The Tavakol–van den Bergh conditions in the theories of gravity and projective changes of Finsler metrics

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Dedicated to Professor Lajos Tamássy on his 70th birthday

In 1986 R. K. TAVAKOL and N. van den BERG [9] were concerned with the constructive-axiomatic development of the theory of gravity given by EHLERS, PIRANI and SCHILD (EPS) which allows a transparent relation to be established between the geometrical structures of the space-time and the observable physical phenomena. They showed that retaining the conformal and the projective structures of space-time does not necessarily reduce its underlying geometry to a Riemannian one, and gave an example of a physically motivated non-Riemannian space-time in wich all EPS conditions hold identically. Recently IAN W. ROXBURGH [8] considered Finsler spaces satisfying the Tavakol–van den Bergh conditions (T–vdB). Both papers are concerned with Finsler spaces having the same geodesic structure as the associated Riemannian space.

The purpose of the first section of the present paper is to show that the T–vdB is likely to reduce its underlying geometry to a Riemannian one sometimes, because the second condition (a2) is too restrictive. It seems to the author that the physical purpose of the metrical generalization will be fairly achieved by introducing the projective change of metric, instead of (a2). The remaining sections are concerned with the theory of projective changes on the basis of A. RAPCSÁK's valuable paper [7], which found the conditions for Finsler metrics to be projective to a given Finsler metric.

§1. The Tavakol–van den Bergh condition

Let $F^n = (M^n, L(x, y))$ be an n–dimensional Finsler space on a differentiable manifold M^n with the fundamental metric function $L(x, y)$ where

$x = (x^i)$ is a local coordinate system and $y = (y^i)$ is sometimes written as $\dot{x}$. The Tavakol–van den Bergh conditions for F^n are given as follows:

   (a1) F^n is equipped with $L(x, y)$ given by

$$(1.1) \qquad\qquad L^2(x, y) = e^{2c(x,y)} a_{ij}(x) y^i y^j \,,$$

where $c(x, y)$ is a positively homogeneous function in y of degree zero and $a_{ij}(x)$ is a (quasi–)Riemannian fundamental tensor.

   (a2) F^n is furnished with a connection such that

$$(1.2) \qquad\qquad G_j{}^i{}_k = \{_j{}^i{}_k\} \,,$$

where $G_j{}^i{}_k$ are coefficients of the Berwald connection and $\{_j{}^i{}_k\}$ are the Christoffel symbols constructed from $a_{ij}(x)$.

   Remark 1. The condition (a1) does not imply $g_{ij} = e^{2c(x,y)} a_{ij}$, where g_{ij} are components $\dot{\partial}_i \dot{\partial}_j (L^2/2)$ of the fundamental tensor of F^n, because the function c is not assumed to be a function of x alone. The tensor g_{ij} above does not give a Finsler metric, but a generalized metric, and the space (M^n, g_{ij}) has been considered by some geometricians [4], [5] and [10]. It is, however, obvious that L defined by (1.1) is certainly a Finslerian fundamental function because of the homogeneity of $c(x, y)$.

$$g_{ij} = e^{2c(x,y)} a_{ij}(x) \dots \quad \text{a generalized metric,}$$
$$L^2 = e^{2c(x,y)} a_{ij}(x) y^i y^i \dots \quad \text{a Finsler metric.}$$

   Remark 2. If we put $\alpha = \sqrt{a_{ij}(x) y^i y^j}$, then (a1) is written as $L = e^{c(x,y)} \alpha$, so that the Finsler metric L may be seen as conformal to the Riemannian α. But *(a1) never specializes L in the true geometrical sense.* In fact, let $L(x, y)$ and $\bar{L}(x, y)$ be two arbitrary Finsler metric functions on the same manifold M^n. Then we get the function $e^{c(x,y)} = \bar{L}(x, y)/L(x, y)$ which is obviously positively homogeneous of degree zero in y, and we have $\bar{L} = e^{c(x,y)} L$. In particular, if M^n admits a Finsler metric $L(x, y)$, then M^n admits also a Riemannian metric, as is well-known. Therefore any Finsler metric $L(x, y)$ may be written as (1.1) without any condition.

