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An inductive definition of higher gap simplified morasses

By ISTVÁN SZALKAI (Veszprém)

Abstract. In this paper we give an inductive definition of higher (finite) gap simplified morasses and prove their existence in L. Our construction continues Velleman's gap-2 definition [Ve8] towards higher (finite) gaps, after revealing the inner structure of and the *hierarchical* connection among these combinatorial set- theoretical structures of *different* gaps.

Our presented variant is different from Ch. Morgan's [Mo] and Jensen's [Je1] ones and has application in [Sz1].

The present paper contains Sections 1 and 5 of the author's Thesis [Sz2] dated 1991.

0. Introduction

Professor R. B. Jensen in 1972 [Je1] first defined structures which he called "gap-β *morasses of height* κ" or shortly "(κ, β) *morasses*" for every regular $\kappa \geq \omega_1$ and for any (finite or infinite) $1 \leq \beta < \kappa$.

In 1984 D. Velleman [Ve2] invented the gap-1 so called "simplified morasses" which possess much simpler structure and applications than Jensen's original ones, and he deduced that "*there exists a simplified gap-1 morass iff there is a Jensen's gap-1 morass*". He in 1987 in [Ve8] went further. He defined the gap-2 simplified morasses and showed the consistency of their existence by forcing. Jensen in the same year in [Je2] gave a direct construction of gap-2 simplified morasses from his original gap-2 morasses.

Ch. Morgan in 1989 in his thesis [Mo] gave a definition of his higher (finite) gap simplified morasses. He constructed these kinds of morasses

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from Jensen's original higher gap morasses. This is because the idea of his definition (based on sets of sequences of ordinals) is closer to Jensen's one than to Velleman's: building of gap-2 morass from gap-1 morass-segments.

The present author in 1987 in [Sz1] and his PhD thesis [Sz2] gave an alternate, inductive definition of higher (finite) gap simplified morasses which are more similar to Velleman's gap-2 definition of [Ve8], and he constructed them from Morgan's higher gap morasses. The gap-1 special case of both variants (Morgan's and Szalkai's), give exactly Velleman's gap-1 simplified morasses. Furhermore, our definition presented in this paper gives also in the gap-2 case precisely Velleman's gap-2 simplified morasses. Our idea is similar to Velleman's idea: we build up higher gap structures from suitable parts of smaller gap ones.

The aim of the present paper is to publish our definition and the contsruction of our inductive higher gap simplified morasses from Morgan's morasses (see Sections 1.b and 3). The present paper is a part of the author's Thesis [Sz2].

In [Sz1], [Sz2] we also discuss several properties and an **application** of our higher gap simplified morasses, and a definition of full linearizing sequences for higher gaps. We think the existence of higher gap simplified morasses with full linearizing sequences can be proved by forcing, similar to the one presented in [Ve8]. We do not know *any* definition of *simplified* morasses of *infinite* height. We intend to construct Morgan's morasses from ours in a forthcoming paper.

Organization of the paper: in Section 1 we give the definitions of Velleman's gap-1 and gap-2 simplified morasses (Section 1.a) and of our inductive higher gap simplified morasses (Section 1.b). In Section 2 we present Morgan's definition. Section 3 contains the construction of our higher gap simplified morasses, from a Morgan's one.

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History

Professor R. B. Jensen in 1972 [Je1] first defined the (κ, β)-morasses for any $1 \leq \beta < \kappa$ and regular $\kappa \geq \omega_1$. He extracted these structures from the fine structure of L while proving the two-cardinal transfer property. These structures allow us to construct objects of size $\kappa^{+\beta}$ in κ steps, in each step building objects of size $< \kappa$ and, in the meantime, to handle each subobject of $\kappa^{+\beta}$ of size $< \kappa$. (This is why this and similar structures play a big role in combinatorial set theory.) For the gap-1 case ($\beta = 1$) definitions see e.g. [CK], [Je0], [De0], [De1], [St1] or [Ve0]. The higher gap definitions are unpublished, see [Je1], [Je2] and [St0].

Jensen proved in [Je0] for $\beta = 1$ and in [Je1] for any $\beta \geq 1$ that these morasses do exist in L for all regular κ. Jensen's original proof for the gap-1 case was simplified by K. Devlin in [De0], [De1]. S. Shelah and L. J. Stanley proved in 1979 that for all $A \subseteq \omega_1$ there is an $(\omega_1, 1)$-morass in $L[A]$, see [SS0] or [De1]. Devlin in [De1] proved that there is an $(\omega_1, 1)$- morass if ω_2 is not an inaccessible cardinal in L. Stanley forced gap-1 morasses in [St1]. P. Komjáth in [Ko] showed that Levy collapsing an inaccessible cardinal to ω_1 there would not be $(\omega_1, 1)$ morass, supposing the consistency of the existence of a Mahlo cardinal.

Many morass-like combinatorial structures have been developed for deciding combinatorial problems and their existence was proven in L. (Coarse morasses by Donder [Do0], quagmires by Burgess [Bu1], Silvers's W_κ principle e.g. in [Bu0] or [Ka0].) These structures and the morasses have many applications in combinatorial set theory, we only refer to [Bu1], [Re], [Mi], [CK], [St0], [St1] and almost all papers of Kanamori and of Velleman. In [HK], [SS2], [SS3] the authors conjectured that their results can also be obtained by morasses. Some of these conjectures were justified by Stanley, Velleman, Morgan in [SVM] and independently by Szalkai in [Sz0] and in [Sz2]. See also Komjáth's paper [Ko]. Further, [Ka0], [Ka1], [Ka2] contain partial survey, while [Sz2] contains a detailed survey of these structures, their existence and their applications.

In the meantime, in the early 80's Shelah, Stanley, Solovay, Velleman and others were looking for Martin's Axiom-like forcing axioms which are valid in L. This was result of a procedure motivated by a question of K. Kunen (see e.g. [Ve0]):

> *"Why are there so many statements which can be shown to be consistent with ZCF by forcing, and which are also true if $V = L$?"*

S. Shelah and L. Stanley in [SS0], [SS1] and independently D. Velleman in [Ve0] obtained Martin's Axiom like forcing axioms which are, in fact equivalent to the existence of gap-1-morasses. (R. Solovay begun similar investigations in 1977 which remained unpublished.)

D. Velleman [Ve2] originally deduced the fact that there exists a simplified gap-1 morass iff there is a Jensen's original gap-1 morass via his forcing principle (which answered Kunen's question), later H. D. Donder in [Do1] gave a direct proof for this fact. Forcing gap-1 simplified morasses is an exercise, similar to forcing Silvers's W_κ in [Bu0].

For many combinatorial problems (see e.g. [Ka0], [Ka1], [SS1], [Ve0]) simplified morasses themselves were not enough, Velleman in [Ve3] defined the notion of *linearizing sequences*, and the notion of *simplified morasses with buit in diamond*. He showed the consistency of their existence for all but not weakly compact regular height by forcing in [Ve3]. H. D. Donder showed the following: *(a) if there is a* $(\kappa, 1)$*-simplified morass with linearizing sequences then* κ *is not weakly compact, (b)* $V = L$ *implies the existence of* $(\kappa, 1)$*-simplified morasses with linearizing sequences for all* κ *not weakly compact cardinal.*

Velleman's simplified morasses provide us easier applications (constructions) since their structure are indeed simpler than Jensen's morasses', and the existence of these two kinds of morasses are equivalent.

Looking for gap-n $(n \geq 1)$ simplified morasses we have to emphasize their main property which is mainly used in applications: in κ many steps, using objects of size $< \kappa$, we can build an object of size κ^+ (if $n = 1$) or of size κ^{+n} (for any n), and so we can fix all $< \kappa$ size subset of κ^{+n}.

Velleman in [Ve8] defined the gap-2 simplified morasses and showed the consistency of their existence by forcing. Jensen in [Je2] gave a direct construction of gap-2 simplified morasses from his original gap-2 morasses. As have we mentioned in the Introduction, Morgan and Szalkai independently defined higher gap simplified morasses. Morgan [Mo] constructed these kinds of morasses from Jensen's original higher gap morasses, his definition can be seen in our Definition 3.1. The present author in [Sz1], [Sz2] gave an alternate, inductive definition of higher (finite) gap simplified morasses, which are more similar to Velleman's gap-2 definition of [Ve8]. The idea: we build higher gap structures from suitable pieces of lower gap ones: using the natural connection among them. Then he constructed them from Morgan's higher gap morasses. The gap-1 special case of both variants (Morgan's and Szalkai's), give exactly Velleman's gap-1 simplified morasses. Furthermore, our definition presented in this paper gives also in the gap-2 case Velleman's gap-2 simplified morasses.

For application of higher-gap morasses we can only refer to [Mo], [Sz1], [Sz2] and [Ve8]. In [Sz2] we also discuss several properties (similar to the ones in [Ve8]) and an application of our higher gap simplified morasses, and a definition of full linearizing sequences for higher gap morasses.

We do not know any forcing axiom equivalent to higher gap morasses.

It is interesting to note that any gap *simplified* morasses (Velleman's, Morgan's and our variants) can be defined of height ω_0 while Jensen's original definition of morasses is meaningful only for height $\geq \omega_1$. Further, Velleman in [Ve5] showed that $(\omega_0, 1)$-simplified morasses do exist in ZCF, and in [Ve9] he showed that his $(\omega_0, 2)$-simplified morasses do exist supposing the existence of a $(\omega_1, 1)$-simplified morass. This latter assumption is necessary since for each $n < \omega_0$ if there exists a (κ, n)-simplified morass, then there must exist (κ^{+s}, m)-simplified morasses where $0 \leq s$, $m < n$ and $m + s \leq n$ (see our Statement 1.12).

Some noncommon notation

$f \circ g$ denotes the *composition* of any functions f and g:
$(f \circ g)(x) = f(g(x))$ for any $x \in \mathrm{Dom}(g)$ s.t. $g(x) \in \mathrm{Dom}(f)$.

$f \restriction H$ is the *restriction* of the function f to any subset H of $\mathrm{Dom}(f)$,

$f''H := \mathrm{Range}\,(f \restriction H)$ is the range of f to the set H for any subset H of $\mathrm{Dom}(f)$

$\mathrm{id}_{\mathcal{M}}$ is the *identity* function for any structure $\mathcal{M}$.[1]

$f^{\rightarrow}$ and $g^{\Rightarrow}$ are short notations of *sequences* and *double sequences*, $\langle f_i : i \leq \theta \rangle$ and $\langle g_{i,j} : i < j \leq \theta \rangle$ resp. if θ is known but any fixed ordinal. These sequences have *length* θ. We denote the *restrictions* of these sequences to ζ, that is the sequences $\langle f_i : i \leq \zeta \rangle$ and $\langle g_{i,j} : i < j \leq \zeta \rangle$, by $f^{\rightarrow} \restriction (\zeta + 1)$ and $g^{\Rightarrow} \restriction (\zeta + 1)$, resp.

$|H|$ denotes the *cardinality* of H, in case H is a set, and the *length* of H, if H is a sequence.

In this paper κ always denotes a regular infinite cardinal, possibly countable.

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[1] This is clear if $\mathcal{M}$ is simply a set. For other relevant structures we will define this notion later in its right place.

1. Definitions

In this chapter we give the definitions of our higher gap simplified morasses. Because of the cumbersome technical details, to warm up let us recall the definitions of Velleman's gap-1 and gap-2 simplified morasses, which will be done in Section 1.a. We present our higher gap definition in Section 1.b.

1.a Gap-1 and -2 simplified morasses

For easier understanding our higher gap morass definition now we state here Velleman's original definitions of gap 1 and gap 2 simplified morasses from [Ve2] and [Ve8], respectively.

In what follows κ always denotes a regular cardinal.

