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Strong convergence theorems for $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$

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Abstract. Multiplier operators on the Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ are investigated and Bernstein's inequality for multi-parameter trigonometric polynomials is verified. We prove that certain means of the partial sums of the multi-parameter trigonometric Fourier series are uniformly bounded operators from $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ to L_p $(1/2 < p \leq 1)$. As a consequence we obtain strong convergence theorems concerning the partial sums. The dual inequalities are also verified and a Marcinkiewicz–Zygmund type inequalities is obtained for the $BMO(\mathbb{T} \times \cdots \times \mathbb{T})$ spaces.

1. Introduction

We introduce the d-dimensional Hardy space $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ by the $L_p(\mathbb{T}^d)$ norm of the non-tangential maximal function of a distribution on $\mathbb{T}^d$. It is known that the trigonometric system is not a basis in $L_1(\mathbb{T})$. Moreover, there exist functions in $H_1(\mathbb{T})$, the partial sums of which are not bounded in $L_1(\mathbb{T})$. Smith [10] and recently Belinskii [1] proved the following strong convergence result for one-parameter trigonometric Fourier series:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|s_k f - f\|_1}{k} = 0$$

where $f \in H_1(\mathbb{T})$ and $s_k f$ denotes the k-th partial sum of the Fourier series. This result for one-parameter Walsh–Fourier series can be found in Simon [9].

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Recently the author [12] generalized this result for two-parameter trigonometric Fourier series by taking the sum over a cone. More exactly, we verified that there exists a constant C depending only on $\alpha > 0$ such that

$$\frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \le k/l \le 2^{\alpha} \\ (k,l) \le (n,m)}} \frac{\|s_{k,l} f\|_1}{kl} \le C \|f\|_{H_1(\mathbb{T}^2)}.$$

Note that the space $H_1(\mathbb{T}^2)$ defined in [12] is different from $H_1(\mathbb{T} \times \mathbb{T})$ used here. With the help of Riesz and conjugate transforms one can show that $\| \cdot \|_{H_1(\mathbb{T}^2)} \le \| \cdot \|_{H_1(\mathbb{T} \times \mathbb{T})}$. We obtained also the convergence result

$$\frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \le k/l \le 2^{\alpha} \\ (k,l) \le (n,m)}} \frac{\|s_{k,l} f - f\|_1}{kl} \to 0 \quad \text{as } n, m \to \infty$$

where $f \in H_1(\mathbb{T}^2)$. The analogous results for two-parameter Walsh–Fourier series can also be found in [12].

In this paper we extend these theorems to the d-dimensional case and prove an even stronger inequality for $f \in H_1(\mathbb{T} \times \cdots \times \mathbb{T})$:

$$\frac{1}{\prod_{i=1}^d \log n_i} \sum_{i=1}^{d} \sum_{k_i=1}^{n_i} \frac{\|s_k f\|_{H_1(\mathbb{T} \times \cdots \times \mathbb{T})}}{\prod_{i=1}^d k_i} \le C \|f\|_{H_1(\mathbb{T} \times \cdots \times \mathbb{T})}$$

where C is an absolute constant. From this it follows easily that

$$\lim_{n \to 0} \frac{1}{\prod_{i=1}^d \log n_i} \sum_{i=1}^{d} \sum_{k_i=1}^{n_i} \frac{\|s_k f - f\|_{H_1(\mathbb{T} \times \cdots \times \mathbb{T})}}{\prod_{i=1}^d k_i} = 0$$

whenever $f \in H_1(\mathbb{T} \times \cdots \times \mathbb{T})$. We extend these results also to $p < 1$, which was unknown even in the one-parameter case.

In the proof we have to use a different method than in [12], we use the multi-parameter Hardy–Littlewood inequality (see Jawerth and Torchinsky [8]) and the fact that the maximal operator of the Cesàro means of a distribution is bounded from $H_p(\mathbb{T} \times \cdots \times \mathbb{T})$ to $L_p(\mathbb{T}^d)$ (see Weisz [13]).

Moreover, we extend Bernstein's inequality to multi-parameter trigonometric polynomials. We investigate also multiplier operators and give a sufficient condition for the multiplier such that the operator is bounded on the Hardy space.