   Remark 3. In the associated Riemannian space (M^n, α) we have the Levi–Civita connection $\{_j{}^i{}_k\}(x)$, from which we get *a Finsler connection* $A\Gamma = (\{_j{}^i{}_k\}(x),\ y^r \{_r{}^i{}_j\}(x),\ 0)$ (Example 9.1 of [2]). On the other hand, we get in F^n the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ determined from $L(x, y)$. Then the condition (a2) asserts that these connections coincide with each other.

Theorem A (TAVAKOL–van den BERGH [9]). *A necessary and suffi-cient condition for a Finsler space $F^n = (M^n, L(x, y))$ to satisfy assump-tions (a1) and (a2) is for the function $c(x, y)$ to satisfy*

$$(1.3) \qquad\qquad c_{;i} = \partial_i c - \dot{\partial}_r c\{i{}^r{}_s\}y^s = 0\,.$$

Remark 4. In (1.3) (;) stands for the h–covariant derivative of $c(x, y)$ with respect to the connection $A\Gamma$ above. Cf. [2], Definition 9.5 and (9.18).

We shall first show Theorem A, based on the following theorem:

Theorem B (OKADA [6]). *The Berwald connection $B\Gamma{=}(G_j{}^i{}_k, G^i{}_j, 0)$ of a Finsler space is uniquely determined from the fundamental function $L(x, y)$ by the following four axioms:*

(1) L–metrical: $L_{;i} = 0$,
(2) $(h)h$–torsion $T_j{}^i{}_k = G_j{}^i{}_k - G_k{}^i{}_j = 0$,
(3)   deflection tensor $D^i{}_j = y^r G_r{}^i{}_j - G^i{}_j = 0$,
(4) $(v)hv$–torsion $P^i{}_{jk} = \dot{\partial}_k G^i{}_j - G_k{}^i{}_j = 0$.

Remark 5. In (1) above $L_{;i} = \partial_i L - (\dot{\partial}_r L)G^r{}_i$ stands for the h–covariant derivative of L with respect to $B\Gamma$, similarly to (1.3), because the connection $A\Gamma$ in (M^n, α) is nothing but the Berwald connection in the space.

Then, if we treat the Berwald connection $B\Gamma$ of a Finsler space $F^n = (M^n, L(x, y))$ and put $e^{c(x,y)} = \bar{L}/L$ for another Finsler space $\bar{F}^n = (M^n, \bar{L}(x, y))$, then $B\Gamma$ is also the Berwald connection of the latter, if and only if $B\Gamma$ satisfies the axiom (1): $\bar{L}_{;i} = 0$ for $\bar{L}$, because the remaining three axioms are satisfied automatically in $\bar{F}^n$. From $\bar{L}_{;i} = e^c c_{;i}L$ we obviously get Theorem A.

We shall consider the integrability condition of (1.3). We have one of the Bianchi identities of $B\Gamma$ as follows:

$$(1.4) \qquad\qquad X^i{}_{;j;k} - X^i{}_{;k;j} = X^r H_r{}^i{}_{jk} - X^i{}_{.r} R^r{}_{jk}\,,$$

for Finslerian vector field $X^i(x, y)$, where $H_r{}^i{}_{jk}$ is the h–curvature tensor and $R^r{}_{jk} = y^s H_s{}^r{}_{jk}$ is the $(v)hv$–torsion tensor of $B\Gamma$. In (1.4) we denote by $(\,;\,,\,\cdot\,)$ the h– and v–covariant derivatives with respect to $B\Gamma$; the latter being only the partial derivative by y^i.

Applying (1.4) to the scalar $c(x, y)$, (a2) implies

$$(1.5) \qquad\qquad c_r R^r{}_{jk} = 0\,,$$

where and throughout the following we shall use c_r and c_{rs} to denote $c_{.r}$ and $c_{.r.s}$ respectively.

Further the Euler theorem on homogeneous functions implies on c_r

(1.6) $$c_r y^r = 0\,.$$

Therefore we have the three equations (1.3), (1.5) and (1.6) for $c(x, y)$ satisfying the Tavakol–van den Bergh conditions.