1.1 Definition [Ve2]. $\mathcal{M} = \langle \overrightarrow{\varphi}, \overset{\Rightarrow}{\mathcal{F}} \rangle$ is a gap-1 **simplified morass of height** κ (or a $(\kappa, 1) - SM$ for short) iff

(0) $\overrightarrow{\varphi} = \langle \varphi_\alpha : \alpha \leq \kappa \rangle$ is an increasing sequence of ordinals φ_α less than κ for $\alpha < \kappa$, $\varphi_\kappa = \kappa^+$, and $\overset{\Rightarrow}{\mathcal{F}} = \langle \mathcal{F}_{\alpha\beta} : \alpha < \beta \leq \kappa \rangle$ where $\mathcal{F}_{\alpha,\beta}$ are nonempty sets of order preserving functions $f : \varphi_\alpha \to \varphi_\beta$ for $\alpha < \beta \leq \kappa$

(1) $(\forall\, \alpha < \beta < \kappa)\ \ |\mathcal{F}_{\alpha\beta}| < \kappa$

(2) $(\forall\, \alpha < \beta < \gamma \leq \kappa)\ \mathcal{F}_{\alpha\gamma} = \mathcal{F}_{\beta\gamma} \circ \mathcal{F}_{\alpha\beta} = \{f \circ g : f \in \mathcal{F}_{\beta\gamma},\, g \in \mathcal{F}_{\alpha\beta}\}$ (composition)

(3) $(\forall\, \alpha < \kappa)\ \mathcal{F}_{\alpha,\alpha+1} = \{\mathrm{id}, h_\alpha\}$ where $\mathrm{id} : \varphi_\alpha \to \varphi_\alpha$ is the identity, and h_α is a **shifting function**: that is for some $\sigma_\alpha < \varphi_\alpha$ (the so called **splitting point**) we have $h_\alpha(\xi) = \xi$ for $\xi < \sigma_\alpha$ and $h_\alpha(\sigma_\alpha + \zeta) = \varphi_\alpha + \zeta$ for $\sigma_\alpha + \zeta < \varphi_\alpha$ (amalgam property)

(4) For every $\alpha \leq \kappa$ limit, $\beta_1, \beta_2 < \alpha$, $f_1 \in \mathcal{F}_{\beta_1\alpha}$, $f_2 \in \mathcal{F}_{\beta_2\alpha}$ there exist a γ: $\beta_1, \beta_2 < \gamma < \alpha$ and $h_1 \in \mathcal{F}_{\beta_1\gamma}$, $h_2 \in \mathcal{F}_{\beta_2\gamma}$ and $g \in \mathcal{F}_{\gamma\alpha}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(5) $(\forall\, \alpha \leq \kappa \text{ limit})\ \ (\forall\, \beta < \alpha)$

$$\varphi_\alpha = \bigcup \{f'' \varphi_\beta : f \in \mathcal{F}_{\beta\alpha}\} \quad \text{(covering property).} \qquad \square$$

Velleman [Ve2] calls these structures *"neat expanded simplified morasses"*, but later on (in [Ve3], [Ve5], [Ve8]) this definition becomes the definition of simplified morasses. We mention again that in Velleman's and in our definition κ may be ω_0, but not in Jensen's original definition. Moreover, the above structures exist in ZFC also for $\kappa = \omega_0$ as it is shown in [Ve5].

Now we turn to gap-2 structures.

If we replace simply $\varphi_\kappa = \kappa^+$ by $\varphi_\kappa = \kappa^{++}$ in (0) then the existence of such structures is inconsistent, so the definition of gap-2 simplified morasses is not so trivial. Velleman's idea is the following. We have to build up κ^{++} from objects of size less than κ in κ steps, which can be done by building up a $(\kappa^+, 1)$ *simplified morass* itself in κ steps, in each step using a part of the final morass of size less than κ, a so called *fake morass*. Of course in this case we have to define also the embeddings between these gap-1 fake morasses. Definition 1.2(vii) gives the definition of gap-2 morasses, but before we need some preliminary definitions. In Definitions 1.2(i) through (vii) $\kappa \geq \omega_0$ is a regular cardinal. These definitions are taken from [Ve8].

1.2 Definition (i). $\mathcal{M} = \langle \overrightarrow{\varphi}, \overset{\Rightarrow}{\mathcal{F}} \rangle$ is a **fake gap-1 morass segment of height** θ and **of size less than** κ iff $\theta < \kappa$, and $\mathcal{M}$ satisfies (0) through (5) of Definition 1.1 with the below modification:

$$\overrightarrow{\varphi} = \langle \varphi_\alpha : \alpha \leq \theta \rangle, \quad \overset{\Rightarrow}{\mathcal{F}} = \langle \mathcal{F}_{\alpha\beta} : \alpha < \beta \leq \theta \rangle, \quad \varphi_\theta < \kappa.$$

We denote the **height** of $\mathcal{M}$ by $\mathrm{ht}(\mathcal{M})$, that is $\mathrm{ht}(\mathcal{M}) = \theta$.

1.2 Definition (ii). Let $\mathcal{M} = \langle \overrightarrow{\varphi}, \overset{\Rightarrow}{\mathcal{G}} \rangle$ and $\mathcal{N} = \langle \overrightarrow{\varphi'}, \overset{\Rightarrow}{\mathcal{G}'} \rangle$ be fake gap-1 morass segments of height θ and θ', resp. Call the function set
$f = \langle f^-, \overrightarrow{f}, \overset{\Rightarrow}{f} \rangle$ an $f : \mathcal{M} \to \mathcal{N}$ **embedding** iff

(1) $f^- : (\theta + 1) \to (\theta' + 1)$ is an order preserving function, $f^-(\theta) = \theta'$

(2) $\overrightarrow{f} = \langle f_\zeta : \zeta \leq \theta \rangle$ where $f_\zeta : \varphi_\zeta \to \varphi_{f^-(\zeta)}$ are order preserving functions for $\zeta \leq \theta$

(3) $\overset{\Rightarrow}{f} = \langle f_{\zeta\xi} : \zeta < \xi \leq \theta \rangle$ where $f_{\zeta\xi} : \mathcal{G}_{\zeta\xi} \to \mathcal{G}'_{f^-(\zeta), f^-(\xi)}$ are functions for $\zeta < \xi \leq \theta$

(4) $f_\zeta(\sigma_\zeta) = \sigma'_{f^-(\zeta)}$ for $\zeta < \theta$ where $\sigma_\zeta \in \varphi_\zeta$ and $\sigma'_{f^-(\zeta)} \in \varphi'_{f^-(\zeta)}$ are the relevant splitting points

(5) $f_{\zeta\eta}(c \circ b) = f_{\xi\eta}(c) \circ f_{\zeta\xi}(b)$ for $b \in \mathcal{G}_{\zeta\xi}$, $c \in \mathcal{G}_{\xi\eta}$, $\zeta < \xi < \eta \leq \theta$

(6) $f_\xi \circ b = f_{\zeta\xi}(b) \circ f_\zeta$ for $b \in \mathcal{G}_{\zeta\xi}$, $\zeta < \xi \leq \theta$.

1.2 Definition (iii). Let $\mathcal{M} = \langle \vec{\varphi}, \vec{\vec{\mathcal{G}}} \rangle$ be an **initial segment** of $\mathcal{M}' = \langle \vec{\varphi}', \vec{\vec{\mathcal{G}}}' \rangle$, that is $\theta < \theta'$, $\varphi'_\zeta = \varphi_\zeta$ for $\zeta \le \theta$ and $\mathcal{G}'_{\zeta\xi} = \mathcal{G}_{\zeta\xi}$ for $\zeta < \xi \le \theta$. Then the embedding $f : \mathcal{M} \to \mathcal{M}'$ is called a **left branching embedding** iff $f^- \restriction \theta = \mathrm{id} \restriction \theta$, $f_\zeta = \mathrm{id} \restriction \varphi_\zeta$ for $\zeta < \theta$, $f_{\zeta\xi} = \mathrm{id} \restriction \mathcal{G}_{\zeta\xi}$ for $\zeta < \xi < \theta$, $f^-(\theta) = \theta'$, $f_\theta \in \mathcal{G}'_{\theta\theta'}$ and (by (vii)6))

$$f_{\zeta\theta}(b) = f_{\zeta\theta}(b) \circ f_\zeta = f_\theta \circ b \qquad \text{for } \zeta < \theta \quad \text{and } b \in \mathcal{G}_{\zeta\theta}.$$

1.2 Definition (iv). $f : \mathcal{M} \to \mathcal{M}'$ is called a **right-branching embedding** *iff* $\mathcal{M} = \langle \vec{\varphi}, \vec{\vec{\mathcal{G}}} \rangle$ is an initial segment of $\mathcal{M}' = \langle \vec{\varphi}', \vec{\vec{\mathcal{G}}}' \rangle$ and for some ordinal $\eta \le \theta$ we have:

(1) $f^- \restriction \eta = \mathrm{id} \restriction \eta$ and $f^-(\eta + \zeta) = \theta + \zeta$ if $\eta + \zeta \le \theta$

(2) $f_\zeta = \mathrm{id} \restriction \varphi_\zeta$ for $\zeta < \eta$ and $f_\eta \in \mathcal{G}'_{\eta\theta}$

(3) $f_{\zeta\xi} = \mathrm{id} \restriction \mathcal{G}_{\zeta\xi}$ for $\zeta < \xi < \eta$
and $f_{\zeta\xi}{}''\mathcal{G}_{\zeta\xi} = \mathcal{G}'_{f^-(\zeta), f^-(\xi)}$ for $\eta \le \zeta < \xi \le \theta$.

η is called the **splitting point** of f.

1.2 Definition (v). If $\mathcal{M}$, $\mathcal{M}'$ are as in (iii), then $\mathcal{F}$ is an **amalgam** iff it contains all left-branching $\mathcal{M} \to \mathcal{M}'$ embeddings (for all $f_\theta \in \mathcal{G}'_{\theta\theta'}$), exactly *one* right-branching embedding, and nothing else.

1.2 Definition (vi). The **composition** $h = g \circ f$ of the embeddings $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{M}' \to \mathcal{M}''$ is straightforward: $h : \mathcal{M} \to \mathcal{M}''$ where $h^- = g^- \circ f^-$, $h_\zeta = g_{f^-(\zeta)} \circ f_\zeta$ for any $\zeta \le \mathrm{ht}(\mathcal{M})$, and $h_{\zeta\xi} = g_{f^-(\zeta), f^-(\xi)} \circ f_{\zeta\xi}$ for any $\zeta < \xi \le \mathrm{ht}(\mathcal{M})$.

Now follows the definition itself:

1.2 Definition (vii). The structure $\mathfrak{M} = \langle \vec{\varphi}, \vec{\vec{\mathcal{G}}}, \vec{\theta}, \vec{\vec{\mathcal{F}}} \rangle$ is called a $(\kappa, 2)$-**simplified morass**, or $(\kappa, 2)$-SM for short, *iff*

(0) (a) $\mathcal{M}_\kappa = \langle \vec{\varphi}, \vec{\vec{\mathcal{G}}} \rangle$ is a $(\kappa^+, 1)$-simplified morass

(b) $\vec{\theta} = \langle \theta_\alpha : \alpha \le \kappa \rangle$ where $\theta_\alpha < \kappa$ for $\alpha < \kappa$, $\theta_\kappa = \kappa^+$, and further the structures $\mathcal{M}_\alpha := \langle \vec{\varphi} \restriction (\theta_\alpha + 1), \vec{\vec{\mathcal{G}}} \restriction (\theta_\alpha + 1) \rangle$ are gap-1 fake morasses of height θ_α, further $\mathcal{M}_\alpha$ is an initial segment of $\mathcal{M}_\beta$ for $\alpha < \beta \le \kappa$.

(c) $\mathcal{M}_\alpha$ are of size less than κ for $\alpha < \kappa$, (that is, $\varphi_\zeta < \kappa$ for $\zeta \le \theta_\alpha$ and $|\mathcal{G}_{\zeta\xi}| < \kappa$ for $\zeta < \xi \le \theta_\alpha$ and for $\alpha < \kappa$).

(d) $\overset{\Rightarrow}{\mathcal{F}} = \langle \mathcal{F}_{\alpha\beta} : \alpha < \beta \leq \kappa \rangle$ where $\mathcal{F}_{\alpha,\beta}$ are sets of $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ embeddings for $\alpha < \beta \leq \kappa$.

(1) $(\forall\, \alpha < \beta < \kappa) \quad |\mathcal{F}_{\alpha\beta}| < \kappa$

(2) $(\forall\, \alpha < \beta < \gamma \leq \kappa) \quad \mathcal{F}_{\alpha\gamma} = \mathcal{F}_{\beta\gamma} \circ \mathcal{F}_{\alpha\beta} = \{f \circ g : f \in \mathcal{F}_{\beta\gamma},\ g \in \mathcal{F}_{\alpha\beta}\}$

(3) $(\forall\, \alpha < \kappa) \quad \mathcal{F}_{\alpha\alpha+1}$ is an amalgam

(4) For $\alpha \leq \kappa$ limit, $\beta_1, \beta_2 < \alpha$, $f_1 \in \mathcal{F}_{\beta_1\alpha}$, $f_2 \in \mathcal{F}_{\beta_2\alpha}$ there exists a γ: $\beta_1, \beta_2 < \gamma < \alpha$ and $h_1 \in \mathcal{F}_{\beta_1\gamma}$, $h_2 \in \mathcal{F}_{\beta_2\gamma}$, $g \in \mathcal{F}_{\gamma\alpha}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(5) $(\forall\, \alpha \leq \kappa$ limit$)$

(a) $\theta_\alpha = \bigcup\{f^{-\,\prime\prime}\theta_\beta : f \in \mathcal{F}_{\beta\alpha}\}$

(b) $(\forall\, \zeta \leq \theta_\alpha)$
$\varphi_\zeta = \bigcup\{f_{\overline{\zeta}}{}^{\prime\prime}\varphi_{\overline{\zeta}} : f^-(\overline{\zeta}) = \zeta$ where $(\exists\beta < \alpha)\ f \in \mathcal{F}_{\beta\alpha}\ \&\ \overline{\zeta} \leq \theta_\beta\}$

(c) $(\forall\, \zeta < \xi \leq \theta_\alpha)$

$\mathcal{G}_{\zeta\xi} = \bigcup\{f_{\overline{\zeta}\,\overline{\xi}}{}^{\prime\prime}\mathcal{G}_{\overline{\zeta}\,\overline{\xi}} : f^-(\overline{\zeta}) = \zeta,\ f^-(\overline{\xi}) = \xi,\ (\exists\beta < \alpha)\ f \in \mathcal{F}_{\beta\alpha}\ \&\ \overline{\zeta}, \overline{\xi} \leq \theta_\beta\}$.

