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Hyers–Ulam stability of the Cauchy functional equation on square-symmetric groupoids

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Abstract. The stability of the functional equation $f(x\diamond y)=f(x)*f(y)$ $(x,y\in X)$ is investigated, where $f:X\to Y$ and $\diamond$, $*$ are square-symmetric operations on the sets X and Y, respectively. The results presented include and generalize the classical theorem of Hyers obtained on the stability of the Cauchy functional equation in 1941.

1. Introduction

The history of the stability theory of functional equations started with the talk of S. M. Ulam held at the Wisconsin University in 1940. In this talk the following problem was raised (see ULAM [Ula64]): *Let $(X,\diamond)$ be a group and $(Y,*)$ be a metric group whose metric is denoted by d. Does, for all positive ε, there exist $\delta>0$, such that if a function $g:G\to H$ satisfies the inequality*

$$d(g(x\diamond y),g(x)*g(y))<\delta \qquad (x,y\in X),$$

then there exists a homomorphism $f:X\to Y$ satisfying

$$d(g(x),f(x))<\varepsilon \qquad (x\in X)?$$

In other words, if $g:X\to Y$ is approximately a homomorphism (that is, if g satisfies the Cauchy equation with small error), then g differs from

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a homomorphism with small error. In case of an affirmative answer, the Cauchy equation is called *stable*. The most frequently quoted result of this subject is the following theorem of D. H. Hyers ([Hye41]) from 1941:

Let X, Y be Banach-spaces, $\varepsilon \geq 0$ and let $g : X \to Y$ satisfy

$$\|g(x+y) - g(x) - g(y)\| \leq \varepsilon \qquad (x, y \in X).$$

Then the limit

$$f(x) := \lim_{n\to\infty} \frac{g(2^n x)}{2^n} \tag{1}$$

exists for all $x \in X$ and the function $f : X \to Y$ is an additive mapping such that

$$\|g(x) - f(x)\| \leq \varepsilon \qquad (x \in X).$$

Moreover, f is the only additive mapping such that $f - g$ is bounded.

The sequence $\big(g(2^n x)/2^n\big)$ on the right hand side of (1) and the above result is often called *Hyers–Ulam sequence*, and the *Hyers–Ulam stability theorem*, respectively.

It is interesting to note that the first stability theorem was discovered in 1925 anticipating the question of Ulam. G. Pólya and G. Szegő in their famous book [PS25, Chapter I, Problem 99] formulated a problem, which, in fact, was equivalent to the stability problem of the Cauchy equation when X and Y are the additive groups of natural and real numbers, respectively.

There are more than 300 research papers born in the investigation of the stability of various functional equations. Excellent overviews and surveys can be found for instance in the papers of Forti [For95], Ger [Ger94b], Székelyhidi [Sze99] and also in the book of Hyers, Isac and Rassias [HIR98].

In this paper, we deal with the stability of the functional equation

$$f(x \diamond y) = f(x) * f(y) \qquad (x, y \in X), \tag{2}$$

where $f : X \to Y$ and $(X, \diamond)$, $(Y, *)$ are groupoids equipped with square-symmetric operations.

A binary operation $\diamond$ defined on X is called *square-symmetric* if

$$(x \diamond y) \diamond (x \diamond y) = (x \diamond x) \diamond (y \diamond y) \qquad (x, y \in X). \tag{3}$$

The groupoid $(X, \diamond)$ is called *square-symmetric* if the operation $\diamond$ is square-symmetric. It is obvious that a commutative semigroup is a square-symmetric groupoid. The converse, however, is not true in general. The role of square-symmetry, and also that of power-symmetry was discovered by RÄTZ [Rat80], and also used by BORELLI-FORTI and FORTI [BFF95].

There are three basically different methods in the stability theory of functional equations. One method is based on an iteration process analogous to using the Hyers–Ulam sequence. Another method is the invariant mean technique developed by SZÉKELYHIDI in [Sze86]. The third method (see e.g. PÁLES [Pal98]) uses sandwich theorems that are generalizations of the Hahn–Banach separation theorem.