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2. Hardy spaces and conjugate functions

For a set $\boldsymbol{X} \neq \emptyset$ let $\boldsymbol{X}^d$ be its Cartesian product taken with itself d-times, moreover, let $\mathbb{T} := [-\pi, \pi)$ and λ be the Lebesgue measure. We briefly write L_p instead of the $L_p(\mathbb{T}^d, \lambda)$ space while the norm (or quasi-norm) of this space is defined by $\|f\|_p := (\int_{\mathbb{T}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leq \infty)$.

For $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$ set $n \cdot x := \sum_{i=1}^d n_i x_i$. Let f be a distribution on $C^\infty(\mathbb{T}^d)$. The nth Fourier coefficient is defined by $\hat{f}(n) := f(e^{-\imath n \cdot x})$ where $\imath = \sqrt{-1}$ and $n \in \mathbb{Z}^d$. In the special case when f is an integrable function then

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath n \cdot x} \, dx.$$

For a distribution f and $z_i := r_i e^{\imath x_i}$ $(0 < r_i < 1)$ let

$$u(z) = u(r_1 e^{\imath x_1}, \ldots, r_d e^{\imath x_d}) := (f * P_{r_1} \times \cdots \times P_{r_d})(x) \qquad (x \in \mathbb{T}^d)$$

where $*$ denotes the convolution and

$$P_r(y) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath k y} = \frac{1 - r^2}{1 + r^2 - 2r \cos y} \qquad (y \in \mathbb{T})$$

is the Poisson kernel. It is easy to show that $u(z)$ is a multi-harmonic function.

Let $0 < \alpha < 1$ be an arbitrary number. We denote by $\Omega_\alpha(x)$ $(x \in \mathbb{T})$ the region bounded by two tangents to the circle $|z| = \alpha$ from $e^{\imath x}$ and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u_\alpha^*(x) := \sup_{z_i \in \Omega_{\alpha_i}(x_i)} |u(z)| \qquad (0 < \alpha_i < 1; \ i = 1, \ldots, d).$$

The *Hardy space* $H_p(\mathbb{T} \times \cdots \times \mathbb{T}) = H_p$ $(0 < p \leq \infty)$ consists of all distributions f for which $u_\alpha^* \in L_p$ and set

$$\|f\|_{H_p} := \|u^*_{1/2, \ldots, 1/2}\|_p \, .$$

The equivalence $\|u_\alpha^*\|_p \sim \|f\|_{H_p}$ $(0 < p \le \infty,\ 0 < \alpha_i < 1)$ was proved in FEFFERMAN, STEIN [4] and GUNDY, STEIN [7].

For a distribution

$$f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{\imath n \cdot x}$$

the *conjugate distributions* are defined by

$$\tilde{f}^{(j_1,\ldots,j_d)} \sim \sum_{n \in \mathbb{Z}^d} \left(\prod_{i=1}^d (-\imath \text{ sign } n_i)^{j_i} \right) \hat{f}(n) e^{\imath n \cdot x} \qquad (j_i = 0, 1).$$

Note that $\tilde{f}^{(0,\ldots,0)} := f$. GUNDY and STEIN [6], [7] verified that if $f \in H_p$ $(0 < p < \infty)$ then all conjugate distributions are also in H_p and

$$\|f\|_{H_p} = \|\tilde{f}^{(j_1,\ldots,j_d)}\|_{H_p} \qquad (j_i = 0, 1). \tag{1}$$

Furthermore (see also CHANG and FEFFERMAN [2], FRAZIER [5], DUREN [3]),

$$\|f\|_{H_p} \sim \sum_{i=1}^d \sum_{j_i=0}^1 \|\tilde{f}^{(j_1,\ldots,j_d)}\|_p, \tag{2}$$

where $\sim$ denotes the equivalences of the spaces and norms.

For a distribution f with Fourier series

$$f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{\imath n \cdot x} \quad \text{let} \quad Pf \sim \sum_{n \in \mathbb{N}^d} \hat{f}(n) e^{\imath n \cdot x}$$

be the Riesz projection. Then $f \in H_p$ if and only if $Pf \in L_p$ and

$$\|f\|_{H_p} \sim \|Pf\|_p \qquad (0 < p < \infty) \tag{3}$$

(see GUNDY and STEIN [6], [7]). Moreover, it is known that $H_p \sim L_p$ $(1 < p < \infty)$.

In this paper the constants C are absolute constants and the constants C_p are depending only on p and may denote different constants in different contexts.

JAWERTH and TORCHINSKY [8] proved the following theorem.