End of Definition 1.2. □

Velleman in [Ve8] forced $(\kappa, 2)$-simplified morasses for any regular $\kappa \geq \omega_0$ while Jensen in [Je2] constructed gap-2 simplified morasses from his original gap-2 morasses which are usually are constructed in L for $\kappa \geq \omega_1$.

Further Velleman proved in [Ve9] the following interesting result in ZFC: *"There exists an $(\omega_0, 2)$-simplified morass* iff *there is an $(\omega_1, 1)$-simplified morass."* (Note that $\mathcal{M}_\kappa$ is always an $(\kappa^+, 1)$-simplified morass.)

Further, every gap-2 morass contains a gap-1 one of the *same* height: it is esy to see, using the notation of Definition 1.2, that $\langle \vec{\theta}, \overset{\Rightarrow}{\mathcal{H}} \rangle$ is a $(\kappa, 1)$-simplified morass, where $\overset{\Rightarrow}{\mathcal{H}} = \langle \mathfrak{h}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle$ and $\mathfrak{h}_{\alpha,\beta} = \{f^- \restriction \theta_\alpha : f \in \mathcal{F}_{\alpha,\beta}\}$ for $\alpha < \beta \leq \kappa$.

1.b Higher gap simplified morasses

In this section we present our definition of higher gap simplified morasses. The definition is by induction on the gap of the morass. More precisely we define several notions in connection with simplified morasses by *simultaneous induction* on their gap in Definition 1.3 through 1.10.

In what follows all morasses are *simplified* ones.

The first of these definitions handles the gap-0 case.

Definition 1.3 (i). $\mathcal{M}$ is a gap-0 **simplified morass segment** (SMS) *iff* $\mathcal{M} = \theta + 1$ is a successor ordinal. The **height** of $\mathcal{M}$ is $\mathrm{ht}(\mathcal{M}) = \theta$.
$\mathcal{M}$ is a $(\kappa, 0)$**-morass** (SM) iff $\mathrm{ht}(\mathcal{M}) = \kappa$ is a regular cardinal.
$\mathcal{M}$ is an **initial segment of** $\mathcal{N}$ iff $\mathcal{M} \leq \mathcal{N}$ (they are ordinals!). We denote this fact by $\mathcal{M} \leq \mathcal{N}$.

(ii) For gap-0 SMS's $\mathcal{M} = \theta+1$ and $\mathcal{N} = \Xi+1$ we say that $f : \mathcal{M} \to \mathcal{N}$ is a **gap-0 embedding** iff f is an order-preserving function from $\mathcal{M}$ to $\mathcal{N}$ and $f(\theta) = \Xi$.

(iii) An embedding $f : \mathcal{M} \to \mathcal{N}$ is called **shift** or **right branching** iff for some ordinal $\sigma < \theta$ we have $f(\xi) = \xi$ for $\xi < \sigma$ and $f(\sigma + \zeta) = \theta + \zeta$ for $\sigma + \zeta \leq \theta$ (and $f(\theta) = \Xi$ of course) where $\mathcal{M} = \theta + 1$ and $\mathcal{N} = \Xi + 1$. In this case σ is called the **splitting point** of f.

(iv) $\mathrm{id}_\theta : \theta \to \theta$ is the well known **identity** function (the **identity embedding on** θ).

(v) A family $\mathcal{F}$ of $\mathcal{M} \to \mathcal{N}$ embeddings is called an **amalgam** iff $\mathcal{F} = \{d, r\}$ where r is a shift and $d \restriction \theta = \mathrm{id}_\theta$ and $d(\theta) = \Xi$, where $\mathcal{M} = \theta + 1$ and $\mathcal{N} = \Xi + 1$. □

In what follows $n < \omega_0$ is a fixed natural number. The Definitions 1.4 through 1.10 below are made *simultaneously* by induction on n, the gap size of our morasses.

Definition 1.4. For any fixed $n < \omega_0$

(i) $\mathcal{M} = \langle \overset{\Rightarrow}{\mathcal{M}}, \overset{\Rightarrow}{\mathcal{F}} \rangle$ is a **gap-**$(n+1)$ **simplified morass segment (SMS) of height** θ (ie. $\mathrm{ht}(\mathcal{M}) = \theta$) iff θ is any ordinal and

(0) $\vec{\mathcal{M}} = \langle \mathcal{M}_i : i \leq \theta \rangle$ is a sequence of gap-n SMS's, $\mathcal{M}_i \leq \mathcal{M}_j$ are initial segments for $i < j \leq \theta$. Further, $\overset{\Rightarrow}{\mathcal{F}} = \langle \mathcal{F}_{ij} : i < j \leq \theta \rangle$ where $\mathcal{F}_{ij}$ is a family of gap-n $\mathcal{M}_i \to \mathcal{M}_j$ embeddings for $i < j \leq \theta$, satisfying the below properties:

(a) $\mathcal{F}_{i,i+1}$ is an amalgam with splitting point $\sigma_i < \mathrm{ht}(\mathcal{M}_i)$ for $i < \theta$

(b) $\mathcal{F}_{ij} = \mathcal{F}_{kj} \circ \mathcal{F}_{ik} = \{f \circ g : f \in \mathcal{F}_{kj},\ g \in \mathcal{F}_{ik}\}$ for $i < k < j \leq \theta$ (composition)

(c) $\mathcal{M}_j = \bigcup \{f'' \mathcal{M}_i : f \in \mathcal{F}_{ij},\ i < j\}$ for $j \leq \theta$ (covering property)

(d) For every $j \leq \theta$ limit, $i < j$ and $f_1, f_2 \in \mathcal{F}_{ij}$ there is a k, $i < k < j$ and there are embeddings $g \in \mathcal{F}_{kj}$ and $h_1, h_2 \in \mathcal{F}_{ik}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(ii) The gap-$(n+1)$ SMS $\mathcal{M} = \langle \vec{\mathcal{M}}, \vec{\vec{\mathcal{F}}} \rangle$ is an **initial segment** of the gap-$(n+1)$ SMS $\mathcal{N} = \langle \vec{\mathcal{N}}, \vec{\vec{\mathcal{G}}} \rangle$, denoted by $\mathcal{M} \leq \mathcal{N}$ iff $\mathrm{ht}(\mathcal{M}) \leq \mathrm{ht}(\mathcal{N})$, $\mathcal{M}_j = \mathcal{N}_j$ and $\mathcal{F}_{ij} = \mathcal{G}_{ij}$ for $i < j \leq \mathrm{ht}(\mathcal{M})$. $\square$

We again stress that all the notions we use are already defined by simultaneous induction on the gap of the morass (that is, on n).

Definition 1.5. *(i)* Let $\mathcal{M} = \langle \vec{\mathcal{M}}, \vec{\vec{\mathcal{F}}} \rangle$ and $\mathcal{N} = \langle \vec{\mathcal{N}}, \vec{\vec{\mathcal{G}}} \rangle$ be gap-$(n+1)$ SMS's such that $\mathcal{M}$ is an initial segment of $\mathcal{N}$, $\mathrm{ht}(\mathcal{M}) = \theta$ and $\mathrm{ht}(\mathcal{N}) = \Xi$. Then $f = \langle f^-, \vec{f}, \vec{\vec{f}} \rangle$ is called an $\mathcal{M} \to \mathcal{N}$ gap-$(n+1)$ **embedding** *iff*

(a) $f^- : (\theta+1) \to (\Xi+1)$ is an order-preserving function, $f^-(\theta) = (\Xi)$

(b) $\vec{f} = \langle f_i : i \leq \theta \rangle$ where $f_i : \mathcal{M}_i \to \mathcal{N}_{f^-(i)}$ are gap-n embeddings for $i \leq \theta$

(c) $\vec{\vec{f}} = \langle f_{ij} : i < j \leq \theta \rangle$ where

$$f_{i,j} : \mathcal{F}_{i,j} \to \mathcal{G}_{f^-(i),f^-(j)}$$

are functions for $i < j \leq \theta$ satisfying properties (d) through (f) below

(d) if $h \in \mathcal{F}_{i,i+1}$ is a (gap-n) shift embedding with splitting point $\sigma_i < \mathrm{ht}(\mathcal{M}_i)$, then

$$f_{i,i+1}(h) \in \mathcal{G}_{f^-(i),f^-(i+1)}$$

is also a shift embedding with splitting point $(f_i)^-(\sigma_i) < \mathrm{ht}(\mathcal{M}_{f^-(i)})$ for all $i < \theta$

(e) $f_{ij}(c \circ b) = f_{kj}(c) \circ f_{ik}(b)$ for all $b \in \mathcal{F}_{ik}$, $c \in \mathcal{F}_{kj}$ and $i < k < j \leq \theta$

(f) $f_j \circ b = f_{ij}(b) \circ f_i$ and

$$\mathrm{Range}\,(f_j \circ b) = \mathrm{Range}\,(f_{ij}(b)) \cap \mathrm{Range}\,(f_j)$$

for all $b \in \mathcal{F}_{ij}$, $i < j \leq \theta$.

(ii) The **identity embedding** $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is defined as $\mathrm{id}_{\mathcal{M}} := \langle f^-, \vec{f}, \vec{\vec{f}} \rangle$ where $f^- = \mathrm{id}_{\mathrm{ht}(\mathcal{M})}$, $f_i = \mathrm{id}_{\mathcal{M}_i}$ and $f_{ij} = \mathrm{id}_{\mathcal{F}_{ij}}$ for $i < j \leq \mathrm{ht}(\mathcal{M})$. $\square$

Definition 1.6. If $f = \langle f^-, \vec{f}, \vec{\vec{f}} \rangle$ is as in Definition 1.5(i) then we define

$$\begin{aligned} f''(\mathcal{M}) &:= \text{Range}\,(f) \\ &:= \text{Range}\,(f^-) \bigcup \cup\{\text{Range}\,(f_i) : i \leq \theta\} \bigcup \cup\{\text{Range}\,(f_{ij}) : i < j \leq \theta\}. \end{aligned}$$

(Here we use disjoint unions.) □

The *composition* and *identity* of gap-$(n+1)$ embeddings can be easily defined and are left to the reader.

Definition 1.7. Let $\mathcal{M} = \langle \vec{\mathcal{M}}, \vec{\vec{\mathcal{F}}} \rangle$ and $\mathcal{N} = \langle \vec{\vec{\mathcal{N}}}, \vec{\vec{\mathcal{G}}} \rangle$ be gap-$(n+1)$ SMS's, $\mathcal{M} \leq \mathcal{N}$ is initial segment, $\text{ht}(\mathcal{M}) = \theta$, $\text{ht}(\mathcal{N}) = \Xi$ and let $f = \langle f^-, \vec{f}, \vec{\vec{f}} \rangle$ be an $\mathcal{M} \to \mathcal{N}$ gap-$(n+1)$ embedding. Then

(i) f is **left branching** iff $f^- \restriction \theta = \text{id}_\theta$, $f^-(\theta) = \Xi$, $f_i = \text{id}_{\mathcal{M}i}$ for $i < \theta$, $f_\theta \in \mathcal{G}_{\theta\Xi}$, $f_{ij} = \text{id}_{\mathcal{F}_{ij}}$ for $i < j < \theta$ and $f_{i\theta}(b) = f_\theta \circ b$ for $i < \theta$, $b \in \mathcal{F}_{i\theta}$.