In a recent paper of PÁLES, VOLKMANN and LUCE [PVL98], a large class of square-symmetric operations was described and lots of former results that can be obtained via the iterative technique have been unified. In this paper, the investigations of [PVL98] will be extended and generalized for mappings whose domain and codomain is also a square-symmetric groupoid. For generalizations with variable right hand side of the results of [PVL98], we refer to a paper of G. H. KIM [Kim01].

2. Main results

The square symmetry of the operation $\diamond : X \times X \to X$ is equivalent to the fact that the function $\sigma_\diamond : X \to X$ defined by

$$\sigma_\diamond(x) := x \diamond x \qquad (x \in X) \tag{4}$$

is an endomorphism of the groupoid $(X, \diamond)$, since, with this notation, (3) can be rewritten as

$$\sigma_\diamond(x \diamond y) = \sigma_\diamond(x) \diamond \sigma_\diamond(y).$$

It is an important situation when the mapping $\sigma_\diamond$ is an automorphism of $(X, \diamond)$, that is when $\sigma_\diamond$ is bijective. Then, for each fixed $x \in X$, the equation $u \diamond u = x$ can be solved uniquely with respect to $u \in X$. A binary operation $\diamond$ such that $\sigma_\diamond$ is an automorphism of $(X, \diamond)$ is called *divisible*, the corresponding groupoid is said to be a *divisible groupoid.*

In order to formulate the results of the paper, we introduce the notion of Lipschitz modulus of a function $\varphi : Y \to Y$ if (Y, d) is a metric space. The *Lipschitz modulus* $\operatorname{Lip}\varphi$ of φ is defined by

$$\operatorname{Lip}\varphi := \sup\left\{ \frac{d(\varphi(x), \varphi(y))}{d(x,y)} \;\Big|\; x, y \in Y,\, x \neq y \right\}.$$

In other words, $L = \operatorname{Lip}\varphi$ is the smallest extended real number such that

$$d(\varphi(x), \varphi(y)) \le L d(x, y) \qquad (x, y \in Y).$$

If $(Y, *)$ is also a groupoid and the operation $*$ is continuous with respect to the topology of the metric space (Y, d), then the triple $(Y, *, d)$ is called a *metric groupoid*.

Now we can formulate the first main result of this paper.

Theorem 1. *Let $(X, \diamond)$ be a square-symmetric groupoid and $(Y, *, d)$ be a complete metric divisible square-symmetric groupoid. Assume that*

$$L := \sum_{n=1}^{\infty} \operatorname{Lip}\bigl(\sigma_*^{-n}\bigr) < \infty. \tag{5}$$

Let $\varepsilon \ge 0$ and assume that $g : X \to Y$ satisfies

$$d(g(x \diamond y), g(x) * g(y)) \le \varepsilon \qquad (x, y \in X). \tag{6}$$

Then, for all $x \in X$, the limit

$$f(x) := \lim_{n\to\infty} \sigma_*^{-n} \circ g \circ \sigma_\diamond^n(x) \tag{7}$$

exists and $f : X \to Y$ is the solution of (2) such that

$$d(g(x), f(x)) \le L\varepsilon \qquad (x \in X). \tag{8}$$

Moreover, f is the only solution of (2) such that the mapping $x \mapsto d(g(x), f(x))$ is bounded on X.

Proof. Define $g_n : X \to Y$ by

$$g_0 := g, \quad g_n := \sigma_*^{-n} \circ g \circ \sigma_\diamond^n \qquad (n \in \mathbb{N}). \tag{9}$$

First we show that, for each fixed $x \in X$, the sequence $\bigl(g_n(x)\bigr)$ is convergent. The metric space being complete, it suffices to show that it is a Cauchy sequence.