Theorem A. *For every distribution $f \in H_p$*

$$\left(\sum_{i=1}^{d}\sum_{|n_i|=0}^{\infty}\frac{|\hat{f}(n)|^p}{\prod_{i=1}^{d}|n_i\vee 1|^{2-p}}\right)^{1/p}\le C_p\|f\|_{H_p}\qquad(0<p\le 2).$$

Denote by $s_n f$ the nth partial sum of the Fourier series of a distribution f, namely,

$$s_n f(x) := \sum_{i=1}^{d}\sum_{k_i=-n_i}^{n_i}\hat{f}(k)e^{\imath k\cdot x}.$$

For $n \in \mathbb{N}^d$ and a distribution f the *Cesàro mean* of order n of the Fourier series of f is given by

$$\sigma_n f := \frac{1}{\prod_{i=1}^{d}(n_i+1)}\sum_{i=1}^{d}\sum_{k_i=0}^{n_i}s_k f = f*(K_{n_1}\times\cdots\times K_{n_d})$$

where

$$K_m(t) := \sum_{|j|=0}^{m}\left(1-\frac{|j|}{m+1}\right)e^{\imath jt}\qquad(m\in\mathbb{N})$$

is the one-dimensional Fejér kernel of order m. It is shown in Zygmund [14] that $K_m \ge 0$ and

$$\int_{\mathbb{T}}K_m(t)\,dt=\pi\qquad(m\in\mathbb{N}). \tag{4}$$

The following result is due to the author [13].

Theorem B. *If $f \in H_p$, then*

$$\|\sup_{n\in\mathbb{N}^d}|\sigma_n f|\,\|_p\le C_p\|f\|_{H_p}\qquad(1/2<p<\infty).$$

3. Strong convergence results

A sequence $(\lambda_k; k\in\mathbb{Z}^d)$ is said to be a *multiplier* and the *multiplier operator* is defined by

$$M_\lambda f(x) := \sum_{k\in\mathbb{Z}^d}\lambda_k\hat{f}(k)e^{\imath k\cdot x}.$$

Let $(\lambda_k; k\in\mathbb{Z}^d)$ be an even sequence of real numbers, i.e. $\lambda_{\epsilon_1 k_1,\ldots,\epsilon_d k_d}=\lambda_k$ for all $\epsilon_i=-1,1$ and $k\in\mathbb{Z}^d$. Suppose that there exists $K\in\mathbb{N}^d$ such that $\lambda_k=0$ if $k_j\geq K_j$ for some $j=1,\ldots,d$. Let

$$\Delta^1\lambda_k:=\sum_{\epsilon_1,\ldots,\epsilon_d\in\{0,1\}}(-1)^{\epsilon_1+\cdots+\epsilon_k}\lambda_{k_1+\epsilon_1,\ldots,k_d+\epsilon_d}$$

be the first and

$$\Delta^2\lambda_k:=\sum_{\epsilon_1,\ldots,\epsilon_d\in\{0,1\}}(-1)^{\epsilon_1+\cdots+\epsilon_k}\Delta^1\lambda_{k_1+\epsilon_1,\ldots,k_d+\epsilon_d}$$

be the second difference of (λ_k).

Lemma 1. *Suppose that (λ_k) is an even multiplier and there exists $K\in\mathbb{N}^d$ such that $\lambda_k=0$ if $k_j\geq K_j$ for some $j=1,\ldots,d$. If $\Lambda:=\sum_{k\in\mathbb{N}^d}\Big(\prod_{i=1}^d(k_i+1)\Big)|\Delta^2\lambda_k|<\infty$ then*

$$\|M_\lambda f\|_{H_p}\leq C_p\Lambda\|f\|_{H_p}\qquad(f\in H_p)$$

for every $1/2<p<\infty$.

Proof. Applying Abel rearrangement twice and Theorem B we get that

$$\|M_\lambda f\|_p=\left\|\sum_{k\in\mathbb{N}^d}\left(\prod_{i=1}^d(k_i+1)\right)\Delta^2\lambda_k\sigma_k f\right\|_p\leq C_p\Lambda\|f\|_{H_p}\quad(1/2<p<\infty).$$