Example 1. TAVAKOL–van den BERGH [9] showed an interesting example satisfying (a1) and (a2). They dealt with the Finslerian g_{ij} corresponding to a *plane wave*:

$$L^2(X, Y, u, v, \dot{X}, \dot{Y}, \dot{u}, \dot{v}) = \alpha(u)\dot{X}^2 + \beta(u)\dot{Y}^2 - 2\dot{u}\dot{v}\,.$$

Here we shall make some comments to derive their result. Putting $(x^i) = (X, Y, u, v)$, the surviving Christoffel symbols are

$$\{_1{}^4{}_1\} = \alpha'/2,\quad \{_2{}^4{}_2\} = \beta'/2,\quad \{_1{}^1{}_3\} = \alpha'/2\alpha,\quad \{_2{}^2{}_3\} = \beta'/2\beta\,.$$

The surviving components of the curvature tensor $R_h{}^i{}_{jk}$:

$$R_1{}^4{}_{13} = \alpha R_3{}^1{}_{13} = \alpha''/2 - (\alpha')^2/4\alpha\,,$$
$$R_2{}^4{}_{23} = \beta R_3{}^2{}_{23} = \beta''/2 - (\beta')^2/4\beta\,.$$

Then (1.3) are written in the form

(1.7)
$$\begin{aligned}
(1)\quad &\partial c/\partial X - c_1(\alpha'/2\alpha)y^3 - c_4(\alpha'/2)y^1 = 0\,,\\
(2)\quad &\partial c/\partial Y - c_2(\beta'/2\beta)y^3 - c_4(\beta'/2)y^2 = 0\,,\\
(3)\quad &\partial c/\partial u - c_1(\alpha'/2\alpha)y^1 - c_2(\beta'/2\beta)y^2 = 0\,,\\
(4)\quad &\partial c/\partial v = 0\,.
\end{aligned}$$

Next (1.5), assuming that both $\alpha''/2 - (\alpha')^2/4\alpha$ and $\beta''/2 - (\beta')^2/4\beta$ do not vanish, are written as

(1.8) $$c_1 y^3 + c_4 \alpha y^1 = 0,\quad c_2 y^3 + c_4 \beta y^2 = 0\,.$$

If we put $P = -c_4/y^3$, then (1.8) are equivalent to $c_1 = P\alpha y^1$, $c_2 = P\beta y^2$ and $c_4 = -Py^3$, so that (1.6) gives

(1.9) $$c_3 = -P\{\alpha(y^1)^2 + \beta(y^2)^2 - y^3 y^4\}/y^3\,.$$

As a consequence it is seen that (1.7) reduce to

(1.7')
$$\begin{aligned}
&\partial c/\partial X = \partial c/\partial Y = \partial c/\partial v = 0\,,\\
&\partial c/\partial u = P\{\alpha'(y^1)^2 + \beta'(y^2)^2\}/2\,.
\end{aligned}$$

Thus we have $c = c(u, y^i)$.

Now, if we consider the function $s(u, y^i) = L^2/(y^3)^2$, then we have

$$(\partial s/\partial u, \ \partial s/\partial y^i) = \{2/P(y^3)^2\}(\partial c/\partial u, \ \partial c/\partial y^i)\,,$$

and, in consequence, *c is a function of s* [9]. Let us remark that $s(u, y^i)$ is a positively homogeneous function of degree zero in y.

Example 2. We shall consider the well-known *Schwarzschild space–time* equipped with the Riemannian metric

(1.10) $$L^2(x, y) = R\dot{t}^2 - \dot{r}^2/R - r^2\dot{\theta}^2 - (r\sin\theta)^2\dot{\varphi}^2\,,$$

where $R = 1 - 2m/r$ with a positive constant m. Putting $(x^i) = (t, r, \theta, \varphi)$, we have the surviving Christoffel symbols:

$$\{_1{}^1{}_2\} = m/Rr^2, \quad \{_1{}^2{}_1\} = mR/r^2, \qquad \{_2{}^2{}_2\} = -m/Rr^2\,,$$
$$\{_3{}^2{}_3\} = -Rr, \qquad \{_4{}^2{}_4\} = -Rr\sin^2\theta, \quad \{_2{}^3{}_3\} = 1/r\,,$$
$$\{_2{}^4{}_4\} = 1/r, \qquad \{_4{}^3{}_4\} = -\sin\theta\cos\theta, \quad \{_3{}^4{}_4\} = \cos\theta/\sin\theta\,.$$