Let $x \in X$ be fixed arbitrarily. Replacing x and y by $\sigma_\diamond^n(x)$ in the inequality (6), we get

$$d\bigl(g \circ \sigma_\diamond^{n+1}(x), \sigma_* \circ g \circ \sigma_\diamond^n(x)\bigr) \le \varepsilon.$$

Thus,

$$\begin{aligned} d\big(g_{n+1}(x), g_n(x)\big) &= d\big(\sigma_*^{-(n+1)} \circ g \circ \sigma_\diamond^{n+1}(x), \sigma_*^{-(n+1)} \circ \sigma_* \circ g \circ \sigma_\diamond^{n}(x)\big) \\ &\le \mathrm{Lip}(\sigma_*^{-(n+1)}) \cdot d\big(g \circ \sigma_\diamond^{n+1}(x), \sigma_* \circ g \circ \sigma_\diamond^{n}(x)\big) \\ &\le \mathrm{Lip}(\sigma_*^{-(n+1)}) \cdot \varepsilon. \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} d\big(g_{n+1}(x), g_n(x)\big) \le \sum_{n=0}^{\infty} \mathrm{Lip}(\sigma_*^{-(n+1)}) \cdot \varepsilon < \infty.$$

The series on the right hand side being convergent, we can see that $\big(g_n(x)\big)$ is a Cauchy sequence. For $k \in \mathbb{N}$, we have

$$d\big(g_{k+1}(x), g(x)\big) \le \sum_{n=0}^{k} d\big(g_{n+1}(x), g_n(x)\big) \le \sum_{n=0}^{k} \mathrm{Lip}(\sigma_*^{-(n+1)}) \cdot \varepsilon \le L\varepsilon.$$

Therefore, by passing the limit $k \to \infty$, we obtain (8) for the limit function f of the sequence (g_n).

It remains to prove that f satisfies (2). Replacing x and y by $\sigma_\diamond^n(x)$ and $\sigma_\diamond^n(y)$, respectively, and using that $\sigma_\diamond$ is an endomorphism of $(X, \diamond)$, we get

$$d\big(g \circ \sigma_\diamond^n(x \diamond y), (g \circ \sigma_\diamond^n(x)) * (g \circ \sigma_\diamond^n(y))\big) \le \varepsilon \qquad (x, y \in X,\ n \in \mathbb{N}).$$

Since σ_* is an automorphism of $(Y, *)$, hence its inverse σ_*^{-1} is also an automorphism. Therefore

$$\begin{aligned} &d\big(g_n(x \diamond y), g_n(x) * g_n(y)\big) \\ &\quad = d\big(\sigma_*^{-n}\big(g \circ \sigma_\diamond^n(x \diamond y)\big), \sigma_*^{-n}\big(g \circ \sigma_\diamond^n(x)\big) * \sigma_*^{-n}\big(g \circ \sigma_\diamond^n(y)\big)\big) \\ &\quad = d\big(\sigma_*^{-n}\big(g \circ \sigma_\diamond^n(x \diamond y)\big), \sigma_*^{-n}\big((g \circ \sigma_\diamond^n(x)) * (g \circ \sigma_\diamond^n(y))\big)\big) \\ &\quad \le \mathrm{Lip}(\sigma_*^{-n}) \cdot d\big(g \circ \sigma_\diamond^n(x \diamond y), (g \circ \sigma_\diamond^n(x)) * (g \circ \sigma_\diamond^n(y))\big) \\ &\quad \le \mathrm{Lip}(\sigma_*^{-n}) \cdot \varepsilon. \end{aligned}$$

The right hand side of this inequality is a null sequence (since it is a the general term of a convergent series), hence

$$d\big(f(x \diamond y), f(x) * f(y)\big) = \lim_{n \to \infty} d\big(g_n(x \diamond y), g_n(x) * g_n(y)\big) = 0 \qquad (x, y \in X).$$

Thus, indeed, $f : X \to Y$ satisfies (2).

To complete the proof of the theorem, assume that f is an arbitrary solution of (2) such that the mapping $x \mapsto d(g(x), f(x))$ is bounded with bound K on X.

Putting $x = y$ in (2), we obtain that $f \circ \sigma_\diamond = \sigma_* \circ f$, whence by induction, it follows that

$$f = \sigma_*^{-n} \circ f \circ \sigma_\diamond^n \qquad (n \in \mathbb{N}).$$

Thus, for $x \in X$, we get

$$\begin{aligned} d(g_n(x), f(x)) &= d(\sigma_*^{-n} \circ g \circ \sigma_\diamond^n(x), \sigma_*^{-n} \circ f \circ \sigma_\diamond^n(x)) \\ &\le \mathrm{Lip}(\sigma_*^{-n}) \cdot d(g \circ \sigma_\diamond^n(x), f \circ \sigma_\diamond^n(x)) \\ &\le \mathrm{Lip}(\sigma_*^{-n}) \cdot K. \end{aligned}$$

The right hand side of this inequality tends to zero, hence

$$f(x) = \lim_{n\to\infty} g_n(x) \qquad (x \in X),$$

i.e., f equals the limit of the sequence (g_n). □

The following result is a simple consequence of Theorem 1.