(ii) f is a **shift** or **right branching** iff for some ordinal $\sigma < \theta$ (the **splitting point** of f) we have

(a) $f^-(i) = i$ for $i < \sigma$

(b) $f^-(\sigma + \zeta) = \theta + \zeta$ for $\sigma + \zeta \leq \theta$

(c) $f_i = \text{id}_{\mathcal{M}_i}$ for $i < \sigma$

(d) $f_{ij} = \text{id}_{\mathcal{F}_{ij}}$ for $i < j < \sigma$

(e) $f_\sigma \in \mathcal{F}_{\sigma\theta}$

(f) $\mathcal{G}_{f^-(i), f^-(j)} = f_{i,j}{}''\mathcal{F}_{i,j}$ for $\sigma \leq i < j \leq \theta$. □

Definition 1.8. A family $\mathcal{F}$ of $\mathcal{M} \to \mathcal{N}$ gap-$(n+1)$ embeddings is called an **amalgam** iff $\mathcal{F}$ contains all possible left branching and exactly one right branching embeddings (shift) and nothing else. □

Definition 1.9. For any gap-0 SMS $\mathcal{M} = \theta + 1$ the **size** of $\mathcal{M}$ is $|\mathcal{M}|$. For gap-$(n+1)$ SMS $\mathcal{M} = \langle \vec{\mathcal{M}}, \vec{\vec{\mathcal{F}}} \rangle$ the size of $\mathcal{M}$ is defined by induction on n as

$$|\mathcal{M}| := |\,\text{ht}(\mathcal{M})| + \sum_{i \leq \text{ht}(\mathcal{M})} |\mathcal{M}_i| + \sum_{i < j \leq \text{ht}(\mathcal{M})} |\mathcal{F}_{ij}|$$

which is a cardinal. □

Definition 1.10. Let κ be any regular cardinal.

(i) The gap-0 SMS $\mathcal{M}$ is called a gap-0 **simplified morass** of height κ, or $(\kappa,0)$-SM for short, iff $\mathcal{M}=\kappa+1$.

(ii) For any finite n the gap-$(n+1)$ SMS $\mathcal{M}=\langle\vec{\mathcal{M}},\vec{\vec{\mathcal{F}}}\rangle$ is a **simplified morass** of height κ, or $(\kappa,n+1)$-SM for short, iff

(a) $\mathrm{ht}(\mathcal{M})=\kappa+1$

(b) $|\mathcal{F}_{ij}|<\kappa$ for $i<j<\kappa$

(c) $|\mathcal{M}_i|<\kappa$ for $i<\kappa$

(d) $\mathcal{M}_\kappa$ is a (κ^+,n) simplified morass.

This is the end of the inductive definition of gap-n simplified morasses. □

We stated that our definition in gap-1 and gap-2 cases covers Velleman's definitions. The careful reader may observe that all our Definitions 1.3 through 1.10 deal with structures of successor length whileVelleman's gap-1 definition does not (see Definition 1.1). This is a technical difference only: from any gap-1 morass we can construct another one, in which each φ_α is a successor ordinal for $\alpha<\kappa$, see e.g. [Ve3].

Below we define some special parts of higher gap simplified morasses (we could call them "skeletons") to state the result that any higher gap simplified morass contains several smaller gap simplified morasses. Other special parts of higher gap morasses are defined in [Sz2]. That work also contains a list of basic properties of higher gap simplified morasses.

All the parts of the below definition are made simultaneously by induction on the gap.

Definition 1.11. (a/i) The t-th **reduct** $\mathrm{red}_t(\mathcal{M})$ of a (κ,n)-simplified morass segment $\mathcal{M}=\langle\vec{\mathcal{M}},\vec{\vec{\mathcal{F}}}\rangle$ for $t\le n$ is the below gap-$(n-t)$ simplified morass segment: $\mathrm{red}_0(\mathcal{M}):=\mathcal{M}$, $\mathrm{red}_n(\mathcal{M}):=\theta+1$, and for $0<t<n$

$$\mathrm{red}_t(\mathcal{M}):=\big\langle\langle\mathrm{red}_t(\mathcal{M}_i):i\le\theta\rangle,\ \langle\mathrm{red}_t(\mathcal{F}_{ij}):i<j\le\theta\rangle\big\rangle$$

where $\vec{\mathcal{M}}=\langle\mathcal{M}_i:i\le\theta\rangle$, $\vec{\vec{\mathcal{F}}}=\langle\mathcal{F}_{ij}:i<j\le\theta\rangle$,

$$\mathrm{red}_t(\mathcal{F}_{ij})=\{\mathrm{red}_t(f):f\in\mathcal{F}_{ij}\}$$

for $i<j\le\theta$, and

(a/ii) The t-th **reduct** $\mathrm{red}_t(f)$ of a gap-n embedding $f : \mathcal{M} \to \mathcal{N}$ between simplified morass segments $\mathcal{M} = \langle \vec{\mathcal{M}}, \vec{\vec{\mathcal{F}}} \rangle$ and $\mathcal{N} = \langle \vec{\mathcal{N}}, \vec{\vec{\mathcal{G}}} \rangle$, $f = \langle f^-, \vec{f}, \vec{\vec{f}} \rangle$ for $0 \le t \le n$ is the below gap-$(n-t)$ embedding:

$$\mathrm{red}_t(f) : \mathrm{red}_t(\mathcal{M}) \to \mathrm{red}_t(\mathcal{N})$$

where $\mathrm{red}_0(f) := f$, $\mathrm{red}_n(f) := f^-$, and for $0 < t < n$

$$\mathrm{red}_t(f) := \langle f^-, \mathrm{red}_t(\vec{f}), \mathrm{red}_t(\vec{\vec{f}}) \rangle$$

where

$$\mathrm{red}_t(\vec{f}) := \langle \mathrm{red}_t(f_i) : i \le \theta \rangle$$

and

$$\mathrm{red}_t(\vec{\vec{f}}) := \langle \mathrm{red}_t(f_{ij}) : i < j \le \theta \rangle$$

where

$$\mathrm{red}_t(f_{ij}) : \mathrm{red}_t(b) \mapsto \mathrm{red}_t(f_{ij}(b))$$

for $b \in \mathcal{F}_{ij}$ and $i < j \le \theta$, and as usual, $\vec{f} = \langle f_i : i \le \theta \rangle$ and $\vec{\vec{f}} = \langle f_{ij} : i < j \le \theta \rangle$.

(b) The $P_{\mathcal{M}}(\kappa^{+s}, m)$**-reducts** of a (κ, n)-simplified morass $\mathcal{M}$ for $0 \le s$, $m \le n$, $m + s \le n$ are defined as follows:

$$P_{\mathcal{M}}(\kappa^{+s}, m) := P_{\mathcal{M}_\kappa}(\kappa^{+s}, m) \text{ for } s \ge 1, \text{ and } P_{\mathcal{M}}(\kappa, m) := \mathrm{red}_{n-m}(\mathcal{M})$$

where $\mathcal{M} = (\vec{\mathcal{M}}, \vec{\vec{\mathcal{F}}})$, $\vec{\mathcal{M}} = \langle \mathcal{M}_i : i \le \kappa \rangle$ and $\vec{\vec{\mathcal{F}}} = \langle \mathcal{F}_{ij} : i < j \le \kappa \rangle$. □

Statement 1.12. If $\mathcal{M}$ is a (κ, n)-simplified morass, then $P_{\mathcal{M}}(\kappa^{+s}, m)$ are (κ^{+s}, m)-simplified morasses for $0 \le s$, $m \le n$, $m + s \le n$.

The statement can be proved by induction, using the inductive definition of our morasses. □

2. Morgan's definition

We shall prove in Theorem 3.7 by using Charles Morgan's [Mo] variant of simplified morasses and his result, that the existence of morasses of his kind are equivalent to the existence of original morasses of R. Jensen.

In this section we present Morgan's definition from [Mo].

First we need some notation, also from [Mo].

2.0 Notation. In what follows $\underline{s}$ is a sequence of ordinals, $\underline{s} = \langle s_1, \dots, s_j \rangle$, the **length** of $\underline{s}$ is $|\underline{s}| := j$.[2] $\varnothing$ is the empty sequence and $|\varnothing| = 0$. We never mix the ordinal ξ and the singleton $\langle \xi \rangle$. $\underline{s}\widehat{\ }\underline{r}$ denotes the concatenation of the sequences $\underline{s}$ and $\underline{r}$. Finally, for any $k \leq |\underline{s}|$ we let $\underline{s} \restriction k := \langle s_1, s_2, \dots, s_k \rangle$ the restriction of $\underline{s}$.[3] Especially we let $\underline{s} \restriction 0 := \varnothing$. Finally we let $\underline{s}\restriction := \underline{s} \restriction (|\underline{s}| - 1)$, which e.g. implies $\langle \xi \rangle \restriction = \varnothing$ for any sigleton $\langle \xi \rangle$. In some complicated formulas we write simply s instead of $\underline{s}$. $\square$

Now we quote verbatim Ch. Morgan's original definition from [Mo]. We only made some highlights, footnotes, and tried to correct all the misprints of the manuscript [Mo].

To understand the forthcoming definition easier let us make some of our own comments before. The morass consists a lot of sequences and functions among them, all arranged in a sense of a hierarchy of n level, where $n < \omega_0$ is fixed, denotes the gap of the morass. To belong to the i'th hierarchy level for a sequence $\underline{s}$, we can not say simply that its length is *at most* $n - i$, we have also restrictions on its entries. This complicated assumption is declared in $(M + i)$ below. This is similar to the notion of "co-ordinates" of our inductive higher gap morasses, defined and used in connection to applications in partition calculus in [Sz2].

We again emphasize that we quote Morgan's definition verbatim.

2.1 Definition [Mo]. Let n be finite and κ be an infinite cardinal. The structure

$$\left\langle \left\langle \theta^i_\alpha : \alpha \leq \kappa^{+i} \right\rangle,\ \left\langle \mathcal{F}^i_{\alpha,\beta} : \alpha \leq \beta \leq \kappa^{+i} \right\rangle : i < n \right\rangle$$

is called a (κ, n)**-Morgan-simplified morass** *iff* the below properties (M+) and (M0) through (M6) hold for any ordinals $i < n$ and $\alpha \leq \beta \leq \kappa^{+i}$ and for any family of functions $f \in \mathcal{F}^i_{\alpha,\beta}$:

$(M+i)$ $\underline{s} = \langle s_1, \dots, s_j \rangle$ is always a sequence satisfying $|\underline{s}| \leq n - (i+1)$, perhaps $\underline{s} = \varnothing$. In case $\underline{s} \neq \varnothing$ we have $s_1 \leq \theta^i_\alpha$ and $s_{k+1} \leq \theta^{i+k}_{s_k}$ for $0 < k < |\underline{s}| = j$.[4] We write $(\underline{s}) \in (M+i)$ if $\underline{s}$ satisfies the above properties.

---

[2] The indexes run from 1 through j, this is only for simplicity, instead of the usual run from 0 through $j - 1$.

[3] Observe that the definition of $s \restriction k$ is not the usual one, too.

[4] For simplicity, in the next section we shall allow to use the symbol s_0 even in the case $\underline{s} = \varnothing$, when s_0 will denote the α fixed there. This also differs from [Mo].

(M0) $f \in \mathcal{F}^i_{\alpha,\beta}$ is a disjoint family of functions:

$$f = \{f\} \cup \bigcup \Big\{ f_{(\underline{s})\sigma} : \sigma \leq \theta^{i+|s|}_{s_j}, j = |\underline{s}|,\ \underline{s} \in (M+i) \Big\}$$
$$\cup \bigcup \Big\{ f_{(\underline{s})\sigma,\tau} : \sigma \leq \tau \leq \theta^{i+|s|}_{s_j},\ j = |\underline{s}|,\ \underline{s} \in (M+i) \Big\}$$

where

$$f : \theta^i_\alpha + 1 \to \theta^i_\beta + 1$$

and

$$f_{(\underline{s})\sigma} : \theta^{i+|s|+1}_\sigma + 1 \to \theta^{i+|s|+1}_{f_{\underline{s}}(\sigma)} + 1$$

are order-preserving functions,

$$f_{(\underline{s})\sigma,\tau} : \mathcal{F}^{i+|s|+1}_{\sigma,\tau} \to \mathcal{F}^{i+|s|+1}_{f_{\underline{s}}(\sigma), f_{\underline{s}}(\tau)}$$

is a function where $f_\varnothing = f$ and $f_{\underline{s}} = f_{(\underline{s}\restriction)s_j}$ for $j = |\underline{s}|$.[5]

Notation. We are rigorous, so we write $f_\varnothing$ instead of f (the single function in the first bracket). This allows us simply to write f (standard) instead of $\boldsymbol{f}$ (boldface).[6]

Further in what follows for $\underline{s} \in (M+i)$, $|\underline{s}| = j < n-(i+1)$ we define $f^{(}$ as

$$f^{(}(\underline{s}) := \big\langle f_\varnothing(s_1), f_{s_1}(s_2), \ldots, f_{\underline{s}\restriction}(s_j) \big\rangle.^7$$

Of course for $f \in \mathcal{F}^i_{\beta,\gamma}$, $g \in \mathcal{F}^i_{\alpha,\beta}$ $\alpha \leq \beta \leq \gamma \leq \kappa^{+i}$ we *define* $f \circ g \in \mathcal{F}^i_{\alpha,\gamma}$ as:

$$f \circ g := \{f_\varnothing \circ g_\varnothing\} \cup \bigcup \Big\{ f_{g^{(}(\underline{s})\hat{}\, g_{\underline{s}}(\sigma)} \circ g_{s\hat{}\sigma} : \sigma \leq \theta^{i+|s|}_{s_j},\ j = |\underline{s}|,\ \underline{s} \in (M+i) \Big\}$$
$$\cup \bigcup \Big\{ f_{(g^{(}(\underline{s}))g_{\underline{s}}(\sigma), g_{\underline{s}}(\tau)} \circ g_{(\underline{s})\sigma,\tau} : \sigma \leq \tau \leq \theta^{i+|s|}_{s_j},\ j = |\underline{s}|,\ \underline{s} \in (M+i) \Big\}$$

---

[5]This defines the functions $f_{\underline{s}}$ for $\underline{s} \in (M+i)$, $|\underline{s}| < n-(i+1)$ by induction on $|\underline{s}| : f_\varnothing := f$ and $f_{\underline{s}} := f_{(\underline{s}\restriction)s_j}$ where $j = |\underline{s}|$, that is $f_{(\underline{s})\sigma} = f_{\underline{s}\hat{}\sigma}$ and $f = f_\varnothing$.
In [Mo] the values $f_{\underline{s}}(\xi)$ are defined similarly as shorthand, but our definition in our footnote (11) for (M4a) some pages below, is different from the present one, since for e.g. $f_\xi \notin \mathcal{F}^{i+1}_{\xi, f(\xi)}$ by the present definition.

