

**Ishikawa and Mann iterative processes
with errors for generalized strongly
nonlinear implicit quasi-variational inequalities**

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Abstract. In this paper, we prove some existence theorems of solutions and convergence theorems of the Ishikawa and Mann iterative sequences with errors for a class of generalized strongly nonlinear implicit quasivariational inequality problems involving Lipschitzian generalized pseudo-contractive mappings in Hilbert spaces. Our results improve and extend the main results of Verma.

1. Introduction

Variational inequality theory has become a rich source of inspiration in pure and applied mathematics. Variational inequalities not only have stimulated new results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. In recent years, variational inequalities have been generalized and applied in various directions. For details, we refer to [1]–[7], [11]–[16], [18] and the references therein.

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Recently, HUANG [3] and [4] constructed some new perturbed Ishikawa and Mann iterative algorithms to approximate the solution of some generalized implicit quasi-variational inclusions (inequalities), which includes many iterative algorithms for variational and quasi-variational inequality problems as special cases.

On the other hand, XU [17] revised the definitions of Ishikawa and Mann iterative processes with errors and studied the convergence problem of Ishikawa and Mann iterative processes with errors for approximating the solution of the nonlinear strongly accretive operator equation.

Inspired and motivated by recent research works [3], [4], [16] and [17], in this paper, we study a class of generalized strongly nonlinear implicit quasivariational inequality problems, which include the nonlinear variational inequality problem of VERMA [16] as a special case. We prove some existence theorems of solutions and convergence theorems of the Ishikawa and Mann iterative sequences with errors for this class of generalized strongly nonlinear implicit quasivariational inequality problems involving Lipschitzian generalized pseudo-contractive mappings in Hilbert spaces. Our results extend and improve Theorems 2.1 and 2.2 of VERMA [16] in the following two aspects:

- (1) Extend the nonlinear variational inequality to the generalized strongly nonlinear implicit quasivariational inequality.
- (2) Replace the Mann iterative process by Ishikawa iterative process with errors.

2. Preliminaries and lemmas

Let H be a real Hilbert space endowed with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, respectively. For a nonempty closed convex subset $M \subset H$, let P_M be the projection of H onto M . Let $K : H \rightarrow 2^H$ be a set-valued mapping with the nonempty closed and convex value, $g : H \rightarrow H$ and $N : H \times H \rightarrow H$ be two nonlinear mappings. We consider the following problem:

Find $g(x) \in K(x)$ such that

$$(2.1) \quad \langle x - N(x, x), y - g(x) \rangle \geq 0$$

for all $y \in K(x)$. The problem (2.1) is called the generalized strongly nonlinear implicit quasivariational inequality problem.

Some special cases of the problem (2.1) are as follows:

(I) If $K(x) = m(x) + K$ for all $x \in H$, where K is a nonempty closed convex subset of H and $m : H \rightarrow H$ is a nonlinear mapping, then the problem (2.1) is equivalent to finding $x \in H$ such that $g(x) - m(x) \in K$ and

$$(2.2) \quad \langle x - N(x, x), y - g(x) \rangle \geq 0$$

for all $y \in m(x) + K$. The problem (2.2) is called the strongly nonlinear implicit quasivariational inequality problem.

(II) If $g = I$, where I is the identity mapping, then the problem (2.1) is equivalent to finding $x \in K(x)$ such that

$$(2.3) \quad \langle x - N(x, x), y - x \rangle \geq 0$$

for all $y \in K(x)$. The problem (2.3) is called the generalized strongly nonlinear quasivariational inequality problem.

(III) If $K(x) = K$ for all $x \in H$, where K is a nonempty closed convex subset of H , then the problem (2.1) is equivalent to finding $x \in H$ such that $g(x) \in K$ and

$$(2.4) \quad \langle x - N(x, x), y - g(x) \rangle \geq 0$$

for all $y \in K$.

(IV) If $N(u, v) = Tu + Sv$ for all $u, v \in H$, where $T, S : H \rightarrow H$ are two nonlinear mappings, then the problem (2.1) is equivalent to finding $x \in H$ such that $g(x) \in K(x)$ and

$$(2.5) \quad \langle x - Tx - Sx, y - g(x) \rangle \geq 0$$

for all $y \in K(x)$.