Corollary 1. *Let $(X, \diamond)$ be a square-symmetric and $(Y, *, d)$ be complete metric divisible square-symmetric groupoid. Assume that there exists a number $0 \le q < 1$ such that*

$$d(u, v) \le q \cdot d(u * u, v * v) \qquad (u, v \in Y). \tag{10}$$

Let $\varepsilon \ge 0$ and let $g : X \to Y$ satisfy (6). Then, for all $x \in X$, the limit (7) exists and the function $f : X \to Y$ is the unique solution of (2) such that

$$d(g(x), f(x)) \le \frac{q}{1-q} \cdot \varepsilon \qquad (x \in X). \tag{11}$$

Proof. The inequality (10) yields that $\mathrm{Lip}(\sigma_*^{-1}) \le q$, whence by induction, we get $\mathrm{Lip}(\sigma_*^{-n}) \le q^n$. Thus the number L defined in Theorem 1 does not exceed $q/(1-q)$. □

The following result is another version of Theorem 1. Here the divisibility of the operation $\diamond$ is assumed.

Theorem 2. *Let $(X, \diamond)$ be a divisible square-symmetric and $(Y, *, d)$ be a complete metric square-symmetric groupoid. Assume that*

$$L := \sum_{n=0}^{\infty} \mathrm{Lip}\big(\sigma_*^n\big) < \infty. \tag{12}$$

Let $\varepsilon \geq 0$ and suppose that $g : X \to Y$ satisfies (6). Then, for all $x \in X$, the limit

$$f(x) := \lim_{n\to\infty} \sigma_*^n \circ g \circ \sigma_\diamond^{-n}(x) \tag{13}$$

exists and $f : X \to Y$ is a solution of (2) such that (8) holds. Moreover, f is the only solution of (2) such that the map $x \mapsto d(g(x), f(x))$ is bounded on X.

Proof. Define the sequence $g_n : X \to Y$ by

$$g_0 := g, \quad g_n := \sigma_*^n \circ g \circ \sigma_\diamond^{-n} \qquad (n \in \mathbb{N}). \tag{14}$$

Arguing as in the proof of Theorem 1, we first prove that $\big(g_n(x)\big)$ is a convergent sequence for all $x \in X$. Let $x \in X$ be fixed. Replacing x and y by $\sigma_\diamond^{-n}(x)$ in (6), we get

$$d\big(g \circ \sigma_\diamond^{-n+1}(x), \sigma_* \circ g \circ \sigma_\diamond^{-n}(x)\big) \leq \varepsilon.$$

Hence

$$\begin{aligned} d\big(g_{n-1}(x), g_n(x)\big) &= d\big(\sigma_*^{n-1} \circ g \circ \sigma_\diamond^{-n+1}(x), \sigma_*^{n-1} \circ \sigma_* \circ g \circ \sigma_\diamond^{-n}(x)\big) \\ &\leq \mathrm{Lip}(\sigma_*^{n-1}) \cdot d\big(g \circ \sigma_\diamond^{-n+1}(x), \sigma_* \circ g \circ \sigma_\diamond^{-n}(x)\big) \\ &\leq \mathrm{Lip}(\sigma_*^{n-1}) \cdot \varepsilon. \end{aligned}$$

Using this estimate and repeating the argument followed in the proof of Theorem 1, we get that $\big(g_n(x)\big)$ is a Cauchy sequence and that its limit function f satisfies (8).

Putting $\sigma_\diamond^{-n}(x)$ and $\sigma_\diamond^{-n}(y)$ into x and y in (6), respectively, we obtain

$$d\big(g_n(x \diamond y), g_n(x) * g_n(y)\big) \leq \mathrm{Lip}(\sigma_*^n) \cdot \varepsilon \qquad (x, y \in X),$$

whence, taking the limit $n \to \infty$, we get that $f : X \to Y$ is a homomorfism, that is, it satisfies (2).