The surviving components of the curvature tensor $R_i{}^h{}_{jk}$ are

$$R_2{}^1{}_{12} = 2R_2{}^3{}_{23} = 2R_2{}^4{}_{24} = -2m/Rr^3\,,$$
$$R_1{}^2{}_{12} = -2R_1{}^3{}_{13} = -2R_1{}^4{}_{14} = -2mR/r^3\,,$$
$$R_3{}^4{}_{34} = 2R_3{}^2{}_{23} = 2R_3{}^1{}_{13} = 2m/r\,,$$
$$R_4{}^3{}_{34} = -2R_4{}^1{}_{14} = -2R_4{}^2{}_{24} = -2m\sin^2\theta/r\,.$$

Then, putting $S_{ij} = c_h R^h{}_{ij}$, we have

(1.11)
$$S_{12} = -(2m/r^3)(c_1 y^2/R + Rc_2 y^1)\,,$$
$$S_{13} = (m/r)(c_1 y^3 + Rc_3 y^1/r^2)\,,$$
$$S_{14} = (m/r)(\sin^2\theta c_1 y^4 + Rc_4 y^1/r^2)\,,$$
$$S_{23} = (m/r)(c_2 y^3 - c_3 y^2/Rr^2)\,,$$
$$S_{24} = (m/r)(\sin^2\theta c_2 y^4 - c_4 y^2/Rr^2)\,,$$
$$S_{34} = (2m/r)(c_4 y^3 - \sin^2\theta c_3 y^4)\,.$$

The three equations $S_{12} = S_{13} = S_{14} = 0$ of (1.5) lead us to

$$c_2 = -c_1 y^2/R^2 y^1, \quad c_3 = -c_1 r^2 y^3/Ry^1, \quad c_4 = -c_1 r^2 \sin^2\theta y^4/Ry^1$$

and the last three, $S_{23} = S_{24} = S_{34} = 0$, are only consequences of the above. The equation (1.6) is immediately written in the form $c_1 L^2/Ry^1 =$

0, so that c_i must be equal to zero and (1.3) implies $c = constant$. Consequently *the Finsler metric must reduce to the Schwarzschild metric.*

Now we shall pay attention to the system of two equations (1.5) and (1.6). They may be regarded as $n(n-1)/2 + 1$ homogeneous linear equations for c_r, $r = 1, \ldots, n$, and in consequence, the rank of the matrix consisting of the coefficients $R^r{}_{jk}$ and y^r must be less than n, if there is a possibility to get a non-trivial solution c_i.

Therefore it may be said that T-vdB is likely to reduce its underlying geometry to a Riemannian one in almost all cases.

Theorem 1. *The Finsler space $F^n = (M^n, L^2 = e^{2c(x,y)}a_{ij}(x)y^iy^j)$ satisfying the T–vdB condition reduces to*

(1) a Riemannian space homothetic to the associated Riemannian space $R^n = (M^n, a_{ij}(x)y^iy^j)$, if R^n is of non-zero constant curvature, or it is two–dimensional with non-zero Gauss curvature,

(2) a locally Minkowski space, if R^n is locally flat.

PROOF of (1). The function c must be constant, as shown by the following theorem ([2], Theorem 26.5):

Theorem C (MATSUMOTO–TAMÁSSY). *Assume that a Finsler space F^n with the fundamental function $L(x,y)$ be of non-zero scalar curvature. If a scalar field $S(x,y)$ on F^n positively homogeneous of degree r in y is h–covariant constant, then S is necessarily equal to sL^r with a constant s.*

Our $c(x,y)$ has $r = 0$ and (1.3) shows that it is h–covariant constant, so that c is necessarily constant.

PROOF of (2). From the assumption of local flatness it follows that there exists a covering by coordinate neighborhoods in each of which the components a_{ij} are all constant, so that the equation (1.3) reduces to $\partial c/\partial x^i = 0$ and we have $c = c(y)$. Therefore the fundamental function L of F^n is a function of y alone, that is, F^n is locally Minkowski.