[6]You can not find these notations in [Mo], we use them for precise discussion. Further, on the basis of the previous footnote we could write $f_{\underline{s}\hat{}\sigma}$ instead of $f_{(\underline{s})\sigma}$, but this will denote another thing, see the footnote (11) of (M4a).

[7][Mo] uses $f^{\downarrow}$ instead of $f^{(}$. We altered the notation for technical reason (our text editor) only.

(M1) $(\forall\, j \leq i < n)\ \ (\forall\, \alpha \leq \beta < \kappa^{+i})$

$$\theta^i_\alpha < \kappa^{+j} \quad \text{and} \quad \theta^i_{\kappa^{+i}} = \kappa^{+(i+1)} \quad \text{and}^8 \quad \left|\mathcal{F}^i_{\alpha,\beta}\right| < \kappa^{+i}$$

(M2) $(\forall\, i < n)\ \ (\forall\, \alpha \leq \beta \leq \kappa^{+i})\ \ \left(\forall\, f \in \mathcal{F}^i_{\alpha,\beta}\right)$

(a) $(\forall\, \underline{s} \in (M+i),\ j = |\underline{s}|)\ \left(\forall\, \xi \leq \sigma \leq \tau \leq \theta^{i+|s|}_{s_j}\right)\ \left(\forall\, b \in \mathcal{F}^{i+|s|+1}_{\xi,\sigma}\right)$ $\left(\forall\ c \in \mathcal{F}^{i+|s|+1}_{\sigma,\tau}\right)$

$$f_{(\underline{s})\xi,\tau}(c \circ b) = f_{(\underline{s})\sigma,\tau}(c) \circ f_{(\underline{s})\xi,\sigma}(b)$$

(b) $(\forall\, \underline{s} \in (M+i),\, j = |\underline{s}|)\ \left(\forall\, \xi \leq \tau \leq \theta^{i+|s|}_{s_j}\right)\ \left(\forall\, b \in \mathcal{F}^{i+|s|+1}_{\xi,\tau}\right)$

$$f_{(\underline{s})\tau} \circ b = f_{(\underline{s})\xi,\tau}(b) \circ f_{(\underline{s})\xi}$$

(c)[9] $(\forall\, k < n-(i+1))\ \ (\forall\, \underline{s} \in (M+i))$:
if $|\underline{s}| = k$, $\xi = s_k$ and the splitting point[10] of $\mathcal{F}^{i+k}_{\xi,\xi+1}$ is σ_ξ, then $f_{(\underline{s})}(\sigma_\xi) = \sigma_{f_{(s\restriction)}(\xi)}$

(M3) $(\forall\, i < n)\ \ (\forall\, \alpha \leq \beta \leq \gamma \leq \kappa^{+i})$

$$\mathcal{F}^i_{\alpha,\gamma} = \mathcal{F}^i_{\beta,\gamma} \circ \mathcal{F}^i_{\alpha,\beta} := \left\{ f \circ g : f \in \mathcal{F}^i_{\beta,\gamma},\ g \in \mathcal{F}^i_{\alpha,\beta} \right\}$$

(M4) $(\forall\, i < n)\ \ (\forall\, \alpha \leq \kappa^{+i})\ \ \mathcal{F}^i_{\alpha,\alpha} = \{\mathrm{id}\}$ *and* $(\forall\, \alpha < \kappa^{+i})\ \ \mathcal{F}^i_{\alpha,\alpha+1} = \{d : d \approx \mathrm{id}\} \cup \{h^i_\alpha\}$ *where* id must be clear, $\approx$ and h^i_α are defined as:

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[8][Mo] Chapter IV omitted the assumption $\left|\mathcal{F}^i_{\alpha,\beta}\right| < \kappa^{+i}$ probably only by accident, since in other parts of [Mo] it is required for 1- and 2- gap morasses.

[9](M+) lists all possible sequences and so the requirement "$\forall\, s \in (+) \ldots$" immediately implies:

$$\text{"}\left(\forall\, r_1 \leq \theta^i_\alpha\right)\left(\forall\, r_2 \leq \theta^{i+1}_{r_1}\right)\left(\forall\, r_3 \leq \theta^{i+2}_{r_2}\right)\ldots\left(\forall\, \xi \leq \theta^{i+k}_{r_k}\right)\ (\exists \underline{s} \in (M+i))$$
$$\underline{s} = \langle r_1, r_2, \ldots, r_k, \xi\rangle\text{".}$$

The original definition [Mo,2c] verbatim is the following:
"$\forall \xi\ \exists s\ \exists k = lh(s)$ such that $\xi = s_{k-1}$ and if σ_ξ is the splitting point of $\mathcal{F}^{i+k}_{\xi,\xi+1}$ then $f_{(s)}(\sigma_\xi) = \sigma_{f_{(s\restriction(k-1))}(\xi)}$." NB. in [Mo] $\underline{s} = \langle s_0, \ldots, s_{k-1}\rangle$ if $|s| = k$.

[10]The splitting point will be defined in (M4b).

(4a) $d \approx \mathrm{id}$ (d **is almost the identity**) for $d \in \mathcal{F}^i_{\alpha,\alpha+1}$, *iff* $d_\varnothing \restriction \theta^i_\alpha = \mathrm{id} \restriction \theta^i_\alpha$ and $d_\xi \in \mathcal{F}^{i+1}_{\xi,d_\varnothing(\xi)}$ for $\xi \leq \theta^i_\alpha$,[11,12]
$d_{(\theta^i_\alpha\,\hat{}\,\underline{s})\xi} = \left(d_{\theta^i_\alpha}\right)_{(\underline{s})\xi}$ for $\theta^i_\alpha\,\hat{}\,\underline{s} \in (M+i)$ and $\xi \leq \theta^{i+|s|+1}_\alpha$,
finally $d_{(\underline{s})\xi} = \mathrm{id}$ for $\underline{s} \in (M+i)$ and $s_0 < \theta^i_\alpha$.

(4b) $f \in \mathcal{F}^i_{\alpha,\alpha+1}$ is a **shift** for any $i < n$, $\alpha \leq \kappa^{+i}$ *iff* for some ordinal $\sigma = \sigma^i_\alpha < \theta^i_\alpha$ (**splitting point**) we have $f_\varnothing \restriction \sigma = \mathrm{id} \restriction \sigma$,
$f_\varnothing(\sigma+\tau) = \theta^i_\alpha + \tau$ for $\sigma+\tau \leq \theta^i_\alpha$, $f_{(\underline{s})\xi} = \mathrm{id}$ if $s_1 < \sigma$ *or* $s = \varnothing$ and $\xi < \sigma$, $f_\sigma \in \mathcal{F}^{i+1}_{\sigma,\theta^i_\alpha}$ (NB. $f_\varnothing(\sigma) = \theta^i_\alpha$),
$f_{(\sigma\,\hat{}\,\underline{s})\xi} = (f_\sigma)_{(\underline{s})\xi}$ and $f_{\xi\zeta}{}''\mathcal{F}^{i+1}_{\xi,\zeta} = \mathcal{F}^{i+1}_{f_\varnothing(\xi),f_\varnothing(\zeta)}$ for $\xi \leq \zeta \leq \theta^i_\alpha$.

In what follows, h^i_α denotes a fixed shift function[13].

(M5) $(\forall\, i < n)\ (\forall\, \alpha \leq \kappa^{+i}, \alpha \text{ limit})\ (\forall\, \beta_0, \beta_1 \leq \alpha)\ \left(\forall\, f_0 \in \mathcal{F}^i_{\beta_0,\alpha}, f_1 \in \mathcal{F}^i_{\beta_1,\alpha}\right)$
$(\exists \gamma : \beta_1, \beta_2 < \gamma < \alpha)\ \left(\exists f_0' \in \mathcal{F}^i_{\beta_0,\gamma}, f_1' \in \mathcal{F}^i_{\beta_1,\gamma}\right)\ \left(\exists g \in \mathcal{F}^i_{\gamma,\alpha}\right)$

$$f_0 = g \circ f_0' \quad \text{and} \quad f_1 = g \circ f_1'$$

(M6) $(\forall\, i < n)\ \ (\forall\, \alpha \leq \kappa^{+i},\ \alpha \text{ limit})$[14]

(6a) $\theta^i_\alpha = \cup\left\{f''_\varnothing \theta^i_\beta : f \in \mathcal{F}^i_{\beta\alpha}, \beta < \alpha\right\}$

(6b) $(\forall\, \underline{s} \in (M+i))\ \left(\forall\, \xi \leq \theta^{i+|\underline{s}|}_{s_j},\ j = |\underline{s}|\right)$

$$\theta^{i+|\underline{s}|+1}_\xi = \cup\big\{f_{(\underline{s}')\xi'}{}''\theta^{i+|\underline{s}'|+1}_{\xi'} : f \in \mathcal{F}^i_{\beta\alpha},\ f_{\underline{s}'}(\xi') = \xi,\ f^{\,(}(\,\underline{s}') \\ = \underline{s},\ \beta < \alpha\big\}$$

(6c) $(\forall\, \underline{s} \in (M+i))\ \left(\forall\, \xi \leq \tau \leq \theta^{i+|\underline{s}|}_{s_j},\ j = |\underline{s}|\right)$

$$\mathcal{F}^{i+|\underline{s}|+1}_{\xi\tau} = \cup\big\{f_{(\underline{s}')\xi'\tau'}{}''\mathcal{F}^{i+|\underline{s}'|+1}_{\xi'\tau'} : f \in \mathcal{F}^i_{\beta\alpha},\ f_{\underline{s}'}(\xi') = \xi,\ f_{\underline{s}'}(\tau') = \tau, \\ f^{\,(}(\,\underline{s}') = \underline{s},\ \beta < \alpha\big\}$$

End of the Definition 2.1. □

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[11] [Mo] does not define neither d_ξ nor $d_{\theta^i_\alpha}$. These may be the following:

Definition. For *any* $i < n$, $\alpha \leq \beta \leq \kappa^{+i}$, $f \in \mathcal{F}^i_{\alpha,\beta}$ and $\tau \leq \theta^i_\alpha$ we let

$$f_\tau := \{f_{(\varnothing)\tau}\} \cup \bigcup\{f_{(\underline{s})\xi} \subset f : s_1 = \tau\} \cup \bigcup\{f_{(\underline{s})\xi\zeta} \subset f : s_1 = \tau\}.$$

Now the requirements $f_\tau \in \mathcal{F}^{i+1}_{\tau,f_\emptyset(\tau)}$ and so $d_\xi \in \mathcal{F}^{i+1}_{\xi,d_\emptyset(\xi)}$ are meaningful.

[12] Using the previous assumption and (4), $d_\xi = \mathrm{id} \in \mathcal{F}^{i+1}_{\xi,\xi}$ must be for $\xi \leq \theta^i_\alpha$.

[13] So $\mathcal{F}^i_{\alpha,\alpha+1}$ contains exactly one shift, denoted by h^i_α.

[14] In [Mo] we find "$\forall\, \alpha < \kappa^{+i} \ldots$" which must be a misprint.

3. On the existence of higher gap inductive morasses

In this selection in Theorem 3.7 we show that the existence of higher gap simplified inductive morasses we defined in Section 1 follows from the existence of morasses defined by professor Ronald Jensen in [Je1]. This immediately gives, that in case $V = L$ for every regular κ and finite $n < \omega_0$ there exist (κ, n)-simplified morasses in our sense.