The proof of the uniqueness of f is now based on the identity

$$f = \sigma_*^n \circ f \circ \sigma_\diamond^{-n} \qquad (n \in \mathbb{N})$$

and can be obtained exactly as for Theorem 1. □

As an application of Theorem 2, the following result can be derived.

Corollery 2. *Let $(X, \diamond)$ be a divisible square-symmetric and $(Y, *, d)$ be a complete metric square-symmetric groupoid. Assume that there exists a number $0 \leq q < 1$ such that*

$$d(u * u, v * v) \leq q \cdot d(u, v) \qquad (u, v \in Y). \tag{15}$$

Let $\varepsilon \geq 0$ and let $g : X \to Y$ satisfy (6). Then, for all $x \in X$, the limit (13) exists and the function $f : X \to Y$ so defined is the unique solution of (2) such that

$$d(g(x), f(x)) \leq \frac{1}{1-q} \cdot \varepsilon \qquad (x \in X). \tag{16}$$

Proof. It follows from (15) that $\mathrm{Lip}(\sigma_*^n) \leq q^n$ for all $n \in \mathbb{N}$. Thus the number L defined in Theorem 2 cannot be greater than $1/(1-q)$. □

Remark 1. The crucial step of the proofs of Theorem 1 and Theorem 2 was to obtain the convergence of the iterating sequences (7) and (13), respectively. These sequences generalize the Hyers–Ulam sequence defined in (1). This will also be transparent when formulating further special cases of these theorems.

Now we consider the case when Y is a Banach space and the operation $*$ is defined via

$$x * y = Ax + By + c \qquad (x, y \in Y) \tag{17}$$

where $A, B : Y \to Y$ are linear operators and $c \in Y$ is a constant vector.

In order to formulate the result, we recall the notion of the *spectral radius* of a linear operator $C : Y \to Y$ which is defined by

$$\rho(C) := \limsup_{n\to\infty} \sqrt[n]{\|C^n\|}.$$

Theorem 3. *Let $(X, \diamond)$ be a square-symmetric groupoid and Y be a Banach space over $\mathbb{K}$ (where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$). In addition, let $A, B : Y \to Y$ be commuting linear operators and $c \in Y$. Assume that one of the following conditions is also valid:*

(i) *$A + B$ is an invertible linear map and $\rho\big((A+B)^{-1}\big) < 1$, or*

(ii) *$(X, \diamond)$ is a divisible groupoid and $\rho(A+B) < 1$.*

Let $\varepsilon \geq 0$ and let $g : X \to Y$ satisfy the functional inequality

$$\|g(x \diamond y) - Ag(x) - Bg(y) - c\| \leq \varepsilon \qquad (x, y \in X). \tag{18}$$

Then there exists a uniquely determined function $f : X \to Y$ such that

$$f(x \diamond y) = Af(x) + Bf(y) + c \qquad (x, y \in X) \tag{19}$$

and

$$\|g(x) - f(x)\| \leq L\varepsilon \qquad (x \in X), \tag{20}$$

where

$$L := \begin{cases} \sum\limits_{n=1}^{\infty} \|(A+B)^{-n}\|, & \text{if condition (i) holds,} \\ \sum\limits_{n=0}^{\infty} \|(A+B)^{n}\|, & \text{if condition (ii) holds.} \end{cases}$$

Proof. Define the operation $*$ by (17). Then, using $AB = BA$, we can see that $*$ is square-symmetric and

$$\sigma_*^n(x) = (A+B)^n x + \left(\sum_{k=0}^{n-1} (A+B)^k\right) c \qquad (x \in Y,\ n \in \mathbb{N}).$$

If condition (i) is valid, then $*$ is divisible and

$$\sigma_*^{-n}(y) = (A+B)^{-n} y - \left(\sum_{k=0}^{n-1} (A+B)^{k-n}\right) c \qquad (y \in Y,\ n \in \mathbb{N}).$$

Hence

$$\mathrm{Lip}(\sigma_*^{-n}) = \|(A+B)^{-n}\| \qquad (n \in \mathbb{N}).$$

Therefore, the series in (5) is convergent (by $\rho\big((A+B)^{-1}\big) < 1$, all but finitely many terms of this series can be majorized by terms of a convergent geometric series). Thus, Theorem 1 can be applied and yields the statement to be proved.