§2. Rapcsák's fundamental theorem

We consider the Berwald connection $B\Gamma = G_j{}^i{}_k, G^i{}_j, 0)$ of a Finsler space $F^n = (M^n, L(x,y))$. Let $g_{ij}(x,y)$ be the fundamental tensor $\dot\partial_i\dot\partial_j F$, $F = L^2/2$. The connection coefficients are given by $G^i{}_j = \dot\partial_j G^i$ and $G_j{}^i{}_k = \dot\partial_k G^i{}_j$, where $G_i = g_{ir}G^r$ are defined as

$$G_i = (y^r\dot\partial_i\partial_r F - \partial_i F)/2 = \{y^r(\dot\partial_i L\partial_r L + L\dot\partial_i\partial_r L) - L\partial_i L\}/2\,.$$

Thus, if we introduce the operator Γ_i for a scalar field $S(x,y)$ as

(2.1) $\qquad \Gamma_i(S) = \{y^r(\dot\partial_i S\partial_r S + S\dot\partial_i\partial_r S) - S\partial_i S\}/2\,,$

then we have $\Gamma_i(L) = G_i$ for the fundamental function L.

Denoting by $(\,;\, \cdot\,)$ the h– and v–covariant derivatives in $B\Gamma$, we have $S_{;i} = \partial_i S - G^r{}_i \dot{\partial}_r S$ and $S_{\cdot i} = \dot{\partial}_i S$. We shall also use the simple symbols $S_i = S_{\cdot i}$ and $S_{ij} = S_{\cdot i \cdot j}$. Substituting $\partial_i S = S_{;i} + S_r G^r{}_i$ in (2.1) and introducing the operator Δ_i as

$$(2.2) \qquad\qquad \Delta_i(S) = S_{;i} - S_{;r \cdot i} y^r,$$

we obtain the expression of $\Gamma_i(S)$ as follows:

$$(2.1') \qquad 2\Gamma_i(S) = S_{;0} S_i - S \Delta_i(S) + 2(S S_{ir} + S_i S_r) G^r.$$

We shall further define the operator $\Delta_{ij} = \dot{\partial}_j \Delta_i$:

$$(2.3) \qquad\qquad \Delta_{ij}(S) = S_{;i \cdot j} - S_{;j \cdot i} - S_{;r \cdot i \cdot j} y^r.$$

We have for $B\Gamma$ the following commutation formulae of covariant differentiations:

$$S_{;i \cdot j} - S_{\cdot j\,;i} = 0, \quad S_{i;j \cdot k} - S_{i \cdot k\,;j} = -S_r G_i{}^r{}_{jk},$$

where the hv–curvature tensor $G_i{}^r{}_{jk}$ of $B\Gamma$ is symmetric in the subscripts and satisfies $G_i{}^r{}_{jk} y^k = 0$. Thus we have in (2.3) $S_{;r \cdot i \cdot j} y^r = S_{i;r \cdot j} y^r = S_{ij;r} y^r$. Consequently we have $\Delta_{ij}(S)$ of the form

$$(2.3'') \qquad\qquad \Delta_{ij}(S) = S_{j;i} - S_{i;j} - S_{ij\,;r} y^r.$$

Now we are concerned with two Finsler spaces $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ on the same underlying manifold M^n. From $\Delta_i(\bar{L}) = \bar{G}_i$ in $\bar{F}^n$ and (2.1') we get

$$2\bar{G}_i = \bar{L}_{;0} \bar{l}_i - \bar{L} \Delta_i(\bar{L}) + 2(\bar{h}_{ir} + \bar{l}_i \bar{l}_r) G^r,$$

where $\bar{l}_i = \bar{L}_{\cdot i}$ and $\bar{h}_{ir} = \bar{L} \bar{L}_{\cdot i \cdot r}$ are the normalized supporting element and the angular metric tensor of $\bar{F}^n$ respectively, so that $\bar{h}_{ir} + \bar{l}_i \bar{l}_r$ is equal to the fundamental tensor $\bar{g}_{ir}$ of $\bar{F}^n$. Then, transvecting the above by $\bar{g}^{ij}$, we obtain the general relation between G^j and $\bar{G}^j$:

$$(2.4) \qquad\qquad 2\bar{G}^j = 2G^j + \bar{L}_{;0} y^j / \bar{L} - \bar{L} \bar{g}^{ij} \Delta_i(\bar{L}).$$

Proposition 1. *We have the relation* (2.4) *between the quantities* G^j *and* $\bar{G}^j$ *of Finsler spaces* F^n *and* $\bar{F}^n$ *on the same underlying manifold.*

We shall consider a change $L \to \bar{L}$ of the metrics. The change is called *projective*, if any geodesic of $F^n = (M^n, L)$ coincides with a geodesic of $\bar{F}^n = (M^n, \bar{L})$ as a set of points and vice versa. Then F^n is said to *be projective to* $\bar{F}^n$ [1], [7]. As is well-known, the necessary and sufficient condition for a projective change is that there exists a positively homogeneous function $P(x, y)$ of degree one in y satisfying

$$(2.5) \qquad\qquad \bar{G}^i = G^i + P y^i.$$

$P(x, y)$ is called the *projective factor*. Therefore we have to examine the last term of (2.4) for a projective change.