We prove Theorem 3.7 by using Charles Morgan's [Mo] variant of simplified morasses defined in the previous section, and his result, that the existence of morasses of his kind are equivalent to the existence of the original morasses of R. Jensen. We construct our morasses using one of Ch. Morgan's morasses of the same gap.

We think however, the existence of morasses even with full linearizing sequences, defined in [Sz1] or in our thesis [Sz2] could be proved by a (complicated) forcing argument, similar to the one in [Ve8].

Now we start to show how to construct our inductive simplified morasses from Morgan's above defined morasses. To avoid confusion call Morgan's morasses (κ, n)**-Morgan-morasses** and *ours* simply (κ, n)**-simplified-morasses** and **fake morasses** or **morass segments**.

3.1 Theorem. *Let $n \leq \omega_0$, κ be a regular cardinal, both fixed, and let*

$$\mathfrak{U} = \Big\langle \big\langle \theta^i_\alpha : \alpha \leq \kappa^{+i} \big\rangle, \big\langle \mathcal{F}^i_{\alpha,\beta} : \alpha \leq \beta \leq \kappa^{+i} \big\rangle : i < n \Big\rangle$$

be a fixed (κ, n)-Morgan-morass. Then we can construct from $\mathfrak{U}$ a (κ, n)-simplified morass (in the sense of Definitions 1.3 through 1.10).

Proof. We refer simply by (M+) and (M0) through (M6) to the parts of Definition 2.1. $\square$

3.2 The construction. By induction on t $(1 \leq t \leq n)$ we construct the gap-$(t-1)$ fake morasses $\mathcal{N}^t_\xi$ and the families $\mathcal{G}^t_{\xi\zeta}$ of gap-$(t-1)$ mappings among them for $\xi < \zeta \leq \kappa^{+(n-t)}$, using $\mathfrak{U}$. Of course we will take care of the assumption $\mathcal{N}^t_\xi \leq \mathcal{N}^t_\zeta$ for $\xi < \zeta \leq \kappa^{+(n-t)}$. Moreover, the structures

$$\mathcal{M}^t_\tau := \Big\langle \big\langle \mathcal{N}^t_\xi : \xi \leq \tau \big\rangle, \big\langle \mathcal{G}^t_{\xi\zeta} : \xi < \zeta \leq \tau \big\rangle \Big\rangle$$

will be gap-t fake morasses for each $\tau \leq \kappa^{+(n-t)}$, and even $\mathcal{M}^t_{\kappa^{+(n-t)}}$ will be a $(\kappa^{+(n-t)}, t)$-morass. This means, that finally in case $t = n$ the structure $\mathcal{M}^n_\kappa$ will be a (κ, n)-morass.

Each function $g \in \mathcal{G}^t_{\xi\zeta}$ will be of from $g = \mathfrak{g}(f)$ for some $f \in \mathcal{F}^{n-t}_{\xi\zeta}$ for all $\xi < \zeta < \kappa^{+(n-t)}$. We mean that

$$\mathfrak{g} : \mathcal{F}^{n-t}_{\xi\zeta} \to \mathcal{G}^t_{\xi\zeta}$$

is a bijection (one-to-one and onto).

To start with, let $\mathcal{M}^0_\tau := \tau + 1$ for $\tau \leq \kappa^{+n}$ and

$$\mathcal{N}^1_\xi := \mathcal{M}^0_{\theta^{n-1}_\xi} = \theta^{n-1}_\xi + 1$$

for $\xi \leq \kappa^{+n-1}$. Let further $\mathcal{G}^1_{\xi\zeta} := \mathcal{F}^{n-1}_{\xi\zeta}$ for $\xi < \zeta \leq \kappa^{+n-1}$ and

$$\mathcal{M}^1_\tau := \Big\langle \langle \mathcal{N}^1_\xi : \xi \leq \tau \rangle, \langle \mathcal{F}^{n-1}_{\xi\zeta} : \xi < \zeta \leq \tau \rangle \Big\rangle$$

for $\tau \leq \kappa^{+n-1}$.

3.3 Statement. $\mathcal{M}^1_\tau$ is a 1-gap fake morass for $\tau \leq \kappa^{+(n-1)}$ and $\mathcal{N}^1_{\kappa^{+(n-1)}}$ is a $\big(\kappa^{+(n-1)}, 1\big)$-morass.

Proof. Using $(M + i)$ we have only $\varnothing \in (M + i)$ since $i = n - 1$. So each element of $\mathcal{G}^1_{\xi\zeta} = \mathcal{F}^{n-1}_{\xi\zeta}$ is of form $\theta^{n-1}_\xi \to \theta^{n-1}_\zeta$ by (M0) (ie. $f = f_\varnothing$ for $f \in \mathcal{F}^{n-1}_{\xi\zeta}$) and these functions are order preserving. By (M1) we have $\theta^{n-1}_\xi < \kappa^{+(n-1)}$ and $\big|\mathcal{F}^{n-1}_{\xi\zeta}\big| < \kappa^{+(n-1)}$ for $\xi < \zeta < \kappa^{+(n-1)}$, and $\theta^{n-1}_{\kappa^{+(n-1)}} = \kappa^{+n}$. By (M3) we have

$$\mathcal{F}^{n-1}_{\xi,\eta} = \mathcal{F}^{n-1}_{\zeta,\eta} \circ \mathcal{F}^{n-1}_{\xi,\zeta}$$

for $\xi < \zeta < \eta \leq \kappa^{+n-1}$.

By (M4) $F^i_{\xi,\xi+1} = \{\mathrm{id}_\approx, h_\xi\}$ and $\mathrm{id}_\approx \restriction \theta^{n-1}_\xi = \mathrm{id} \restriction \theta^{n-1}_\xi$, $\mathrm{id}_\approx(\theta^{n-1}_\xi) = \theta^{n-1}_{\xi+1}$, $h_\xi \restriction \sigma_\xi = \mathrm{id} \restriction \sigma_\xi$ (the ordinary identity) and $h_\xi(\sigma_\xi + \tau) = \theta^{n-1}_\xi + \tau$ for $\sigma_\xi + \tau \leq \theta^{n-1}_\xi$ for some $\sigma_\xi < \theta^{n-1}_\xi$ and for each $\xi < \kappa^{+(n-1)}$. (M5) ensures the amalgam and (M6) the covering property. $\square$

The inductive step

Suppose that we have constructed the gap-$(t-1)$ fake morasses $\mathcal{N}^t_\xi$ for some fixed t $(1 \leq t < n)$ and the families $\mathcal{G}^t_{\xi\zeta}$ of embeddings of type $\mathcal{N}^t_\xi \to \mathcal{N}^t_\zeta$ have already been defined for all $\xi < \zeta \leq \kappa^{+(n-t)}$ such that $\mathcal{N}^t_\xi \leq \mathcal{N}^t_\zeta$ for $\xi < \zeta \leq \kappa^{+(n-t)}$. Suppose further, that the structures

$$\mathcal{M}^t_\tau := \Big\langle \langle \mathcal{N}^t_\xi : \xi \leq \tau \rangle, \langle \mathcal{G}^t_{\xi\zeta} : \xi < \zeta \leq \tau \rangle \Big\rangle$$

are gap-t fake morasses for each $\tau \leq \kappa^{+(n-t)}$ and moreover that $\mathcal{M}^t_{\kappa^{+(n-t)}}$ is a $\left(\kappa^{+(n-t)}, t\right)$-morass. Recall further, that $\mathcal{G}^t_{\xi\zeta} = \mathfrak{g}''\mathcal{F}^{n-t}_{\xi\zeta}$ for $\xi < \zeta \leq \kappa^{+(n-t)}$ for some function $\mathfrak{g} : \mathcal{F}^{n-t}_{\xi\zeta} \to \mathcal{G}^t_{\xi\zeta}$ by our inductive construction.

Let now define $\mathcal{N}^{t+1}_\alpha := \mathcal{M}^t_{\theta^{n-(t+1)}_\alpha}$ for $\alpha \leq \kappa^{+n-(t+1)}$. Next, first of all we have to define the elements of $\mathcal{G}^{t+1}_{\alpha\beta}$, that is for each (Morgan's) function $f \in \mathcal{F}^{n-(t+1)}_{\alpha\beta}$ we have to define our embedding $\mathfrak{g}(f) : \mathcal{N}^{t+1}_\alpha \to \mathcal{N}^{t+1}_\beta$ (in the sense of Definition 1.5) for all $\alpha < \beta \leq \kappa^{+n-(t+1)}$. Clearly we will then take $\mathcal{G}^{t+1}_{\alpha\beta} := \left\{\mathfrak{g}(f) : f \in \mathcal{F}^{n-(t+1)}_{\alpha\beta}\right\}$.

Let $\alpha < \beta \leq \kappa^{+n-(t+1)}$ and $f \in \mathcal{F}^{n-(t+1)}_{\alpha\beta}$ be fixed, and let further $i = n-(t+1)$. First, for all sequence $\underline{s} \in (M+i)$, $|\underline{s}| \leq t$ we define the *set* of functions $\mathfrak{g}_0(f, \underline{s})$ by *descending* induction on $|\underline{s}|$.

For $\underline{s} = \langle s_1, \ldots, s_t\rangle$ we put $\mathfrak{g}_0(f, \underline{s}) := (\mathfrak{g}_0(f, \underline{s}))^- := \left\{f_{(\underline{s}\restriction)s_t}\right\}$.

For $|\underline{s}| < t$, $\underline{s} = \langle s_1, \ldots, s_j\rangle$ we let

$$\mathfrak{g}_0(f, \underline{s}) := (\mathfrak{g}_0(f, \underline{s}))^- \cup (\mathfrak{g}_0(f, \underline{s}))^{\rightarrow} \cup (\mathfrak{g}_0(f, \underline{s}))^{\Rightarrow}$$

where

$$(\mathfrak{g}_0(f, \underline{s}))^- := \left\{f_{(\underline{s}\restriction)s_j}\right\},$$
$$(\mathfrak{g}_0(f, \underline{s}))^{\rightarrow} := \left\{\mathfrak{g}_0(f, \underline{s}^\frown\langle\xi\rangle) : \xi \leq \theta^{n-(t+1)+j}_{s_j}\right\}$$

and

$$(\mathfrak{g}_0(f, \underline{s}))^{\Rightarrow} = \left\{(\mathfrak{g}_0(f, \underline{s}))_{\xi\zeta} : \xi < \zeta \leq \theta^{n-(t+1)+j}_{s_j}\right\}$$

where

$$(\mathfrak{g}_0(f, \underline{s}))_{\xi\zeta} : \mathcal{G}^{t-|s|}_{\xi,\zeta} \to \mathcal{G}^{t-|s|}_{h(\xi),h(\zeta)} \qquad (h := (\mathfrak{g}_0(f, \underline{s}))^-)$$
$$\mathfrak{g}(b) \mapsto \mathfrak{g}\big(f_{(\underline{s})\xi,\zeta}(b)\big) \quad \text{for} \quad \xi < \zeta \leq \theta^{n-(t+1)+j}_{s_j}.$$

Now we are able to define $\mathfrak{g}(f) := \mathfrak{g}_0(f, \varnothing)$ where $(\mathfrak{g}_0(f, \varnothing)) := f_\varnothing$. This defines $\mathfrak{g}(f)$ for all $f \in \mathcal{F}^{n-(t+1)}_{\alpha\beta}$ and so we can let

$$\mathcal{G}^{t+1}_{\alpha\beta} := \left\{\mathfrak{g}(f) : f \in \mathcal{F}^{n-(t+1)}_{\alpha\beta}\right\}.$$

Remarks. Recall, that by induction we have $\mathcal{G}_{\xi\zeta}^{t-|\underline{s}|} = \mathfrak{g}''\mathcal{F}_{\xi\zeta}^{n-(t-|\underline{s}|)}$ and

$$f_{(\underline{s})\xi,\zeta}(b) \in \mathcal{G}_{h(\xi),h(\zeta)}^{t-|s|} = \mathfrak{g}''\mathcal{F}_{h(\xi),h(\zeta)}^{n-(t-|s|)}$$

for $b \in \mathcal{G}_{\xi\zeta}^{t-|\underline{s}|}$ and for all possible $\xi < \zeta$ since $|\underline{s}| < t$ i.e. $t - |\underline{s}| > 0$. So the above definition is meaningful, $(\mathfrak{g}_0(f,\underline{s}))_{\xi\zeta}$ is defined on the whole set $\mathcal{G}_{\xi,\zeta}^{t-|s|}$.