If condition (ii) is satisfied then the assumptions of Theorem 2 are valid since $\diamond$ is divisible and using

$$\mathrm{Lip}(\sigma_*^n) = \|(A+B)^n\| \qquad (n \in \mathbb{N}),$$

the inequality $\rho(A+B) < 1$ yields that (12) is convergent. Now the statement follows from that of Theorem 2. $\square$

The most important particular case of the above theorem is when the operators A, B can be obtained via multiplication by scalars from $\mathbb{K}$.

Corollary 3. *Let $(X, \diamond)$ be a square-symmetric groupoid and Y be a Banach space over $\mathbb{K}$. Let $a, b \in \mathbb{K}$ and $c \in Y$. Assume that one of the following conditions is satisfied:*

(i) $|a+b| > 1$ *or*

(ii) $(X, \diamond)$ *is divisible and* $|a+b| < 1$.

Let $\varepsilon \geq 0$ and let $g : X \to Y$ satisfy

$$\|g(x \diamond y) - ag(x) - bg(y) - c\| \leq \varepsilon \qquad (x, y \in X). \tag{21}$$

Then there exists a uniquely determined function $f : X \to Y$ such that

$$f(x \diamond y) = af(x) + bf(y) + c \qquad (x, y \in X) \tag{22}$$

and

$$\|g(x) - f(x)\| \leq \frac{\varepsilon}{\big||a+b| - 1\big|} \qquad (x \in X). \tag{23}$$

Proof. Let $Ax := ax$, $Bx := bx$ for $x \in Y$. Then the conditions of Theorem 3 are trivially satisfied. In the case when (i) holds, $A+B$ is invertible, $\rho\big((A+B)^{-1}\big) = |a+b|^{-1} < 1$ and

$$L = \sum_{n=1}^{\infty} |a+b|^{-n} = \frac{1}{|a+b| - 1}.$$

If (ii) is valid, then $\rho(A+B) = |a+b| < 1$ and

$$L = \sum_{n=0}^{\infty} |a+b|^{n} = \frac{1}{1 - |a+b|}.$$

Thus (23) follows from (20). $\square$

Remark 2. (1) The special case $a = b = 1$, $c = 0$ of the above corollary yields the Hyers stability theorem described in the introduction for functions defined on square-symmetric groupoids. In this generality, this result is due to Páles, Volkmann, and Luce [PVL98].

(2) In the case $0 < a+b \neq 1$, the estimate obtained in (23) is the best possible. Indeed, define

$$g(x) = \frac{c + \varepsilon e}{1 - (a+b)} \qquad (x \in X),$$

where $e \in Y$ is an arbitrary unit vector. Then g satisfies (21), furthermore, the function f defined by

$$f(x) = \frac{c}{1 - (a+b)} \qquad (x \in X)$$

solves (22) and

$$\|g(x) - f(x)\| = \frac{\varepsilon}{|a+b-1|} \qquad (x \in X).$$

In order to obtain a further consequences of Theorem 1 and Theorem 2, we now consider the case when $(Y, \cdot)$ is a semigroup with neutral element and the operation $*$ is defined via

$$x * y = H(x, y) \qquad (x, y \in Y), \tag{24}$$

where $H : Y \times Y \to Y$ is a *first-order homogeneous function*, that is,

$$\begin{aligned} H(u \cdot x, u \cdot y) &= u \cdot H(x, y), \\ H(x \cdot u, y \cdot u) &= H(x, y) \cdot u \end{aligned} \qquad (u, x, y \in Y).$$

If $(Y, \cdot)$ is also a metric space with the metric d then we define $u \in Y$ its *d-norm* by

$$\|u\|_d := \sup\left\{ \frac{d(u \cdot x, u \cdot y)}{d(x, y)} \;\Big|\; x, y \in Y,\ x \neq y \right\}.$$