This together with (1) implies that

$$\|(M_\lambda f)^{\sim(j_1,\ldots,j_d)}\|_p=\|M_\lambda\tilde f^{(j_1,\ldots,j_d)}\|_p\leq C_p\Lambda\|\tilde f^{(j_1,\ldots,j_d)}\|_{H_p}=C_p\Lambda\|f\|_{H_p}$$

for $j_i=0,1$ and $1/2<p<\infty$. The equivalence (2) proves now the lemma. □

Let us consider the function

$$v(t):=\begin{cases}1 & \text{if } |t|<1\\ 2-|t| & \text{if } 1\leq|t|\leq 2\\ 0 & \text{if } |t|>2\end{cases}$$

and the multiplier operator V_{2N} defined by

$$V_{2N}f(x) := \sum_{k\in\mathbb{N}^d}\left(\prod_{i=1}^d v\left(\frac{k_i}{N_i}\right)\right)\hat{f}(k)e^{\imath k\cdot x}.$$

Lemma 2. *If $1/2 < p < \infty$ then*

$$\|V_{2N}f\|_{H_p} \le C_p\|f\|_{H_p} \qquad (f\in H_p).$$

PROOF. Let $\lambda_k := \prod_{i=1}^d v(\frac{k_i}{N_i})$. It is easy to see that $\Delta^2\lambda_k = \prod_{i=1}^d \times \Delta^2 v(\frac{k_i}{N_i})$ and so we have

$$\sum_{k\in\mathbb{N}^d}\left(\prod_{i=1}^d(k_i+1)\right)|\Delta^2\lambda_k| = 3^d$$

which proves the result. □

Now we extend the well known Bernstein's inequality from one- to multi-parameter trigonometric polynomials.

Lemma 3. *Let f be a trigonometric polynomial in the i-th variable of order N_i. If $I\subset\{1,\dots,d\}$, then for every $1\le p<\infty$*

$$\left\|\left(\prod_{i\in I}\partial_i\right)f\right\|_p \le C\left(\prod_{i\in I}N_i\right)\|f\|_p\,.$$

PROOF. Let us define

$$\phi_{N_i,i\in I}(y) := \prod_{i\in I}\left(K_{N_i-1}(y_i)(e^{\imath N_i y_i}+e^{-\imath N_i y_i})\right) \qquad (y=(y_i,\ i\in I)).$$

Then by (4), $\|\phi_{N_i,i\in I}\|_1 = C$ and

$$\phi_{N_i,i\in I}(y) = \sum_{i\in I}\sum_{|k_i|=0}^{N_i-1}\left(\prod_{i\in I}\left(1-\frac{|k_i|}{N_i}\right)\left(e^{\imath(k_i+N_i)y_i}+e^{\imath(k_i-N_i)y_i}\right)\right).$$

It is easy to see that $\hat{\phi}_{N_i,i\in I}(k) = \prod_{i\in I}\frac{k_i}{N_i}$ for $-N_i\le k_i\le N_i$. Then

$$\phi_{N_i,i\in I}*f = -\frac{(\prod_{i\in I}\partial_i)f}{\prod_{i\in I}N_i}$$

proves the lemma. □

Our main result is the following

Theorem 1. *If $f \in H_p$ and $1/2 < p \leq 1$ then*

$$\sup_{N_i \geq 2} \left(\frac{1}{\prod_{i=1}^d \log N_i} \right)^{[p]} \sum_{i=1}^{d} \sum_{k_i=1}^{N_i} \frac{\|s_k f\|_{H_p}^p}{\prod_{i=1}^d k_i^{2-p}} \leq C_p \|f\|_{H_p}^p$$

where $[p]$ denotes the integer part of p.

Proof. To avoid some technical difficulties, we prove the theorem for two parameters, only. By (3), it is enough to show that

$$\sup_{N,M \geq 2} \left(\frac{1}{\log N \log M} \right)^{[p]} \sum_{k=1}^{N} \sum_{l=1}^{M} \frac{\|s_{k,l}(Pf)\|_p^p}{(kl)^{2-p}} \leq C_p \|Pf\|_p^p$$

whenever $f \in H_p$ and $1/2 < p \leq 1$.