(V) If $K(x) = K$ for all $x \in H$, where K is a nonempty closed convex subset of H , $g = I$ and $N(u, v) = Tu$ for all $u, v \in H$, where $T : H \rightarrow H$ is a nonlinear mapping, then the problem (2.1) is equivalent to finding $x \in H$ such that

$$(2.6) \quad \langle x - Tx, y - x \rangle \geq 0$$

for all $y \in K$, which is the nonlinear variational inequality considered recently by VERMA [16].

Now, we recall the following two iterative processes due to ISHIKAWA [8] and MANN [10], respectively.

(I) Let K be a nonempty convex subset of H and $T : K \rightarrow X$ be a mapping. The sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the Ishikawa iteration process, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying some conditions.

(II) In particular, if $\beta_n = 0$ for all $n \geq 0$, then $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the Mann iteration process.

Recently, LIU [9] introduced the concepts of Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings as follows:

(III) For a nonempty subset K of a Banach space X and a mapping $T : K \rightarrow X$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the Ishikawa iteration process with errors. Here $\{u_n\}$ and $\{v_n\}$ are two summable sequences in X (i.e., $\sum_{n=0}^{\infty} \|u_n\| < +\infty$ and $\sum_{n=0}^{\infty} \|v_n\| < +\infty$) and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying certain restrictions.

(IV) In particular, if $\beta_n = 0$ and $v_n = 0$ for all $n \geq 0$, the sequences $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + u_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the Mann iteration process with errors, where $\{u_n\}$ is a summable sequence in X and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying certain restrictions.

However, in a recent paper [17], Xu pointed out that the definitions of LIU [9] are against the randomness of the errors and revised the definitions of LIU [9] as follows:

(V) Let K be a nonempty convex subset of a Banach space X and $T : K \rightarrow X$ be a mapping. For any given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n = \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the Ishikawa iteration process with errors, where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K , $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ are six sequences in $[0,1]$ satisfying the conditions

$$\alpha_n + \beta_n + \gamma_n = 1, \quad \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$$

for $n = 0, 1, 2, \dots$

(VI) In particular, if $\hat{\beta}_n = \hat{\gamma}_n = 0$ for $n = 0, 1, 2, \dots$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the Mann iteration process with errors.

Clearly, the Ishikawa and Mann iteration processes are all special cases of the Ishikawa process with errors.

For our main results, we need the following lemmas:

Lemma 2.1 ([2]). *If $K \subset H$ is a closed convex subset and $x \in H$ is a given point, then $z \in K$ satisfies the inequality*

$$\langle x - z, y - x \rangle \geq 0$$

for all $y \in K$ if and only if

$$(2.7) \quad x = P_K z,$$

where P_K is the projection of H onto K .

Lemma 2.2 ([2]). *The mapping P_K defined by (2.7) is nonexpansive, that is,*

$$\|P_K u - P_K v\| \leq \|u - v\|$$

for all $u, v \in H$.

Lemma 2.3 ([2]). *If $K(u) = m(u) + K$ and $K \subset H$ is a closed convex subset, then for any $u, v \in H$, we have*

$$(2.8) \quad P_{K(u)} v = m(u) + P_K(v - m(u)).$$

Lemma 2.4 ([9]). *Let a_n, b_n and c_n be three nonnegative real sequences satisfying*

$$(2.9) \quad a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$$

for $n = 0, 1, 2, \dots$ and

$$t_n \in [0, 1], \quad \sum_{n=0}^{+\infty} t_n = +\infty, \quad b_n = o(t_n), \quad \sum_{n=0}^{+\infty} c_n < +\infty.$$

Then $\lim_{n \rightarrow +\infty} a_n = 0$.

By Lemma 2.1, we know that the generalized strongly nonlinear implicit quasivariational inequality problem (2.1) has a unique solution if and only if the mapping $F : H \rightarrow H$ defined by

$$F(x) = x - g(x) + P_{K(x)}[g(x) - t(x - N(x, x))]$$

has a unique fixed point, where $t > 0$ is a constant.

3. Main results

In this section, we prove some existence theorems of solutions and convergence theorems of the Ishikawa and Mann iterative processes with errors for the generalized strongly nonlinear implicit quasivariational inequality problems (2.1) and (2.2) involving Lipschitzian generalized pseudocontractions in Hilbert spaces.

First, we give some definitions.

Definition 3.1. A mapping $T : H \rightarrow H$ is said to be *generalized pseudo-contractive* if there exists a constant $r > 0$ such that

$$(3.1) \quad \|Tx - Ty\|^2 \leq r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2$$

for all $x, y \in H$.