Clearly, $\|u\|_d = \operatorname{Lip} \varphi_u$, where $\varphi_u(x) = u \cdot x$ $(x \in Y)$. It is easy to see that the above norm is submultiplicative, i.e.,

$$\|u \cdot v\|_d \leq \|u\|_d \cdot \|v\|_d \qquad (u, v \in Y).$$

The *d-spectral radius* of an element $u \in Y$ is defined by

$$\rho_d(u) := \limsup_{n\to\infty} \sqrt[n]{\|u^n\|_d}\,.$$

Theorem 4. *Let $(X,\diamond)$ be a square-symmetric groupoid and let $(Y,\cdot,d)$ be a complete metric semigroup with neutral element e. Let $H: Y\times Y\to Y$ be a continuous first-order homogeneous function. Assume that one of the following conditions is satisfied:*

(i) *$H(e,e)\in Y$ is invertible and $\rho_d\big(H(e,e)^{-1}\big)<1$, or*

(ii) *$(X,\diamond)$ is a divisible groupoid and $\rho_d(H(e,e))<1$.*

Let $\varepsilon\geq 0$ and let $g: X\to Y$ satisfy

$$d\big(g(x\diamond y), H(g(x),g(y))\big)\leq\varepsilon \qquad (x,y\in X). \tag{25}$$

Then there exists a uniquely determined function $f: X\to Y$ such that

$$f(x\diamond y) = H(f(x),f(y)) \qquad (x,y\in X) \tag{26}$$

and

$$d(g(x),f(x))\leq L\varepsilon \qquad (x\in X), \tag{27}$$

where

$$L := \begin{cases} \sum\limits_{n=1}^{\infty} \|H(e,e)^{-n}\|_d, & \textit{if condition (i) holds,}\\ \sum\limits_{n=0}^{\infty} \|H(e,e)^{n}\|_d, & \textit{if condition (ii) holds.}\end{cases}$$

Proof. Define $*$ by (24). Then, as it was proved in [PVL98], $*$ is square-symmetric. For convenience and completeness we repeat the proof of [PVL98]. Using the homogeneity of H repeatedly, we get, for $x,y\in X$,

$$\begin{aligned}(x*x)*(y*y) &= H(H(x,x),H(y,y)) = H(H(x\cdot e,x\cdot e),H(y\cdot e,y\cdot e))\\ &= H(x\cdot H(e,e), y\cdot H(e,e)) = H(x,y)\cdot H(e,e)\\ &= H(H(x,y)\cdot e, H(x,y)\cdot e) = H(H(x,y),H(x,y))\\ &= (x*y)*(x*y).\end{aligned}$$

Therefore, $*$ is square-symmetric. Furthermore, $\sigma_*(x) = H(x,x) = H(e\cdot x, e\cdot x) = H(e,e)\cdot x$, thus,

$$\sigma_*^n(x) = H(e,e)^n\cdot x \qquad (x\in Y,\ n\in\mathbb{N}).$$

If condition (i) is satisfied, then

$$\sigma_*^{-n}(x) = H(e,e)^{-n} \cdot x \qquad (x \in Y,\, n \in \mathbb{N}),$$

whence

$$\mathrm{Lip}(\sigma_*^{-n}) = \|H(e,e)^{-n}\|_d \qquad (n \in \mathbb{N}).$$

Using this equality, the statement of this theorem follows from that of Theorem 1.

In the case when (ii) is valid the statement is the consequence of Theorem 2. □

Finally, we consider the case when Y is a closed multiplicative subsemigroup of the real or complex field with unit element. The result obtained in this case is due to Páles, Volkmann, and Luce [PVL98].

Corollary 4. *Let* $(X,\diamond)$ *be a square-symmetric groupoid and* Y *be a closed multiplicative subsemigroup of* $\mathbb{K}$ *such that* $1\in\mathbb{K}$. *Let* $H\colon Y\times Y\to Y$ *be a continuous first-order homogeneous function. Assume that one of the following conditions is satisfied:*

(i) $\frac{1}{H(1,1)} \in Y$ *and* $|H(1,1)| > 1$, *or*

(ii) $(X,\diamond)$ *is a divisible groupoid and* $|H(1,1)| < 1$.