Let us write $(\mathfrak{g}_0(f,\underline{s}))_\xi$ instead of $\mathfrak{g}_0(f,\underline{s}^\frown\langle\xi\rangle)$ which allows us, as usual, to write

$$(\mathfrak{g}_0(f,\underline{s}))^{\to} = \left\{(\mathfrak{g}_0(f,\underline{s}))_\xi : \xi \leq \theta_{s_j}^{n-(t+1)+j}\right\}.$$

Why does it work?

In Lemma 3.6 we will show that the elements of $\mathcal{G}_{\alpha,\beta}^{t+1}$ are $\mathcal{N}_\alpha^{t+1} \to \mathcal{N}_\beta^{t+1}$ *embeddings* for $\alpha < \beta \leq \kappa^{+n-(t+1)}$. Before that we need some technical lemmas.

3.4 Lemma. *$\mathfrak{g}(f) = f$ for $f \in \mathcal{F}_{\alpha\beta}^{n-(t+1)}$, $\alpha < \beta \leq \kappa^{+n-(t+1)}$, $t < n$.*

Proof. f and $\mathfrak{g}(f)$ both are disjoint unions of functions either order preserving ones from ordinals to ordinals, or of functions mapping from and into sets of such functions, etc.

The exact proof is by induction on t. The case $t = 1$ is OK by definition (see just before Statement 3.3).

Let now $\alpha < \beta \leq \kappa^{+n-(t+1)}$, $f \in \mathcal{F}_{\alpha\beta}^{n-(t+1)}$ and $1 \leq t < n$ be given and fixed. We have to show that $\mathfrak{g}(f) \in \mathcal{G}_{\alpha\beta}^{t+1}$ contains exactly of the functions $f_\varnothing$, $f_{(\underline{s})\xi}$ and $f_{(\underline{s})\xi\zeta}$ where $\underline{s} \in (M+i)$, $\xi < \zeta \leq \theta_{s_j}^{i+|s|}$, $j = |\underline{s}|$, $i = n - (t+1)$. By the definition of $\mathfrak{g}(f)$, $\mathfrak{g}(f)$ is the disjoint union of the function sets $(\mathfrak{g}(f,\underline{s}))^-$, $(\mathfrak{g}(f,\underline{s}))^{\to}$ and $(\mathfrak{g}(f,\underline{s}))^{\Rightarrow}$ for $s \in (M+i)$. Examining the definition of $\mathfrak{g}(f)$ we can see that the function sets $(\mathfrak{g}(f,\underline{s}))^-$ contain the functions $f_{(\underline{s})\xi}$ which collect the function sets $(\mathfrak{g}(f,\underline{s}))^{\to}$, this can be proved by descending induction on $|\underline{s}|$. Similarly, the function sets $(\mathfrak{g}(f,\underline{s}))^{\Rightarrow}$ contain of the functions $f_{(\underline{s})\xi\zeta}$ since by the induction on t we have $\mathfrak{g}(b) = b$ for $b \in \mathcal{F}_{\xi\zeta}^k$ and $k \leq t$. The functions $f_{(\underline{s})\xi\zeta}$ are collected again by the function sets $(\mathfrak{g}(f,\underline{s}))^{\to}$. Finally $\mathfrak{g}(f)^- = f_\varnothing$. $\square$

Corollaries. $(\mathfrak{g}(f,\underline{s}))_{\xi\zeta} = f_{(\underline{s})\xi,\zeta}$ immediately follow for all possible $\underline{s}$, ξ, ζ, and similarly

$$f_\xi = \left\{f_{(\varnothing)\xi}\right\} \cup \bigcup \Big\{ f_{(\underline{s})} \cup f_{(\underline{s})\sigma} \cup f_{(\underline{s})\sigma,\tau} : \sigma < \tau \leq \theta^{i+|s|}_{s_j}, \\ j = |\underline{s}|,\ \underline{s} \in (M+i),\ s_1 = \xi \Big\}$$

for $\xi \leq \theta^{n-(t+1)}_\alpha$. Moreover in the above equality we should not bother what kind of function does f_ξ mean: in the sense of Definition 1.5 or the above $\mathfrak{g}(f)_\xi$ or Morgan's function (defined in footnote for (M4a)). In what follows we will use this equality without any remark. Further, we will distinguish f and $\mathfrak{g}(f)$, $\mathcal{F}^{n-t}_{\alpha\beta}$ and $\mathcal{G}^t_{\alpha\beta}$ only in critical cases.

In order to prove Theorem 3.1 we need only one further statement.

3.5 Statement. The structures

$$\mathcal{M}^{t+1}_\tau := \Big\langle \big\langle \mathcal{N}^{t+1}_\alpha : \alpha \leq \tau \big\rangle, \big\langle \mathcal{G}^{t+1}_{\alpha\beta} : \alpha < \beta \leq \tau \big\rangle \Big\rangle$$

are gap-$(t+1)$ fake morasses for all ordinal $\tau \leq \kappa^{+n-(t+1)}$, and $\mathcal{M}^{t+1}_{\kappa^{+n-(t+1)}}$ is a $\big(\kappa^{+n-(t+1)}, t+1\big)$ morass.

Proof. We have to show that $\mathcal{M}^{t+1}_\tau$ satisfies the requirement of Definition 1.4. We prove this by induction on t. (The proof runs through the next 5 pages.)

Statement 3.3 proved the case $t = 1$. Now let us consider the critical points of Definition 1.4. (As we indicated, $(M+i)$ and (M0) through (M6) refers to the points of Definition 3.1.)

1.4.0): $\mathcal{N}^{t+1}_\alpha \leq \mathcal{N}^{t+1}_\beta$ for $\alpha < \beta \leq \kappa^{+(n-t)}$ hold by the construction. For the other half part of 1.4.0) (that is that the elements $\mathcal{G}^{t+1}_{\alpha\beta}$ are $\mathcal{N}^{t+1}_\alpha \to \mathcal{N}^{t+1}_\beta$ *embeddings*) we need the below lemma. □

3.6 Lemma. $\mathfrak{g}(f)$ *are gap-t* $\mathcal{N}^{t+1}_\alpha \to \mathcal{N}^{t+1}_\beta$ embeddings *for all* $f \in \mathcal{F}^{n-(t+1)}_{\alpha\beta}$, $\alpha < \beta \leq \kappa^{+n-(t+1)}$ *and* $t < n$.

Proof. We prove the statement simultaneously for all fixed α, β and any sequence $\underline{s} \in (M+i)$ by induction on $|\underline{s}|$ (and an outer induction on t).

To be more precise we show that

$$\mathfrak{g}(f,\underline{s}) : \mathcal{N}^{t-|s|+1}_{s_j} \to \mathcal{N}^{t-|s|+1}_{s_j}$$

are gap-$(t-|\underline{s}|)$-embeddings for all $\underline{s} \in (M+i)$ where $j=|\underline{s}| \le t$, $f \in \mathcal{F}_{\alpha\beta}^{n-(t+1)+|\underline{s}|}$ and $\alpha<\beta\le\kappa^{+n-(t+1)}$.

For $|\underline{s}|=0$ we have $\mathfrak{g}(f)=\mathfrak{g}(f,\varnothing)$.

For the inductive step we distinguish two cases. Before, for simplicity let $i:=n-(t+1)$.

In case $|s|=t$ we know that

$$\mathfrak{g}(f,\underline{s})=f_{(\underline{s}\restriction)s_t}:\theta_{s_t}^{n-1}+1\to\theta_{f_{s\restriction}(s_t)}^{n-1}+1$$

are order preserving functions, that is gap-0 morass embeddings, moreover $\theta_{s_t}^{n-1}+1=\mathcal{N}_{s_t}^1$ and $\theta_{f_{s\restriction}(s_t)}^{n-1}+1=\mathcal{N}_{f_{s\restriction}(s_t)}^1$.

In case $|\underline{s}|<t$ we have

$$\mathfrak{g}(f,\underline{s})=(\mathfrak{g}(f,\underline{s}))\cup(\mathfrak{g}(f,\underline{s}))\cup(\mathfrak{g}(f,\underline{s}))$$

where

$$\mathfrak{g}(f,\underline{s})^-=f_{(\underline{s}\restriction)s_j}:\theta_{s_j}^{i+|s|}+1\to\theta_{f_{\underline{s}\restriction}(s_j)}^{i+|s|}+1$$

are order preserving functions by (M0).

Further $\theta_{s_j}^{i+|s|}+1=\mathrm{ht}\left(\mathcal{N}_{s_j}^{t-|s|+1}\right)$ and $j=|\underline{s}|$ and

$$\mathfrak{g}(f,\underline{s})_{\xi\zeta}:\mathcal{G}_{\xi,\zeta}^{t-|s|}\to\mathcal{G}_{h(\xi),h(\zeta)}^{t-|s|}$$

where $h=(\mathfrak{g}(f,\underline{s}))^-$,

$$\mathfrak{g}(f,\underline{s})_\xi=\mathfrak{g}(f,\underline{s}^\frown\langle\xi\rangle):\mathcal{N}_\xi^{t-|s|}\to\mathcal{N}_{h(\xi)}^{t-|s|}$$

where $h=(\mathfrak{g}(f,\underline{s}^\frown\langle\xi\rangle))^-$ are gap-$(t-|\underline{s}|-1)$ morass embeddings by the induction hypothesis. This means that

$$\mathfrak{g}(f,\underline{s}):\mathcal{N}_{s_j}^{t-|s|+1}\to\mathcal{N}_{h(s_j)}^{t-|s|+1}$$

is indeed a gap-$(t-|s|)$ morass embedding if $j=|\underline{s}|$ and $h=\mathfrak{g}(f,\underline{s})^-$, again by the induction hypothesis, assuming that the requirements d)–f) of Definition 1.5 hold.

We prove these requirements now.

1.5.d) Let

$$\mathcal{G}_{\xi,\xi+1}^{t-|s|}:=\{b: b \text{ is left branching embedding}\}\cup\{k\}$$

where $k : \mathcal{N}_\xi^{t-|s|} \to \mathcal{N}_{\xi+1}^{t-|s|}$ is right branching embedding with splitting point

$$\sigma_\xi < \theta_\xi + 1 = \theta_\xi^{n-t+|\underline{s}|} + 1 = \mathrm{ht}\left(\mathcal{N}_\xi^{t-|\underline{s}|}\right)$$

and that $\xi < \theta_{s_j}^{n-t+|\underline{s}|} + 1 = \mathrm{ht}\left(\mathcal{N}_{s_j}^{t-|\underline{s}|+1}\right)$.

(Recall that $|\underline{s}| \lneqq t$ now.) Now we have to show that the splitting point of the right branching embedding $\ell \in \mathcal{G}_{z,z+1}^{t-|s|}$ is

$$\sigma_z = \mathfrak{g}(f,\underline{s})_\xi^-(\sigma_\xi) < \theta_z^{n-t+|\underline{s}|} + 1 = \mathrm{ht}\left(\mathcal{N}_z^{t-|\underline{s}|}\right)$$

where $z = \mathfrak{g}(f,\underline{s})^-(\xi)$. Using Statement 3.4 and (M4b) we know that σ_ξ is also the splitting point of the shift function of $\mathcal{F}_{\xi,\xi+1}^{n-t+|s|}$. Now applying (M2c) for the sequence $\underline{r} := \underline{s}^\frown\langle\xi\rangle \in (M+i)$ we get that the splitting point of $\mathcal{F}_{z,z+1}^{n-t+|s|}$ is

$$\sigma_z = f_{(\underline{r})}(\sigma_\xi) = f_{(\underline{s}^\frown\langle\xi\rangle)}(\sigma_\xi) = \mathfrak{g}(f,\underline{s})_\xi^-(\sigma_\xi),$$

and moreover

$$z = f_{(\underline{r}\upharpoonright)}(\xi) = f_{(\underline{s})}(\xi) = \mathfrak{g}(f,\underline{s})^-(\xi).$$

This implies that σ_z is also the splitting point of $\ell \in \mathcal{G}_{z,z+1}$.

1.5.e): Using Statement 3.4, the definition of $\mathfrak{g}(f,\underline{s})$ and (M2a) we have

$$\begin{aligned}
\mathfrak{g}(f,\underline{s})_{\xi\vartheta}(b\circ c) &= \mathfrak{g}(f,\underline{s})_{\xi\vartheta}(\mathfrak{g}(b\circ c)) = \mathfrak{g}\big(f_{(\underline{s})\xi\vartheta}(b\circ c)\big)\\
&= \mathfrak{g}\big(f_{(\underline{s})\zeta\vartheta}(b)\circ f_{(\underline{s})\xi\zeta}(c)\big) = \mathfrak{g}\big(f_{(\underline{s})\zeta\vartheta}(b)\circ f_{(\underline{s})\xi\zeta}(c)\big)\\
&= \mathfrak{g}\big(f_{(\underline{s})\zeta\vartheta}(b)\big)\circ\mathfrak{g}\big(f_{(\underline{s})\xi\zeta}(c)\big) = \mathfrak{g}(f,\underline{s})_{\zeta\vartheta}(\mathfrak{g}(b))\circ\mathfrak{g}(f,\underline{s})_{\xi\zeta}(\mathfrak{g}(c))\\
&= \mathfrak{g}(f,\underline{s})_{\zeta\vartheta}(b)\circ\mathfrak{g}(f,\underline{s})_{\xi\zeta}(c)
\end{aligned}$$

hold for all $\xi < \zeta < \vartheta < \theta_{s_j}^{n-t+|\underline{s}|} + 1$, $b \in \mathcal{G}_{\zeta,\vartheta}^{t-|s|}$ and $c \in \mathcal{G}_{\xi,\zeta}^{t-|s|}$.

