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On some properties of simplices in spaces of constant curvature

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Abstract. Let $k \geq 2$ and $d \geq k+2$. Martini [MH93] proved for $k = 2$ that a d-dimensional simplex S_d in spaces of constant curvature is regular if the 2-faces of S_d are congruent. We shall prove analogous theorems for $k = 3$ and $k = 4$ and show that a similar statement for $d = k+1$ is false.

1. Introduction

1.1. A tetrahedron S_3 is called isosceles if all 2-faces of S_3 are congruent. Isosceles tetrahedra have interesting properties in Euclidean space R^3. The following analogous equivalent statements for a tetrahedron S_3 in 3-dimensional spherical and hyperbolic space was proved by the author [HJ69, HJNL96].

1.1.1. The tetrahedron S_3 is an isosceles tetrahedron.

1.1.2. The 2-faces of S_3 have equal areas.

1.1.3. The stereoangles at the vertices of S_3 are congruent.

1.1.4. The measures of the above stereoangles are equal.

1.1.5. The circumcircles of the 2-faces of S_3 have the same radius.

Let O, K and L denote the midpoint of the circumscribed ball, the midpoint of the inscribed ball and the centroid of S_3.

1.1.6. At least two of the points O, K and L coincide.

1.1.7. The perimeters of the faces of S_3 are equal.

1.1.8. The perimeters of the vertex figures of S_3 are equal.

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Bui Van Dung [BVD84] proved the majority of the above properties for tetrahedra in hyperbolic 3-space whose vertices are ideal.

1.2. H. Martini [MH93] proved that the following four properties of a d-simplex in spaces of constant curvature (R^d, S^d, H^d) for $d \geq 4$ are equivalent.

1.2.1. The simplex S_d is regular.

1.2.2. The 2-faces of S_d have equal areas.

1.2.3. The 2-faces of S_d are congruent.

1.2.4. The measures of the four stereoangles of each 3-face of S_d are equal to each other.

As a consequence of the results in [HJ69] and [HJNL96] it is sufficient to show that the statements 1.2.1 and 1.2.3 are equivalent. These results imply the equivalences of the statements according to 1.1.5–1.1.8.

Analogous results were derived in [FPMH90] for Euclidean simplices.

Let us consider a d-simplex S_d in R^d, S^d, H^d. An edge of length a is called an *a-edge*. Let $(0, 1, 2, \ldots, n)$ denote the simplex with vertices $0, 1, 2, \ldots, n$.

2. Results

The 3-faces of S_d are congruent isosceles tetrahedra in cases 1.2.2–1.2.4. The result of Martini can be formulated as follows. If the 3-faces of S_d are congruent isosceles tetrahedra for $d \geq 4$, then S_d is regular.

Theorem 1. *If the 3-faces of a d-simplex S_d in d-dimensional spaces of constant curvature for $d \geq 5$ are congruent, then S_d is regular.*

Proof. It is sufficent to prove the statement for $d = 5$. The number of the edges of S_5 is 15.

Let $01 = a$. There exists an a-edge in (2345) too. Let $23 = a$. Then (0123) has two skew a-edges. Hence (1245) has also two skew a-edges. There are two possibilities.

α) $12 = 45 = a$. It follows from this that the simplex (0123) has 3 joining a-edges and the number of the a-edges of S_5 is at least 4. The simplex (0345) contains also at least 3 a-edges. Hence S_5 has at least 6 a-edges.

β) $25 = 14 = a$ $(15 = 24 = a)$. The simplex S_5 contains at least 4 a-edges. There are at least 2 a-edges in (0345), thus that S_5 has at least 6 a-edges.

If S_5 is not regular, then there exists a b-edge with $a \neq b$. By similar arguments the number of the b-edges is at least 6.

The role of a and b can be interchanged hence the number of the a-edges and the b-edges is equal. Keeping in mind that the number of the edges of S_5 is 15, the number of the a-edges is at most 5. But this is a contradiction to α) and β) and the theorem is proved. □

Theorem 2. *If the 4-faces of a d-simplex S_d in d-dimensional spaces of constant curvature for $d \geq 6$ are congruent, then S_d is regular.*

Proof. It is sufficent to prove the statement for $d = 6$. The number of the edges of S_6 is 21. We prove that the number of the a-edges is at least 8.

Let $01 = a$. The simplex (23456) has also an a-edge. Let $23 = a$. Then (01234) has two skew a-edges. Hence (03456) has also two skew a-edges. In this case there is an a-edge among the edges with endpoints 4 or 5 or 6 and 0 or 3. Let $04 = a$. Then (01234) has at least 3 a-edges, two of them intersect (01 and 04) and the third (23) is in a screw position in regard to the intersecting a-edges. It follows that the simplex (12456) has also at least 3 a-edges of preceding types. Then there is an a-edge among the edges with endpoints 1, 2 or 4. The simplex (01234) has also a fourth a-edge and these a-edges joint to one another. For example let $12 = a$. The simplex (12456) contains also at least 4 a-edges, 12 and at least three other a-edges (e.g. $16 = 65 = 54 = a$). It follows that S_6 has at least 7 a-edges. But in this case there are at least 5 a-edges in (01456), and the same holds in (01234), too. The fifth a-edge is different from the preceding a-edges. Hence the number of the a-edges of S_6 is at least 8.

It can be proved analogously to the proof of Theorem 1 that S_6 contains at most 7 a-edges. But this is a contradiction to the number 8 and the theorem is proved. □

Conjecture. Let $k \geq 5$ and $d \geq k + 2$. If the k-faces of a d-simplex S_d in d-dimensional spaces of constant curvature are congruent, then S_d is regular.

Remark. The conjecture is false for $d = k + 1$. Let the graph of the simplex S_{k+1} be a regular $(k+2)$-gon. The edges of S_{k+1} which correspond to the equal sides or diagonals of the $(d + 1)$-gon, are equal. If we omit a vertex, then we get the graph of a k-face. It is clear that the k-faces of S_{k+1} are congruent but the simplex S_{k+1} is not regular.

There exists such a simplex S_{k+1} of the above type in R^{k+1} whose congruent examples are the faces of a tesselation in R^{k+1}.

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