Let $\varepsilon \geq 0$ *and let* $g : X \to Y$ *satisfy*

$$|g(x \diamond y) - H(g(x), g(y))| \leq \varepsilon \qquad (x, y \in X). \tag{28}$$

Then there exists a uniquely determined function $f : X \to Y$ *such that* (26) *holds and*

$$|g(x) - f(x)| \leq \frac{\varepsilon}{||H(1,1)| - 1|} \qquad (x \in X). \tag{29}$$

Proof. Applying the previous theorem and the argument followed when showing Corollary 3, the proof can be carried out easily. □

Remark 3. (1) We note that the main results – Theorem 1 and Theorem 2 – of this paper were motivated by the above corollary observing that operations of the form $x * y = H(x,y)$ (where H is a first-order homogeneous function) are square-symmetric.

(2) The above results does not yield the stability of (26) in the case $|H(1,1)| = 1$, thus e.g., the stability of the Jensen functional equation obtained by Kominek [Kom89] does not follow from our results directly.

(3) The estimate obtained in (29) is the best possible if $Y = \mathbb{K}$ and $H(1,1)$ is a positive number different from 1. Indeed, the function g given by

$$g(x) = \frac{\varepsilon}{H(1,1) - 1} \qquad (x \in X)$$

satisfies the stability inequality (28), the function $f \equiv 0$ solves (26) and

$$|g(x) - f(x)| = \frac{\varepsilon}{|H(1,1) - 1|} \qquad (x \in X).$$

To illustrate Corollary 4, we consider the functional equations

$$f\big(x + y - \sqrt[3]{x^3 + y^3}\,\big) = f(x) + f(y) - \sqrt[3]{f^3(x) + f^3(y)} \qquad (x, y \in \mathbb{R}) \tag{30}$$

and

$$f\big(\sqrt[3]{x^3 + x^2y + xy^2 + y^3}\,\big) = \sqrt[3]{f^3(x) + f^2(x)f(y) + f(x)f^2(y) + f^3(y)} \qquad (x, y \in \mathbb{R}). \tag{31}$$

The equation (30) is of the form (26) if we take $x \diamond y = x * y = H(x,y) = x + y - \sqrt[3]{x^3 + y^3}$ $(x, y \in \mathbb{R})$, $X = Y = \mathbb{R}$. Since H is homogeneous, the operations $*$ and $\diamond$ are square-symmetric. Due to $H(1,1) = 2 - \sqrt[3]{2} < 1$ and $H(1,1) \neq 0$. we have that $\diamond$ is divisible, therefore, condition (ii) of Corollary 4 holds. Thus (30) is stable, more precisely, if $\varepsilon \geq 0$ and $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$\big|g(x + y - \sqrt[3]{x^3 + y^3}\,) - g(x) - g(y) + \sqrt[3]{g^3(x) + g^3(y)}\,\big| \leq \varepsilon \quad (x, y \in \mathbb{R}),$$

then there exists $f : \mathbb{R} \to \mathbb{R}$ such that (30) holds and

$$|g(x) - f(x)| \leq \frac{\varepsilon}{\sqrt[3]{2} - 1} \qquad (x \in \mathbb{R}).$$

The equation (31) also reduces to (26) if we take $x \diamond y = x * y = H(x,y) = \sqrt[3]{x^3 + x^2y + xy^2 + y^3}$ $(x, y \in \mathbb{R})$, $X = Y = \mathbb{R}$. The function H is homogeneous, too, therefore $*$ and $\diamond$ are square-symmetric operations. $H(1,1)=\sqrt[3]{4}>1$, thus condition (i) of Corollary 4 is satisfied. Therefore (30) is stable, that is, if $\varepsilon\ge 0$ and $g:\mathbb{R}\to\mathbb{R}$ satisfies

$$\left|g\left(\sqrt[3]{x^3+x^2y+xy^2+y^3}\,\right)-\sqrt[3]{g^3(x)+g^2(x)g(y)+g(x)g^2(y)+g^3(y)}\,\right|\le\varepsilon \qquad (x,y\in\mathbb{R}),$$

then there exists a function $f:\mathbb{R}\to\mathbb{R}$ that solves (31) and satisfies

$$|g(x)-f(x)|\le\frac{\varepsilon}{\sqrt[3]{4}-1}\qquad (x\in\mathbb{R}).$$

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