From (2.5) we have

(2.6) $$\bar{G}^i_{\ j} = G^i_{\ j} + P_j y^i + P\delta^i_j, \quad P_j = P_{\cdot j},$$

and the metricity condition $\bar{L}_{;\bar{i}} = 0$ in $\bar{F}^n$ is written in the form

$$\partial_i \bar{L} - \bar{l}_r (G^r_{\ i} + P_i y^r + P\delta^r_j) = \bar{L}_{;i} - (\bar{L}P)_{\cdot i} = 0,$$

from $\bar{L}_{\cdot i} = \bar{l}_i$, $\bar{l}_r y^r = \bar{L}$ and $\bar{L}_{;i} = \partial_i \bar{L} - \bar{L}_{\cdot i} G^r_{\ i}$. Consequently we see $\Delta_i(\bar{L}) = (\bar{L}P)_{\cdot i} - (\bar{L}P)_{\cdot r \cdot i} y^r$ which is equal to zero from the homogeneity of $(\bar{L}P)_{\cdot i}$.

Conversely, $\Delta_i(\bar{L}) = 0$ leads us to the form (2.5) from (2.4), where we have $P = \bar{L}_{;0}/2\bar{L}$.

We shall show other forms of the condition $\Delta_i(\bar{L}) = 0$. From (2.3′) we have

$$\Delta_{ij}(\bar{L}) = \bar{l}_{j;i} - \bar{l}_{i;j} - \bar{l}_{ij;r} y^r,$$

where we put $\bar{l}_{ij} = \bar{l}_{i \cdot j}$. In the above the term $\bar{l}_{j;i} - \bar{l}_{i;j}$ is skew-symmetric, while $\bar{l}_{ij;r} y^r$ is symmetric, so that the condition $\Delta_i(\bar{L}) = 0$ implies

(2.7)   (1) $\bar{l}_{j;i} - \bar{l}_{i;j} = 0,$ (2) $\bar{l}_{ij;r} y^r = 0.$

It is, however, shown that (2) is a consequence of (1). In fact, it follows from the commutation formula that (1) may be written as $\bar{L}_{;i \cdot j} - \bar{L}_{;j \cdot i} = 0$ and

$$\bar{l}_{ij;r} y^r = \bar{l}_{i \cdot j;r} y^r = (\bar{l}_{i;r \cdot j} + \bar{l}_s G_i^{\ s}_{\ rj}) y^r = \bar{L}_{;r \cdot i \cdot j} y^r = \bar{L}_{;i \cdot r \cdot j} y^r,$$

which is equal to zero from the homogeneity of $\bar{L}_{;i \cdot j}$, so that we have (2).

Conversely, from $\bar{L}_{;i \cdot j} - \bar{L}_{;j \cdot i} = 0$ we immediately get $\Delta_i(\bar{L}) = 0$ by transvecting by y^j.

Therefore we obtain the following fundamental theorem on projective change of metrics:

Theorem D (RAPCSÁK [**7**]). *A Finsler space $F^n = (M^n, L)$ is projective to a Finsler space $\bar{F}^n = (M^n, \bar{L})$, if and only if $\bar{L}$ satisfies one of the following three conditions:*

$$(1) \quad \Delta_i(\bar{L}) = \bar{L}_{;i} - \bar{L}_{;r \cdot i} y^r = 0,$$

$$(2) \quad \bar{l}_{j;i} - \bar{l}_{i;j} = 0, \qquad (3) \quad \bar{L}_{;i \cdot j} - \bar{L}_{;j \cdot i} = 0.$$

Then the projective factor P is given by $P = \bar{L}_{;0}/2\bar{L}$.

Remark 6. In the three conditions above only the nonlinear connection $(G^i{}_j)$ of $B\Gamma$ appears. It is common to the Cartan connection $C\Gamma = (\Gamma^{*i}_j{}_k, G^i{}_j, C_j{}^i{}_k)$. [2].

