|z|=1\}$.

## 2. The functional equation (2)

We have first the following result concerning the functional equation (2):

**Theorem 1.** *A function $f:K\to K$ is a continuous solution of the functional equation*

$$f(x.f(y)) = f(x).f(y) \qquad (x,y\in K) \tag{2}$$

*if, and only if $f=0$ or $f=1$*

*or*

(i) $f(x)=ax (x\in K)$*where $a$ is an arbitrary element of $K$*

*or*

- *in the case $K=\mathbb{R}$*

(ii) $f(x)=\mathrm{Sup}(-bx,cx)$ $(x\in\mathbb{R})$ *where $b$ and $c$ are arbitrary nonnegative real numbers satisfying $c>-b$*

- *in the case $K=\mathbb{C}$ and if $f(\mathbb{C})\setminus\{0\}$ is connected*

(iii) $$f(x)=\begin{cases} 0 & \text{if } x=0 \text{ or if } x\neq 0 \\ & \text{and } \dfrac{x}{|x|^{1+i\beta}}\in\mathcal{N} \\ \left(|x|\psi\left(\dfrac{x}{|x|^{1+i\beta}}\right)\right)^{1+i\beta} & \text{if } x\neq 0 \text{ and } \dfrac{x}{|x|^{1+i\beta}}\notin\mathcal{N} \end{cases}$$

*where $\beta$ is an arbitrary real number, $\psi : U \to [0,+\infty[$ is an arbitrary continuous function and $\mathcal{N}=\psi^{-1}(0)$.*

*Remark.* A simple example of a continuous function $\psi : U \to [\,0,+\infty\,[$ for which $\mathcal{N}$ is not empty is the following: $\psi(x)=|1-x|$ $(x\in U)$.

Proof of Theorem 1. The only constant solutions of (2) are $f=0$ and $f=1$.

Let $f : K\to K$ be a nonconstant continuous solution of (2). By taking $x=y=0$ in (2), we obtain $f(0)=0$ or $f(0)=1$. If we suppose $f(0)=1$, we see, by taking $x=0$ in (2), that $f$ is identically equal to 1, which is not the case. Therefore, we have

(3) $$f(0)=0.$$

Let us define $N=f^{-1}(0)$ and $M=f(K)\setminus\{0\}=f(K\setminus N)$.

$M$ is a subgroup of $(\mathbb{C}\setminus\{0\},\,.\,)$. Namely, if $f(y)\neq 0$, we get by taking $\frac{x}{f(y)}$ in place of $x$ in (2): $\frac{f(x)}{f(y)}=f\left(\frac{x}{f(y)}\right)$.

Let us prove now that $M$ is closed in $\mathbb{C}\setminus\{0\}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $K\setminus N$ such that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $z$ in $\mathbb{C}\setminus\{0\}$. We have by (2): $f\left(\frac{x_0}{z}\,.\,f(x_n)\right)=f\left(\frac{x_0}{z}\right).f(x_n)$. Using the continuity of $f$, we get $f(x_0)=f\left(\frac{x_0}{z}\right)z$ . Since $f(x_0)\neq 0$, we have $f\left(\frac{x_0}{z}\right)\neq 0$. Since $M$ is a subgroup of $(\mathbb{C}\setminus\{0\},\,.\,)$, we deduce that $z$ belongs to $M$. Therefore, $M$ is closed in $(\mathbb{C}\setminus\{0\},\,.)$.

So, $M=f(K)\setminus\{0\}$ *is a closed subgroup of* $(\mathbb{C}\setminus\{0\},\,.)$.

Let us consider the mapping $h$ defined by: $h(x)=e^x$ $(x\in\mathbb{C})$. $h$ is a continuous homomorphism from the additive group $(\mathbb{C},+)$ onto the multiplicative group $(\mathbb{C}\setminus\{0\},\,.)$. Therefore, $h^{-1}(M)$ is a closed additive subgroup of $(\mathbb{C},+)$. We deduce that we have the following possibilities (cf. [1]):

(i) $h^{-1}(M) = a\mathbb{R}$;

(ii) $h^{-1}(M) = a\mathbb{Z}$ where $a$ is some nonzero complex number;

(iii) $h^{-1}(M) = a\mathbb{Z} + b\mathbb{Z}$;

(iv) $h^{-1}(M) = a\mathbb{R} + b\mathbb{Z}$ where $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$;

(v) $h^{-1}(M) = \mathbb{C}$;

(vi) $h^{-1}(M) = \{0\}$.