1.5.f): The proof of Statement 3.4 also gives $\mathfrak{g}(f,\underline{s})_\zeta = f_{(s)\zeta}$ for all possible f, $\underline{s}$ and ζ. Now Statement 3.4 and (M2b) give 1.5.f).

This concludes the proof of the inductive step and so the proof of Lemma 3.6. □

Now we turn back to the proof of Statement 3.5. So far we have proved that the elements of $\mathcal{G}_{\alpha,\beta}^{t+1}$ are $\mathcal{N}_\alpha^{t+1} \to \mathcal{N}_\beta^{t+1}$ embeddings for $\alpha < \beta \leq \kappa^{+n-(t+1)}$. Now we have to show that the structures

$$\mathcal{M}_\tau^{t+1} := \left\langle \left\langle \mathcal{N}_\alpha^{t+1} : \alpha \leq \tau \right\rangle, \left\langle \mathcal{G}_{\alpha\beta}^{t+1} : \alpha \leq \tau \right\rangle \right\rangle$$

are gap-$(t+1)$ fake morasses for $\tau \leq \kappa^{+n-(t+1)}$. We have to show that $\mathcal{M}_\tau^{t+1}$ satisfies the requirements of Definition 1.4. In what follows, we will often use Statement 3.4 without mentioning it.

1.4.0): We have already proved this just before and in Lemma 3.6.

1.4.a): We have

$$\mathcal{G}^{t+1}_{\alpha,\alpha+1} = \left\{\mathfrak{g}(d) : d \approx \mathrm{id},\, d \in \mathcal{F}^i_{\alpha,\alpha+1}\right\} \cup \left\{\mathfrak{g}(h^i_\alpha) : h^i_\alpha \in \mathcal{F}^i_{\alpha,\alpha+1}\right\}$$

where $i = n-(t+1)$.

First we show that the set

$$\left\{\mathfrak{g}(d) : d \approx \mathrm{id},\, d \in \mathcal{F}^i_{\alpha,\alpha+1}\right\}$$

gives all the **left branching** embeddings $\mathcal{N}_\alpha^{t+1} \to \mathcal{N}_\beta^{t+1}$. By (M4a) we have

$$d^- \restriction \theta = d_\varnothing \restriction \theta = \mathrm{id} \restriction \theta$$

for $\theta = \theta^i_\alpha$, and by the footnote for (M4a) we have $d_\xi = \mathrm{id} \in \mathcal{F}^{i+1}_{\xi,\xi}$, that is $d_\xi = \mathrm{id} \restriction \mathcal{N}^t_\xi$ for $\xi < \theta$. Furthermore by the definition of $\mathfrak{g}(d)$ we also have

$$d^{(1.4.\text{ Def.})}_{\theta^i_\alpha} = d^{(\text{footnote (M4a)})}_{\theta^i_\alpha} \in \mathcal{F}^{i+1}_{\theta^i_\alpha,\theta^i_{\alpha+1}} = \mathcal{G}^t_{\theta^i_\alpha,\theta^i_{\alpha+1}}.$$

Finally $d_{\xi\zeta}(b) = b$ for each $b \in \mathcal{G}^t_{\xi\zeta}$ by 1.5.f) and the above results. So $\mathfrak{g}(d)$ is indeed a $\mathcal{N}_\alpha^{t+1} \to \mathcal{N}_\beta^{t+1}$ left branching embedding if $d \approx \mathrm{id}$, $d \in \mathcal{F}^i_{\alpha,\alpha+1}$. The fact, that all the left branching $\mathcal{N}_\alpha^{t+1} \to \mathcal{N}_\beta^{t+1}$ embeddings are of form $\mathfrak{g}(d)$ where $d \approx \mathrm{id}$ and $d \in \mathcal{F}^i_{\alpha,\alpha+1}$, is trivial.

Now we turn to the **right branching** elements of $\mathcal{G}^{t+1}_{\alpha,\alpha+1}$, that is we show that $\mathfrak{g}(h^i_\alpha) : \mathcal{N}_\alpha^{t+1} \to \mathcal{N}_\beta^{t+1}$ is a shift (right branching embedding), where $h^i_\alpha \in \mathcal{F}^i_{\alpha,\alpha+1}$ is the function from (M4b) and $i = n-(t+1)$. This means that we have to show that $\mathfrak{g}(h^i_\alpha)$ satisfies the requirements of Definition 1.7.ii)[15].

In what follows, for simplicity, we write h instead of h^i_α and θ instead of θ^i_α.

1.7.ii) a) and b) are trivial since $\mathfrak{g}(h)^- = h_\varnothing$.

[15]Recall that $\mathcal{F}^i_{\alpha,\alpha+1}$ contains exactly *one* function $h^i_\alpha \in \mathcal{F}^i_{\alpha,\alpha+1}$ in (M4b).

1.7.ii) c): By (M4b) $h_{(\underline{s})\xi} = \mathrm{id}$ and by (M2b) $h_{(\underline{s})\xi\zeta} = \mathrm{id}$ for all $\underline{s} \in (M+i)$, $\xi \le \zeta \le s_j$, assuming either $s_1 < \sigma$ or ($s = \varnothing$ and $\xi \le \zeta < \sigma$). Then, by the Corollary of Statement 3.4 we have $\mathfrak{g}(h)_\xi = \mathrm{id}$ for $\xi < \sigma$.

1.7.ii) d): we proved above that $h_{\xi\zeta} = \mathrm{id}$ for $\xi \le \zeta < \sigma$.

1.7.ii) e) and f): arguing as in c) we know that $h_\sigma \in \mathcal{F}^{i+1}_{\sigma,\theta} = \mathcal{G}^t_{\sigma,\theta}$ and $h_{\xi\zeta}{}''\mathcal{F}^{i+1}_{\xi,\zeta} = \mathcal{F}^{i+1}_{h^-(\xi),h^-(\zeta)}$ for $\sigma \le \xi < \zeta \le \theta$. So $\mathfrak{g}(h)$ is a shift. So we have proved that $\mathcal{G}^{t+1}_{\alpha,\alpha+1}$ is an amalgam, and so we proved 1.4.a).

1.4.b): By (M3) and Statement 3.4.

1.4.c): Let $\alpha \le \kappa^{+i}$ be limit where $i = n-(t+1)$. Using Statement 3.4 and (M6a) we have

$$\mathrm{ht}\left(\mathcal{N}^{t+1}_\alpha\right) = \theta^i_\alpha + 1 = \bigcup \left\{ f^-{}''\mathrm{ht}\left(\mathcal{N}^{t+1}_\beta\right) : f \in \mathcal{G}^{t+1}_{\beta\alpha}, \beta < \alpha \right\}$$

since $\mathrm{ht}\left(\mathcal{N}^{t+1}_\beta\right) = \theta^i_\beta + 1$ and $f^-\left(\theta^i_\beta\right) = \theta^i_\alpha$ for $f \in \mathcal{G}^{t+1}_{\beta\alpha}$ and $\beta < \alpha$. Now by (M6c)

$$\mathcal{G}^t_{\xi\tau} = \bigcup \left\{ f_{\xi',\tau'}{}''\mathcal{G}^t_{\xi'\tau'} : f \in \mathcal{G}^{t+1}_{\beta\alpha},\ f^-(\xi') = \xi,\ f^-(\tau') = \tau,\ \beta < \alpha \right\}$$

for $\xi < \tau \le \theta^i_\alpha$.

By the construction of $\mathcal{N}^t_\xi$ for $\xi \le \theta^i_\alpha$, $\mathcal{N}^t_\xi$ are the disjoint unions of the sets

$$\mathrm{ht}\left(\mathcal{N}^j_\rho\right) = \theta^{n-j}_\rho + 1 \quad \text{and} \quad \mathcal{G}^k_{\sigma\tau} = \mathcal{F}^{n-k}_{\sigma\tau}$$

for $1 \le k < t$, $1 \le j \le t$, $\sigma \le \tau \le \mathrm{ht}\left(\mathcal{N}^j_\rho\right) = \theta^{n-j}_\rho + 1$ where $\rho \le \mathrm{ht}\left(\mathcal{N}^t_\xi\right) = \theta^{n-t}_\xi + 1$ in case $j = t$, and $\rho \le \mathrm{ht}\left(\mathcal{N}^{j+1}_{r_{j+1}}\right) = \theta^{n-j-1}_\rho$ in case $j < t$, and further $r_t = \mathrm{ht}\left(\mathcal{N}^t_\xi\right) = \theta^{n-t}_\xi + 1$ and $r_{j-1} = \mathrm{ht}\left(\mathcal{N}^j_{r_j}\right) = \theta^{n-j}_{r_j} + 1$.

So, by (M6b) and (M6c) we know that $\mathcal{N}_\xi$ are covered by the ranges of the functions $f_{(\underline{s}')\xi'}$ and $f_{(\underline{s}')\xi'\tau'}$ where especially $f(s'_1) = \xi$. But $f_{\xi'}$ is the union of these functions, where $f \in \mathcal{G}^t_{\beta\alpha}$, $\beta < \alpha$, $f^-(\xi') = \xi$. $\mathcal{N}^t_{\xi'}$ and $\mathcal{N}^t_\xi$ have similar structures, so

$$\mathcal{N}^t_\xi = \bigcup \left\{ f_{\xi'}{}''\mathcal{N}^t_{\xi'} : f \in \mathcal{G}^{t+1}_{\beta\alpha},\ f^-(\xi') = \xi,\ \beta < \alpha \right\}$$

where $\xi \le \theta^i_\alpha$. This concludes the proof of 1.4.c).

1.4.d): use (M5) and Statement 3.4.

So far we showed that $\mathcal{M}^{t+1}_\tau$ are gap-$(t-1)$ fake morasses for all $\tau \le \kappa^{+n-(t+1)}$.

Now we show that $\mathcal{M}^{t+1}_{\kappa^{+n-(t+1)}}$ is a $\big(\kappa^{+n-(t+1)}, t+1\big)$-morass, that is it satisfies also Definition 1.10. (ii).

1.10. (ii)a): By the construction.

1.10. (ii)b): By (M1) and $\mathcal{G}^{t+1}_{\beta\alpha} = \mathcal{F}^{n-(t+1)}_{\alpha\beta}$ for $\alpha < \beta \le \kappa^{+n-(t+1)}$.

1.10. (ii)c): The structure of $\mathcal{N}^{t+1}_{\alpha}$ and of $\mathcal{N}^{t}_{\xi}$ for $\alpha \le \kappa^{+n-(t+1)}$ and $\xi \le \theta^{i}_{\alpha}$ are similar (see the proof of 1.27.c)). So, using (M1) we have that $\mathrm{size}(\mathcal{N}^{t+1}_{\alpha}) < \kappa^{+n-(t+1)}$ for $\alpha \le \kappa^{+n-(t+1)}$ since $\kappa^{+n-(t+1)}$ is a regular cardinal.

1.10. (ii)d): By the inductive hypothesis on t.

So we proved Statement 3.5. $\square$

This concludes the proof of Theorem 3.1.

Theorem 3.1 and Morgan's Theorem 7 in [Mo Ch. IV.] (which says that the existence of a (κ, m)-Morgan-morass is equivalent to the existence of a (κ, m)-Jensen-morass for every $m < \omega_0$ and regular $\kappa \ge \omega_1$) imply the below theorem:

3.7 Theorem. *If there is a (κ, m)-Jensen morass then there exists a (κ, m)-simplified morass (in our sense) for $m < \omega_0$ and κ regular.* $\square$

Using the results of [Je1], which ensure the existence of (κ, m)-Jensen-morasses for *any* ordinal m and regular $\kappa \ge \omega_1$ we get:

3.8 Theorem. *In case $V = L$ there exist (κ, m)-simplified morass (in the sense of Definitions 1.3–1.10) for all $m < \omega_0$ and κ regular.* $\square$

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ISTVÁN SZALKAI
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
UNIVERSITY OF VESZPRÉM
H–8201 VESZPRÉM, P.O. BOX 158
HUNGARY

E-mail: szalkai@almos.vein.hu

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