It is easy to check that (3.1) is equivalent to

$$(3.2) \quad \langle Tx - Ty, x - y \rangle \leq r\|x - y\|^2.$$

For $r = 1$ in (3.1), we get the usual concept of the pseudo-contractivity of T introduced by BROWDER and PETRYSHYN in [1], i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2.$$

Definition 3.2. A mapping $N : H \times H \rightarrow H$ is said to be

(1) *generalized pseudo-contractive with respect to the first argument* if there exists a constant $p > 0$ such that

$$\langle N(u, \cdot) - N(v, \cdot), u - v \rangle \leq p\|u - v\|^2$$

for all $u, v \in H$.

(2) *Lipschitz continuous with respect to the first argument* if there exists a constant $s > 0$ such that

$$\|N(u, \cdot) - N(v, \cdot)\| \leq s\|u - v\|$$

for all $u, v \in H$.

In a similar way, we can define Lipschitz continuity of N with respect to the second argument.

Definition 3.3. Let $K : H \rightarrow 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H . The projection $P_{K(x)}$ is said to be Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$\|P_{K(x)}z - P_{K(y)}z\| \leq \xi\|x - y\|$$

for all $x, y, z \in H$.

Remark 3.1. In many important applications, $K(u)$ has the following form:

$$K(u) = m(u) + K,$$

where $m : H \rightarrow H$ is a single-valued mapping and K is a nonempty closed convex subset of H . If m is Lipschitz continuous with constant λ , then it follows from Lemma 2.3 that $P_{K(x)}$ is Lipschitz continuous with the Lipschitz constant $\mu = 2\lambda$.

We now give the main results of this paper.

Theorem 3.1. *Let H be a real Hilbert space and $K : H \rightarrow 2^H$ be a set-valued mapping with the nonempty closed and convex value. Let a mapping $N : H \times H \rightarrow H$ be generalized pseudo-contractive with respect to the first argument (with constant r) and Lipschitz continuous with respect to the first and second arguments (with constants s and η , respectively). Let $g : H \rightarrow H$ be a mapping such that $I - g$ is Lipschitz continuous (with constant δ). Suppose that $P_{K(u)}$ is Lipschitz continuous (with constant ξ). Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in H and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ be six sequences in $[0, 1]$ satisfying the following conditions:*

- (1) $\alpha_n + \beta_n + \gamma_n = 1, \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1, \quad n \geq 0,$
- (2) $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \hat{\beta}_n = \lim_{n \rightarrow \infty} \hat{\gamma}_n = 0,$
- (3) $\sum_{n=0}^{+\infty} \beta_n = \infty, \quad \sum_{n=0}^{+\infty} \gamma_n < \infty.$

If the following conditions hold:

$$(3.3) \quad \left| t - \frac{\eta(h-1) + (1-r)}{1-2r+s^2-\eta^2} \right| < \frac{\sqrt{[\eta(h-1) + (1-r)]^2 - (1-2r+s^2-\eta^2)(2-h)h}}{1-2r+s^2-\eta^2},$$

$$(3.4) \quad \eta(h-1) + (1-r) > \sqrt{(1-2r+s^2-\eta^2)(2-h)h},$$

$$(3.5) \quad s > r + \frac{\eta^2}{2}, \quad h + t\eta < 1, \quad h = \xi + 2\delta,$$

then there exists a unique $x \in H$ satisfying the generalized strongly nonlinear implicit quasivariational inequality (2.1) and

$$x_n \rightarrow x \quad (n \rightarrow \infty),$$

where $\{x_n\}$ is the Ishikawa iteration process with errors defined as follows:

$$(3.6) \quad \begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \{y_n - g(y_n) \\ \quad + P_{K(y_n)}[g(y_n) - ty_n + tN(y_n, y_n)]\} + \gamma_n u_n, \\ y_n = \hat{\alpha}_n x_n + \hat{\beta}_n \{x_n - g(x_n) \\ \quad + P_{K(x_n)}[g(x_n) - tx_n + tN(x_n, x_n)]\} + \hat{\gamma}_n v_n \end{cases}$$

for $n = 0, 1, 2, \dots$

PROOF. We first prove that the generalized strongly nonlinear implicit quasivariational inequality (2.1) has a unique solution. By Lemma 2.1, it is sufficient to prove the mapping defined by

$$F(x) = x - g(x) + P_{K(x)}[g(x) - t(x - N(x, x))]$$

has a unique fixed point in H .