Since $f$ is continuous and satisfies (3), $M$ is not reduced to 1 and the case (vi) cannot occur. Also, since $M \cup \{0\} = f(\mathbb{C})$ is connected, the cases (ii) and (iii) cannot occur.

• *In the case $K = \mathbb{R}$,*
only the cases (i) and (iv) can occur with $a \in \mathbb{R}$ and $\mathrm{Im}(b) \in \pi\mathbb{Z}$ ($\mathrm{Im}(b)$ denotes the imaginary part of $b$). We deduce either $f(\mathbb{R}) = \mathbb{R}$ or $f(\mathbb{R}) = [0, +\infty[$.

In the first case, we have by using (2): $f(zx) = z\ f(x)$ $(x, z \in \mathbb{R})$. This leads to the solutions (i) by taking $x = 1$.

In the second case, by using (2), we have for all $x > 0$:

$$f(x) = f(1)\ x \quad \text{and} \quad f(-x) = (-f(-1))\ (-x)$$

which leads to the solutions (ii).

• *Let us consider now the case $K = \mathbb{C}$.*

*If $h^{-1}(M) = \mathbb{C}$, we have $M = \mathbb{C} \setminus \{0\}$.*
We have by (2): $f(zx) = z\ f(x)$ $(x, z \in \mathbb{C})$. With $x = 1$, we deduce the solutions (i) of (2).

*If $h^{-1}(M) = a\mathbb{R}$ where $a$ is some nonzero complex number, we have* $M = \{e^{\lambda a};\ \lambda \in \mathbb{R}\}$.

Let us write: $a = \alpha + i\beta$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

If $\alpha = 0$, we have $M = U$. This case cannot occur since, by the continuity of $f$, $f(\mathbb{C}) = M \cup \{0\}$ is connected.

Therefore, we have $\alpha \neq 0$. For each $x$ in $\mathbb{C} \setminus N$, there exists a unique real number $\lambda$ such that $f(x) = e^{\lambda a}$, namely $\lambda = \frac{\ln(|f(x)|)}{\alpha}$. Therefore, we have

$$f(x) = e^{\frac{a}{\alpha}\ln(|f(x)|)} = |f(x)|^{1+i\frac{\beta}{\alpha}} \qquad (x \in \mathbb{C} \setminus N). \tag{4}$$

(2) implies: $f(x.e^{\lambda a}) = e^{\lambda a}.f(x)$ $(\lambda \in \mathbb{R};\ x \in \mathbb{C})$ and therefore

$$|f(x.e^{\lambda a})| = e^{\lambda \alpha}.|f(x)| \quad (\lambda \in \mathbb{R}; x \in \mathbb{C}). \tag{5}$$

Now, using (5), we may write for all $x$ in $\mathbb{C} \setminus \{0\}$:

$$|f(x)| = \left| f\left( \frac{x}{|x|^{1+i\frac{\beta}{\alpha}}}\, e^{\frac{a}{\alpha}\ln(|x|)} \right) \right| = |x| \, \left| f\left( \frac{x}{|x|^{1+i\frac{\beta}{\alpha}}} \right) \right|.$$

If we define the function $\psi : U \to [0, +\infty[$ by $\psi(x) = |f(x)|$ $(x \in U)$, we see by (4) that $f$ has the expression (iii) given in Theorem 1.

Conversely, it is easy to see that all functions of the form (iii) are continuous solutions of (2) since, for $y$ in $\mathbb{C} \setminus \{0\}$ such that $\frac{y}{|y|^{1+i\beta}} \notin \mathcal{N}$, we have $f(y) = |f(y)|^{1+i\beta}$.

*If $h^{-1}(M) = a\mathbb{R} + b\mathbb{Z}$ where $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$, we have $M = \{e^{\lambda a + nb};\ n \in \mathbb{Z},\ \lambda \in \mathbb{R}\}$.*

Therefore, for each $x$ in $\mathbb{C} \setminus N$, there exist $n$ in $\mathbb{Z}$ and $\lambda$ in $\mathbb{R}$ such that:

$$f(x) = e^{\lambda a + nb}. \tag{6}$$

Let us write $a = \alpha + i\alpha'$; $b = \beta + i\beta'$ with $\alpha$, $\alpha'$, $\beta$, $\beta'$ in $\mathbb{R}$.