It is easy to see that

$$\sum_{k=1}^{N} \sum_{l=1}^{M} \frac{\|s_{k,l}(Pf)\|_p^p}{(kl)^{2-p}} \leq \sum_{k=1}^{2N} \sum_{l=1}^{2M} \frac{\|s_{k,l}(V_{2N,2M}(Pf))\|_p^p}{(kl)^{2-p}}.$$

For fixed x and y, the (k,l)-th Fourier coefficient of

$$\sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}\big(V_{2N,2M}(Pf)\big)(x,y) e^{\imath kt} e^{\imath lu}$$

is $s_{k,l}(V_{2N,2M}(Pf))(x,y)$. Then we can apply Theorem A and (3) to obtain

$$\sum_{k=1}^{2N} \sum_{l=1}^{2M} \frac{|s_{k,l}(V_{2N,2M}(Pf))(x,y)|^p}{(kl)^{2-p}}$$
$$\leq C_p \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \sum_{k=1}^{2N} \sum_{l=1}^{2M} s_{k,l}\big(V_{2N,2M}(Pf)\big)(x,y) e^{\imath kt} e^{\imath lu} \right|^p dt\, du.$$

Using the notation

$$a_{n,m} := v\left(\frac{n}{N}\right) v\left(\frac{m}{M}\right) \hat{f}(n,m) e^{\imath nx} e^{\imath my}$$

we have

$$\sum_{k=1}^{2N}\sum_{l=1}^{2M} s_{k,l}\big(V_{2N,2M}(Pf)\big)(x,y)e^{\imath kt}e^{\imath lu} = \sum_{k=1}^{2N}\sum_{l=1}^{2M}\sum_{n=1}^{k}\sum_{m=1}^{l} a_{n,m}e^{\imath kt}e^{\imath lu}$$

$$= \sum_{n=1}^{2N}\sum_{m=1}^{2M} a_{n,m}\frac{e^{\imath(2N+1)t}-1}{e^{\imath t}-1}\frac{e^{\imath(2M+1)u}-1}{e^{\imath u}-1}$$

$$+ \sum_{n=1}^{2N}\sum_{m=1}^{2M} a_{n,m}\frac{e^{\imath(2N+1)t}-1}{e^{\imath t}-1}\frac{1-e^{\imath mu}}{e^{\imath u}-1}$$

$$+ \sum_{n=1}^{2N}\sum_{m=1}^{2M} a_{n,m}\frac{1-e^{\imath nt}}{e^{\imath t}-1}\frac{e^{\imath(2M+1)u}-1}{e^{\imath u}-1}$$

$$+ \sum_{n=1}^{2N}\sum_{m=1}^{2M} a_{n,m}\frac{1-e^{\imath nt}}{e^{\imath t}-1}\frac{1-e^{\imath mu}}{e^{\imath u}-1}$$

$$= (A)+(B)+(C)+(D).$$

Recall that for the Dirichlet kernel

$$D_N(t) := \frac{1}{2}e^{-\imath Nt}\frac{e^{\imath(2N+1)t}-1}{e^{\imath t}-1}$$

we have

$$\|D_N\|_1 \sim \log N \quad \text{and} \quad |D_N(t)| \le \frac{C}{t} \qquad (N\in\mathbb{N})$$

(see e.g. Torchinsky [11]). Applying this, Lemma 2 and (3) we conclude that

$$\int_{\mathbb{T}^4} |(A)|^p\,dt\,du\,dx\,dy = \int_{\mathbb{T}^2}\left|\frac{e^{\imath(2N+1)t}-1}{e^{\imath t}-1}\right|^p \left|\frac{e^{\imath(2M+1)u}-1}{e^{\imath u}-1}\right|^p dt\,du$$

$$\times \|V_{2N,2M}(Pf)\|_p^p \le \begin{cases} C\log N\log M\|Pf\|_1 & \text{if } p=1,\\ C_p\|Pf\|_p^p & \text{if } 1/2<p<1.\end{cases}$$

For the second term we obtain

$$\int_{\mathbb{T}^4} |(B)|^p \, dt \, du \, dx \, dy = \int_{\mathbb{T}} \left| \frac{e^{\imath(2N+1)t} - 1}{e^{\imath t} - 1} \right|^p \, dt$$

$$\times \int_{\mathbb{T}} \left| \frac{1}{e^{\imath u} - 1} \right|^p \int_{\mathbb{T}^2} \left| V_{2N,2M}(Pf)(x,y) - V_{2N,2M}(Pf)(x,y+u) \right|^p dx \, dy \, du,$$

which can be estimated by $C_p \|Pf\|_p^p$ if $1/2 < p < 1$ and, moreover, if $p = 1$ then by

$$C \log N \int_{|u|<1/M} \frac{1}{|u|} \int_{\mathbb{T}^2} \left| \int_0^u \partial_2 V_{2N,2M}(Pf)(x, y+w) \, dw \right| dx \, dy \, du$$

$$+ C \log N \int_{|u| \geq 1/M} \frac{1}{|u|} \|V_{2N,2M}(Pf)\|_1 \, du =: (B_1) + (B_2).$$