Let u, v be two arbitrary points in H . From Lemma 2.2 and the Lipschitz continuity of $P_{K(u)}$ and $I - g$, we have

$$(3.7) \quad \begin{aligned} \|F(u) - F(v)\| &= \|u - g(u) + P_{K(u)}[g(u) - t(u - N(u, u))] \\ &\quad - \{v - g(v) + P_{K(v)}[g(v) - t(v - N(v, v))]\| \\ &\leq \|u - g(u) - (v - g(v))\| \\ &\quad + \|P_{K(u)}[g(u) - t(u - N(u, u))] - P_{K(u)}[g(v) - t(v - N(v, v))]\| \\ &\quad + \|P_{K(u)}[g(v) - t(v - N(v, v))] - P_{K(v)}[g(v) - t(v - N(v, v))]\| \\ &\leq 2\|u - g(u) - (v - g(v))\| \\ &\quad + \|[u - t(u - N(u, u))] - [v - t(v - N(v, v))]\| + \xi\|u - v\| \\ &= 2\delta\|u - v\| + \|(1 - t)(u - v) + t(N(u, u) - N(v, v))\| + \xi\|u - v\| \\ &\leq \|(1 - t)(u - v) + t(N(u, u) - N(v, u))\| \\ &\quad + t\|N(v, u) - N(v, v)\| + (\xi + 2\delta)\|u - v\|. \end{aligned}$$

Since N is generalized pseudo-contractive with respect to the first argument and Lipschitz continuous with respect to the first and second arguments, respectively, we get

$$\begin{aligned}
 (3.8) \quad & \| (1-t)(u-v) + t(N(u,u) - N(v,u)) \|^2 = (1-t)^2 \|u-v\|^2 \\
 & + 2t(1-t) \langle u-v, N(u,u) - N(v,u) \rangle + t^2 \|N(u,u) - N(v,u)\|^2 \\
 & \leq [(1-t)^2 + 2t(1-t)r] \|u-v\|^2 + t^2 s^2 \|u-v\|^2 \\
 & = [(1-t)^2 + 2t(1-t)r + t^2 s^2] \|u-v\|^2
 \end{aligned}$$

and

$$(3.9) \quad \|N(v,u) - N(v,v)\| \leq \eta \|u-v\|.$$

It follows from (3.7)~(3.9) that

$$(3.10) \quad \|F(u) - F(v)\| \leq k \|u-v\|$$

for all $u, v \in H$, where

$$k = \sqrt{(1-t)^2 + 2t(1-t)r + t^2 s^2} + t\eta + h, \quad h = \xi + 2\delta.$$

From (3.3)~(3.5), we know that $0 < k < 1$ and so F has a unique fixed point $x \in H$, which is a unique solution of the generalized strongly non-linear implicit quasivariational inequality problem (2.1).

Now, we prove that $\{x_n\}$ converges to x . In fact, it follows from (3.6) and $x = F(x)$ that

$$\begin{aligned}
 (3.11) \quad & \|x_{n+1} - x\| = \|\alpha_n x_n + \beta_n \{y_n - g(y_n) \\
 & + P_{K(y_n)}[g(y_n) - ty_n + tN(y_n, y_n)]\} + \gamma_n u_n - x\| \\
 & = \|\alpha_n x_n + \beta_n \{y_n - g(y_n) + P_{K(y_n)}[g(y_n) - ty_n + tN(y_n, y_n)]\} \\
 & + \gamma_n u_n - (\alpha_n x + \beta_n \{x - g(x) + P_{K(x)}[g(x) - tx + tN(x, x)]\} \\
 & + \gamma_n x)\| \\
 & \leq \alpha_n \|x_n - x\| + \beta_n \|F(y_n) - F(x)\| + \gamma_n \|u_n - x\|.
 \end{aligned}$$

From (3.10) and (3.11), it follows that

$$(3.12) \quad \|x_{n+1} - x\| \leq \alpha_n \|x_n - x\| + k\beta_n \|y_n - x\| + \gamma_n \|u_n - x\|.$$