If $\alpha = 0$ we have by (6) for $x$ in $\mathbb{C} \setminus N$:

$$|f(x)| = e^{n\beta}. \tag{7}$$

By the continuity of $f$ and (3), the set $\{|f(x)|; x \in \mathbb{C}\}$ is an interval of $\mathbb{R}$ containing 0. By (7), this set is included in $\{0\} \cup \{e^{n\beta}; n \in \mathbb{Z}\}$ which is a discrete set of points. Since $f$ is not identically zero, this case cannot occur.

Therefore, we have $\alpha \neq 0$. Using (6), we get $\lambda = \frac{1}{\alpha}(\ln(|f(x)|) - n\beta)$ and $\operatorname{Arg}(f(x)) - n\beta' - \lambda\alpha' \in 2\pi\mathbb{Z}$.

We deduce: $\operatorname{Arg}(f(x)) - n\left(\beta' - \beta\frac{\alpha'}{\alpha}\right) - \frac{\alpha'}{\alpha}\ln(|f(x)|) \in 2\pi\mathbb{Z}$.

Therefore, for each $x$ in $\mathbb{C} \setminus N$, there exists $n$ in $\mathbb{Z}$ such that:

$$f(x) = |f(x)|^{1+i\frac{\alpha'}{\alpha}} \quad e^{in\left(\beta' - \beta\frac{\alpha'}{\alpha}\right)}. \tag{8}$$

Note that, since $\{a,b\}$ is a basis of the real vector space $\mathbb{C}$, we have $\beta' - \beta\frac{\alpha'}{\alpha} \neq 0$. *Since* $f(\mathbb{C}) \setminus \{0\}$ *is connected*, the set $\{f(x)|f(x)|^{-1-i\frac{\alpha'}{\alpha}};$ $x \in \mathbb{C} \setminus N\}$ is a connected subset of $U$. By (8), this set is included in $\{e^{in(\beta'-\beta\frac{\alpha'}{\alpha})}\ ; n \in \mathbb{Z}\}$ which is a discrete set of points of $U$. Therefore, there exists some $n_0$ in $\mathbb{Z}$ such that we have:

$$f(x) = |f(x)|^{1+i\frac{\alpha'}{\alpha}}\ e^{in_0(\beta'-\beta\frac{\alpha'}{\alpha})} \quad (x \in \mathbb{C} \setminus N). \tag{9}$$

Since $f(\mathbb{C}) \setminus \{0\}$ is a multiplicative subgroup of $(\mathbb{C} \setminus \{0\}, .)$, we have for all $x$ in $\mathbb{C} \setminus N$ and for all $p$ in $\mathbb{Z}$:

$$f(x)^p = |f(x)|^{p(1+i\frac{\alpha'}{\alpha})}\ e^{in_0 p(\beta'-\beta\frac{\alpha'}{\alpha})} = |f(x)|^{p(1+i\frac{\alpha'}{\alpha})}\ e^{in_0(\beta'-\beta\frac{\alpha'}{\alpha})}.$$

We deduce: $n_0(\beta' - \beta\frac{\alpha'}{\alpha}) \in 2\pi\mathbb{Z}$.

From (9), we get: $M = e^{a\mathbb{R}}$ which is the previous case.

This completes the proof of Theorem 1.

*Remark.* Among the continuous solutions $f : \mathbb{C} \to \mathbb{C}$ such that $f(\mathbb{C}) \setminus \{0\}$ is connected, there are those such that $\mathbb{C} \setminus f^{-1}(0)$ is connected. Let us prove that *the solutions of (2) of the form (iii) such that* $\mathbb{C} \setminus f^{-1}(0)$ *is connected correspond to the functions* $\psi : U \to [0, +\infty[$ *such that* $U \setminus \mathcal{N}$ *is connected*.