It is easy to see that $(B_2) \leq C \log N \log M \|Pf\|_1$. By Lemma 3,

$$(B_1) \leq C \log N \|V_{2N,2M}(Pf)\|_1 \leq C \log N \|Pf\|_1.$$

The estimation of (C) is similar. Let us consider (D).

$$\int_{\mathbb{T}^4} |(D)|^p \, dt \, du \, dx \, dy = \int_{\mathbb{T}^2} \left| \frac{1}{e^{\imath t} - 1} \right|^p \left| \frac{1}{e^{\imath u} - 1} \right|^p \int_{\mathbb{T}^2} \big| V_{2N,2M}(Pf)(x,y)$$

$$- V_{2N,2M}(Pf)(x, y+u) - V_{2N,2M}(Pf)(x+t, y)$$

$$+ V_{2N,2M}(Pf)(x+t, y+u) \big|^p \, dx \, dy \, dt \, du.$$

This can be estimated by $C_p \|Pf\|_p^p$ if $1/2 < p < 1$. In case $p = 1$ we split the integral with respect to t and u into the integrals over the sets $\{|t| < 1/N, |u| < 1/M\}$, $\{|t| < 1/N, |u| \geq 1/M\}$, $\{|t| \geq 1/N, |u| < 1/M\}$ and $\{|t| \geq 1/N, |u| \geq 1/M\}$ and we denote these integrals by (D_1), (D_2), (D_3) and (D_4), respectively. Applying Bernstein's inequality we obtain

$$(D_1) \leq C \int_{|t|<1/N} \int_{|u|<1/M} \frac{1}{|tu|}$$

$$\times \int_{\mathbb{T}^2} \left| \int_0^t \int_0^u \partial_1 \partial_2 V_{2N,2M}(Pf)(x+v, y+w) \, dv \, dw \right| dx \, dy \, dt \, du \leq C \|Pf\|_1.$$

Similarly,

$$(D_2) \le C \int_{|t|<1/N} \int_{|u|\ge 1/M} \frac{1}{|tu|} \int_{\mathbb{T}^2} \Big| \int_0^t \partial_1 V_{2N,2M}(Pf)(x+v,y) - \partial_1 V_{2N,2M}(Pf)(x+v,y+u)\,dv \Big|\,dx\,dy\,dt\,du \le C \log M \|Pf\|_1.$$

(D_3) can be estimated in the same way. For (D_4) we have simply

$$(D_4) \le C \log N \log M \|Pf\|_1,$$

which finishes the proof of Theorem 1. □

The set of the trigonometric polynomials is dense in H_p, so by the usual density argument we can easily verify the next consequence (cf. WEISZ [12]).

Corollary 1. *If $f \in H_p$ and $1/2 < p \le 1$ then*

$$\lim_{N\to\infty} \left(\frac{1}{\prod_{i=1}^d \log N_i} \right)^{[p]} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{\|s_k f - f\|_{H_p}^p}{\prod_{i=1}^d k_i^{2-p}} = 0.$$

Since $\|\cdot\|_p \le \|\cdot\|_{H_p}$, we get

$$\lim_{N\to\infty} \frac{1}{\prod_{i=1}^d \log N_i} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{\|s_k f - f\|_1}{\prod_{i=1}^d k_i} = 0$$

whenever $f \in H_1$, which was proved by SMITH [10] in the one-parameter case.

We now give the dual inequality to Theorem 1, which is a Marcinkiewicz–Zygmund type inequality for the BMO space, where BMO is the dual of H_1. Since the proof is similar to that of Theorem 3 in WEISZ [12], we omit it.

Theorem 2. *If g^k $(k \in \mathbb{N}^d)$ are uniformly bounded in BMO then*

$$\sup_{N_i \ge 2} \left\| \frac{1}{\prod_{i=1}^d \log N_i} \sum_{i=1}^d \sum_{k_i=1}^{N_i} \frac{s_k g^k}{\prod_{i=1}^d k_i} \right\|_{BMO} \le C \sup_{k\in\mathbb{N}^d} \|g^k\|_{BMO}.$$

Note that the corresponding results for multi-parameter Walsh–Fourier series are still unknown.

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