Similarly, we have

$$\begin{aligned}
 (3.13) \quad & \|y_n - x\| = \|\hat{\alpha}_n x_n + \hat{\beta}_n \{x_n - g(x_n) + P_{K(x_n)}[g(x_n) - tx_n \\
 & \quad + tN(x_n, x_n)]\} + \hat{\gamma}_n v_n - x\| \\
 & = \|\hat{\alpha}_n(x_n - x) + \hat{\beta}_n \{x_n - g(x_n) + P_{K(x_n)}[g(x_n) - tx_n \\
 & \quad + tN(x_n, x_n)] - (x - g(x) + P_{K(x)}[g(x) - tx + tN(x, x)])\} \\
 & \quad + \hat{\gamma}_n(v_n - x)\| \\
 & \leq \hat{\alpha}_n \|x_n - x\| + \hat{\beta}_n \|F(x_n) - F(x)\| + \hat{\gamma}_n \|v_n - x\| \\
 & \leq \hat{\alpha}_n \|x_n - x\| + k\hat{\beta}_n \|x_n - x\| + \hat{\gamma}_n \|v_n - x\| \\
 & = (\hat{\alpha}_n + k\hat{\beta}_n) \|x_n - x\| + \hat{\gamma}_n \|v_n - x\|.
 \end{aligned}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned}
 (3.14) \quad & \|x_{n+1} - x\| \leq \alpha_n \|x_n - x\| + k\beta_n \|y_n - x\| + \gamma_n \|u_n - x\| \\
 & \leq (\alpha_n + k\hat{\alpha}_n\beta_n + k^2\beta_n\hat{\beta}_n) \|x_n - x\| + k\beta_n\hat{\gamma}_n \|v_n - x\| + \gamma_n \|u_n - x\|.
 \end{aligned}$$

Let

$$d = \max\{\sup_n \|u_n - x\|, \sup_n \|v_n - x\|\}.$$

Then $d < \infty$ and (3.14) implies that

$$(3.15) \quad \|x_{n+1} - x\| \leq A_n \|x_n - x\| + b_n + c_n,$$

where

$$A_n = \alpha_n + k\beta_n\hat{\alpha}_n + k^2\beta_n\hat{\beta}_n, \quad b_n = kd\beta_n\hat{\gamma}_n, \quad c_n = d\gamma_n.$$

Since $0 < k < 1$, it follows from (1)~(3) that

$$(3.16) \quad A_n = \alpha_n + k\beta_n\hat{\alpha}_n + k^2\beta_n\hat{\beta}_n \leq 1 - \beta_n + k\beta_n = 1 - (1 - k)\beta_n.$$

From (3.15), (3.16) and Lemma 2.4, we know that $\{x_n\}$ converges to the solution x . This completes the proof. \square

Remark 3.2. Theorem 3.1 extends and improves Theorems 2.1 and 2.2 of VERMA [16] in the following two aspects:

- (1) Extend the nonlinear variational inequality problem (2.6) to the generalized strongly nonlinear implicit quasivariational inequality problem (2.1).

- (2) Replace the Mann iterative process by Ishikawa iterative process with errors.

From Theorem 3.1 and Remark 3.1, we have the following result:

Theorem 3.2. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H . Let a mapping $N : H \times H \rightarrow H$ be generalized pseudo-contractive with respect to the first argument (with constant r) and Lipschitz continuous with respect to the first and second arguments (with constants s and η , respectively). Suppose that a mapping $m : H \rightarrow H$ is Lipschitz continuous (with constant $\mu > 0$) and $g : H \rightarrow H$ is a mapping such that $I - g$ is Lipschitz continuous (with constant δ). Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in H and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ be six sequences in $[0, 1]$ satisfying the conditions (1)~(3) of Theorem 3.1. If the conditions (3.3)~(3.5) hold for $\xi = 2\mu$, then there exists a unique $x \in H$ satisfying the strongly nonlinear implicit quasivariational inequality (2.2) and*

$$x_n \rightarrow x \quad (n \rightarrow \infty),$$

where $\{x_n\}$ is the Ishikawa iteration process with errors defined as follows:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \{y_n - g(y_n) + m(y_n) \\ \quad + P_K[g(y_n) - ty_n + tN(y_n, y_n) - m(y_n)]\} + \gamma_n u_n, \\ y_n = \hat{\alpha}_n x_n + \hat{\beta}_n \{x_n - g(x_n) + m(x_n) \\ \quad + P_K[g(x_n) - tx_n + tN(x_n, x_n) - m(x_n)]\} + \hat{\gamma}_n v_n \end{cases}$$

for $n = 0, 1, 2, \dots$

From Theorems 3.1 and 3.2, we have the following results:

Theorem 3.3. *Let H , K , N , g and $P_{K(u)}$ be the same as in Theorem 3.1. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\hat{\alpha}_n\}$ and $\{\hat{\beta}_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions:*

- (1) $\alpha_n + \beta_n = 1$, $\hat{\alpha}_n + \hat{\beta}_n = 1$, $n \geq 0$,
- (2) $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \hat{\beta}_n = 0$,
- (3) $\sum_{n=0}^{+\infty} \beta_n = \infty$.

If the conditions (3.3)~(3.5) hold, then there exists $x \in H$ satisfying the generalized strongly nonlinear implicit quasivariational inequality problem (2.1) and

$$x_n \rightarrow x \quad (n \rightarrow \infty),$$

where the Ishikawa iteration process $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \{y_n - g(y_n) + P_{K(y_n)}[g(y_n) - ty_n + tN(y_n, y_n)]\}, \\ y_n = \hat{\alpha}_n x_n + \hat{\beta}_n \{x_n - g(x_n) + P_{K(x_n)}[g(x_n) - tx_n + tN(x_n, x_n)]\} \end{cases}$$

for $n = 0, 1, 2, \dots$.

Theorem 3.4. Let H, K, N, g and m be the same as in Theorem 3.2. Let $\{\alpha_n\}, \{\beta_n\}, \{\hat{\alpha}_n\}$ and $\{\hat{\beta}_n\}$ be four sequences in $[0, 1]$ satisfying the conditions (1)~(3) of Theorem 3.3. If the conditions (3.3)~(3.5) hold for $\xi = 2\mu$, then there exists $x \in H$ satisfying the strongly nonlinear implicit quasivariational inequality problem (2.2) and

$$x_n \rightarrow x \quad (n \rightarrow \infty),$$

where the Ishikawa iteration process $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \{y_n - g(y_n) + m(y_n) \\ \quad + P_K[g(y_n) - ty_n + tN(y_n, y_n) - m(y_n)]\}, \\ y_n = \hat{\alpha}_n x_n + \hat{\beta}_n \{x_n - g(x_n) + m(x_n) \\ \quad + P_K[g(x_n) - tx_n + tN(x_n, x_n) - m(x_n)]\} \end{cases}$$

for $n = 0, 1, 2, \dots$.

Theorem 3.5. Let H, K, N, g and $P_{K(u)}$ be the same as in Theorem 3.1. Let $\{u_n\}$ be a bounded sequence in H and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:

$$(1) \quad \alpha_n + \beta_n + \gamma_n = 1, \quad n \geq 0,$$

$$(2) \quad \lim_{n \rightarrow \infty} \beta_n = 0,$$

$$(3) \quad \sum_{n=0}^{+\infty} \beta_n = \infty, \quad \sum_{n=0}^{+\infty} \gamma_n < \infty.$$

If the conditions (3.3)~(3.5) hold, then there exists a unique $x \in H$ satisfying the generalized strongly nonlinear implicit quasivariational inequality (2.1) and

$$x_n \rightarrow x \quad (n \rightarrow \infty),$$

where $\{x_n\}$ is the Mann iteration process with errors defined as follows:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \{x_n - g(x_n) \\ \quad + P_{K(x_n)}[g(x_n) - tx_n + tN(x_n, x_n)]\} + \gamma_n u_n \end{cases}$$

for $n = 0, 1, 2, \dots$

Theorem 3.6. Let H, K, N, g and m be the same as in Theorem 3.2. Let $\{u_n\}$ be a bounded sequence in H and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the conditions (1)~(3) of Theorem 3.5. If the conditions (3.3)~(3.5) hold for $\xi = 2\mu$, then there exists a unique $x \in H$ satisfying the strongly nonlinear implicit quasivariational inequality (2.2) and

$$x_n \rightarrow x \quad (n \rightarrow \infty),$$

where $\{x_n\}$ is the Mann iteration process with errors defined as follows:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \{x_n - g(x_n) + m(x_n) \\ \quad + P_K[g(x_n) - tx_n + tN(x_n, x_n) - m(x_n)]\} + \gamma_n u_n \end{cases}$$

for $n = 0, 1, 2, \dots$

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