Let $N = f^{-1}(0)$ and let us define the continuous function $\varphi_\beta : \mathbb{C} \setminus \{0\} \to U$ by:

$$\varphi_\beta(x) = \frac{x}{|x|^{1+i\beta}} \qquad (x \in \mathbb{C} \setminus \{0\}).$$

We may remark that, for each $z$ in $U$, we have $\varphi_\beta^{-1}(\{z\}) = \{z\lambda^{1+i\beta};$ $\lambda \in ]0, +\infty[\}$ which moreover proves that $\varphi_\beta$ is onto.

We have $U \setminus \mathcal{N} = \varphi_\beta(\mathbb{C} \setminus N)$. Therefore, if $\mathbb{C} \setminus N$ is connected, $U \setminus \mathcal{N}$ is connected since $\varphi_\beta$ is continuous.

Conversely, we have $\mathbb{C} \setminus N = \varphi_\beta^{-1}(U \setminus \mathcal{N}) = \{z\ \lambda^{1+i\beta}; z \in U \setminus \mathcal{N},$ $\lambda \in ]0, +\infty[\}$. Since the function $(z, \lambda) \in U \times ]0, +\infty[ \longrightarrow z\lambda^{1+i\beta} \in \mathbb{C} \setminus \{0\}$ is continuous, $\mathbb{C} \setminus N$ is connected if $U \setminus \mathcal{N}$ is connected.

So, $\mathbb{C} \setminus N$ is connected if, and only if, $U \setminus \mathcal{N}$ is connected, which proves the assertion.

## 3. The functional equation (1)

Using Theorem 1, we may prove now the following result concerning the functional equation (1):

**Theorem 2.** *A function $f : K \to K$ is a continuous solution of the functional equation:*

$$f(x.f(y)) = f(y.f(x)) \qquad (x, y \in K) \tag{1}$$

*if, and only if either $f$ is a constant function,*

*or*

(i) $f(x) = ax$ $(x \in K)$ *where $a$ is an arbitrary element of $K$,*

*or*

- *in the case $K = \mathbb{R}$ either*

(ii) $f(x) = \mathrm{Sup}(-bx, cx)$ $(x \in \mathbb{R})$ *where $b$ and $c$ are arbitrary nonnegative real numbers satisfying $c > -b$,*

*or*

(iii) $f(x) = d|x|$ $(x \in \mathbb{R})$ *where $d$ is an arbitrary negative real number,*

- *in the case $K = \mathbb{C}$ and if $f(\mathbb{C}) \setminus \{0\}$ is connected either*

(iv)
$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ a|x|\theta(|x|) & \text{if } x \neq 0 \end{cases}$$

*where $a$ is an arbitrary positive real number and $\theta : ]0, +\infty[ \to U$ is an arbitrary continuous function,*

*or*

(v)
$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or} \\ & \text{if } x \neq 0 \text{ and } \dfrac{x}{|x|^{1+i\beta}} \in \mathcal{N} \\ e^{\frac{2ip\pi}{n}} \left( |x| \psi \left( \dfrac{x}{|x|^{1+i\beta}} \right) \right)^{1+i\beta} & \text{if } x \neq 0 \text{ and } \dfrac{x}{|x|^{1+i\beta}} \notin \mathcal{N} \end{cases}$$

*where $p$ is an arbitrary integer in $\mathbb{Z}$, $n$ is an arbitrary positive integer, $\beta$ is an arbitrary real number, $\psi : U \to [0, +\infty[$ is an arbitrary* *continuous function, which satisfies:* $\psi(e^{\frac{2i\pi}{n}}x) = \psi(x)$ $(x \in U)$ *in the case* $p \neq 0$*, and* $\mathcal{N} = \psi^{-1}(0)$.

Proof. Let $f : K \to K$ be a nonconstant continuous solution of the functional equation (1).

We see, by taking $y = 0$ in (1), that $f$ satisfies: $f(0) = 0$.

Let us define: $N = f^{-1}(0)$.

$f$ satisfies the following functional equation:

$$f(x.f(y.f(z))) = f(x.f(y).f(z)) \qquad (x, y, z \in K) \tag{10}$$

since we have by (1):

$$\begin{aligned} f(x.f(y.f(z))) &= f(y.f(x).f(z)) = f(z.f(y.f(x))) \\ &= f(z.f(x.f(y))) = f(x.f(y).f(z)). \end{aligned}$$

Let us prove that this implies

$$|f(yf(z))| = |f(y)|\,|f(z)| \qquad (y, z \in K). \tag{11}$$

Namely, if $y$ or $z$ belongs to $N$, we have by (10): $f(x.f(y.f(z))) = 0$ for all $x$ in $K$. This implies $f(y.f(z)) = 0$ since $f$ is not identically zero.

If $y$ and $z$ do not belong to $N$, we may define the continuous function $h : (K \setminus N) \times (K \setminus N) \to K$ by

$$h(y, z) = \frac{f(y.f(z))}{f(y).f(z)} \qquad (y, z \in K \setminus N). \tag{12}$$

By replacing $x$ by $\frac{x}{f(y).f(z)}$ in (10), we have

$$f(x) = f(x.h(y, z)) \qquad (x \in K;\ y, z \in K \setminus N). \tag{13}$$

If we had $h(y, z) = 0$ for some $y$ and $z$ in $K \setminus N$, this would imply that $f$ is identically zero, which is not the case. Therefore, $h$ takes its values in $K \setminus \{0\}$. We deduce from (13):

$$f(x) = f(x.(h(y, z))^n) \qquad (x \in K;\, y, z \in K \setminus N;\, n \in \mathbb{Z}). \tag{14}$$

If $|h(y, z)| \neq 1$ for some $y$ and $z$ in $K \setminus N$, the continuity of $f$ at 0 would imply that $f$ is identically zero, which is not the case. Therefore, we have: $|h(y, z)| = 1$ for all $y$ and $z$ in $K \setminus N$. This proves (11).

1) *Let us suppose first that there exist $y$ and $z$ in $K \setminus N$ such that*
*in the case $K = \mathbb{R}$, $h(y,z) = -1$*
*in the case $K = \mathbb{C}$, $h(y,z)$ is not a root of 1.*

In the case $K = \mathbb{R}$, we deduce from (13) that $f$ is an even function. In the case $K = \mathbb{C}$, $\{(h(y,z))^n\}_{n\in\mathbb{Z}}$ is dense in $U$. The continuity of $f$ and (14) imply: $f(\lambda x) = f(x)$ $(x \in \mathbb{C};\ \lambda \in U)$. Therefore, for all $x$ in $\mathbb{C} \setminus \{0\}$, we may write: $f(x) = f(|x|e^{i\,\mathrm{Arg}(x)}) = f(|x|)$.

So, in both cases, we have: $f(x) = f(|x|)$ $(x \in K)$.

Let us define the continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ by:

$$\varphi(|x|) = |f(|x|)| = |f(x)| \qquad (x \in K).$$

By (11), we have for all $x$ and $y$ in $K$:

$$\begin{aligned}\varphi(|x|.\varphi(|y|)) &= |f(|x|.|f(|y|)|)| = |f(x.f(y))| \\ &= |f(x)|.|f(y)| = \varphi(|x|).\varphi(|y|).\end{aligned}$$

So, $\varphi : [0, +\infty[ \to [0, +\infty[$ is a continuous solution of the functional equation (2). By considering the proof of Theorem 1, we see that we have either $\varphi = 0$ or $\varphi = 1$ or $\varphi(|x|) = a|x|$ $(x \in K)$ where $a$ is some positive real number.

$\varphi = 0$ or $\varphi = 1$ cannot hold since $f$ is not constant.
We deduce $|f(x)| = a|x|$ $(x \in K)$ where $a$ is some positive real number. Therefore, $f$ vanishes only at 0 in this case and we have

$$f(x) = |f(x)|\frac{f(x)}{|f(x)|} = a|x|\frac{f(|x|)}{|f(|x|)|} \qquad (x \in K \setminus \{0\})$$

i.e. $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ a|x|\theta(|x|) & \text{if } x \neq 0 \end{cases}$, where $a$ is some positive real number and $\theta : ]0, +\infty\,[ \to U$ is some continuous function.

Conversely, it is easy to verify that all functions of this form are continuous solutions of the functional equation (1).

If $K = \mathbb{R}$, this gives the solutions (ii) with $b = c$ and the solutions (iii) of (1).

If $K = \mathbb{C}$, this gives the solutions (iv) of (1).

2) *Let us suppose now that, for all $y$ and $z$ in $K \setminus N$, $h(y,z)$ is*
*either equal to 1 in the case $K = \mathbb{R}$*

*or a root of 1 in the case $K = \mathbb{C}$.*

In the case $K = \mathbb{C}$, we suppose that $f(\mathbb{C}) \setminus \{0\}$ is connected. In this case, $h(y, z)$ belongs to $e^{2i\pi\mathbb{Q}}$ for all $y$ and $z$ in $\mathbb{C} \setminus N$. By (1), we have

$$(15) \qquad h(y,z) = \frac{f(yf(z))}{f(y)f(z)} = \frac{f(zf(y))}{f(y)f(z)} = h(z,y) \qquad (y, z \in \mathbb{C} \setminus N).$$

Let us fix $y$ in $\mathbb{C} \setminus N$. Since $f$ is continuous and $f(\mathbb{C}) \setminus \{0\}$ is connected, $h(y, \mathbb{C} \setminus N) = \left\{ \frac{f(y\ t)}{tf(y)}; t \in f(\mathbb{C}) \setminus \{0\} \right\}$ is a connected part of $e^{2i\pi\mathbb{Q}}$, which is totally disconnected. Therefore, there exists a unique element $\lambda(y)$ of $e^{2i\pi\mathbb{Q}}$ such that we have $h(y, z) = \lambda(y)$ $(z \in \mathbb{C} \setminus N)$.

By using (15), we get $h(y, z) = \lambda(y) = h(z, y) = \lambda(z)$ $(y, z \in \mathbb{C} \setminus N)$. We deduce that $h$ is a constant function.

So, in both cases, by (11), (12) and (13), there exists $\lambda$ in $e^{2i\pi\mathbb{Q}}$ ($\lambda = 1$ if $K = \mathbb{R}$) such that we have:

$$(16) \qquad \begin{cases} f(xf(y)) = \lambda f(x)f(y) & (x, y \in K), \\ f(\lambda x) = f(x) & (x \in K). \end{cases}$$

Let us define $g(x) = \lambda f(x)$ $(x \in K)$. By (16), $g : K \to K$ is a nonconstant continuous solution of the functional equation (2). By applying Theorem 1, we get

either the continuous solutions (i) of (1),

or, in the case $K = \mathbb{R}$, the solutions (ii) of (1),

or, in the case $K = \mathbb{C}$, since $g(\mathbb{C}) \setminus \{0\}$ is connected,

$$f(x) = \frac{1}{\lambda} \left( |x| \psi \left( \frac{x}{|x|^{1+i\beta}} \right) \right)^{1+i\beta} \qquad (x \in \mathbb{C} \setminus N)$$

where $\beta$ is an arbitrary real number and $\psi : U \to [0, +\infty[$ is an arbitrary continuous function.

If $\lambda$ is not 1, we have: $\lambda = e^{\frac{2ip\pi}{n}}$ where $p$ and $n$ are some nonzero relatively prime integers, $n > 0$. $G = \left\{ \frac{2\pi}{n}(mn + kp); (m, k) \in \mathbb{Z}^2 \right\}$ is a subgroup of $(\mathbb{R}, +)$. By (16), we have $\psi(e^{ig}x) = \psi(x)$ $(x \in U; g \in G)$. Since $p$ and $n$ are relatively prime, there exists $(m, k) \in \mathbb{Z}^2$ such that $mn + kp = 1$. Therefore, $G$ is of the form $G = \frac{2\pi}{n}\mathbb{Z}$ and $\psi$ satisfies $\psi(e^{\frac{2i\pi}{n}}x) = \psi(x)$ $(x \in U)$.

We deduce then the continuous solutions (iii) of (1) by noticing that $\lambda^k f$ is also a solution of (1) for all $k$ in $\mathbb{Z}$.

This completes the proof of Theorem 2.

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NICOLE BRILLOUËT–BELLUOT
ECOLE CENTRALE DE NANTES
DÉPARTEMENT D'INFORMATIQUE ET DE MATHÉMATIQUES
1 RUE DE LA NOË, 44 072 NANTES CEDEX 03
FRANCE

*E-mail*: Nicole.Belluot@ec-nantes.fr

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