Applying (1) and (3) of Theorem D to the form $\bar{L} = e^{c(x,y)}L$ of $\bar{L}$, we get

Theorem 2. *A Finsler space $F^n = (M^n, L)$ is projective to a Finsler space $\bar{F}^n = (M^n, \bar{L} = e^{c(x,y)}L)$, if and only if $c(x,y)$ satisfies one of the following two conditions:*

$$(1) \quad L(c_{;i} - c_{;r\cdot i}y^r) = c_{;0}(Lc_i + l_i)\,,$$
$$(2) \quad Lc_{;i\cdot j} + c_{;i}(Lc_j + l_j) - (i/j) = 0\,.$$

Then the projective factor P is given by $P = c_{;0}/2$.

Thus the condition (1.3) of the T–vdB is eased into (1) or (2) above. Throughout the following, as in (2) above, the symbol (i/j) stands for the term(s) obtained from the preceding term(s) by interchanging the indices i, j.

§3. Generalization of the integrability condition

The integrability condition (1.5) of (1.3) is directly given by the Ricci identity (1.4), but it is not easy to write the corresponding condition for the equations of Theorem D. We can, however, derive interesting equations from them.

First we consider the equation (1) of Theorem D. We get $\bar{L}_{;i;j} - \bar{L}_{;r\cdot i;j}y^r = 0$, which gives

$$(3.1) \qquad \bar{L}_{;i;j} - \bar{L}_{;r\cdot i;j}y^r - (i/j) = 0\,.$$

It follows from (1.4) that

$$\bar{L}_{;r\cdot i;j} = \bar{L}_{\cdot i;r;j} = \bar{l}_{i;r;j} = \bar{l}_{i;j;r} - \bar{l}_h H_i{}^h{}_{rj} - \bar{l}_{ih}R^h{}_{rj}\,.$$

Thus (3.1) may be written in the form

$$(3.1') \quad -\bar{l}_r R^r{}_{ij} - (\bar{l}_{i;j} - \bar{l}_{j;i})_{;0} +$$
$$+ \bar{l}_h(H_i{}^h{}_{0j} - H_j{}^h{}_{0i}) + (\bar{l}_{ih}R^h{}_{0j} - \bar{l}_{jh}R^h{}_{0i}) = 0\,.$$

It is well-known [2] that the h–curvature tensor $H_i{}^h{}_{jk}$ satisfies the identities

$$(3.2) \qquad
\begin{array}{ll}
(1) \quad H_0{}^h{}_{ij} = R^h{}_{ij}, & (2) \quad H_k{}^h{}_{ij} = R^h{}_{ij\cdot k}\,, \\
(3) \quad H_i{}^h{}_{jk} + (i,j,k) = 0\,, &
\end{array}$$

where (i, j, k) denotes the two terms obtained from the preceding term(s) by cyclic permutation of i, j, k. Then (1) and (3) imply

$$(3.3) \qquad R^h{}_{jk} = H_j{}^h{}_{0k} - H_k{}^h{}_{0j}.$$

Therefore (2) of Theorem D and (3.3) show that $(3.1')$ can finally be written in the form

$$(3.4) \qquad \bar{l}_{hi} R^h{}_{0j} - (i/j) = 0.$$

On the other hand, following Rapcsák [7], we have from (2) of Theorem D

$$(\bar{l}_{j;i} - \bar{l}_{i;j})_{;k} + (i, j, k) = -(\bar{l}_{i;j;k} - \bar{l}_{i;k;j}) - (i, j, k) = 0.$$

Thus (1.4) and (3) of (3.2) lead us to

$$(3.5) \qquad \bar{l}_{hi} R^h{}_{jk} + (i, j, k) = 0.$$

We get (3.4) and (3.5) for the projective change, but it is easy to show that these are equivalent to each other. In fact, the transvection of (3.5) by y^k implies (3.4) and the differentation of (3.4) by y^k implies (3.5) because of (3.2). Consequently we obtain

Proposition [7]. *If a Finsler space $F^n = (M^n, L)$ is projective to a Finsler space $\bar{F}^n = (M^n, \bar{L})$, then the angular metric tensor $\bar{h}_{ij} = \bar{L}\bar{l}_{ij}$ of $\bar{F}^n$ must satisfy one of the equations*

$$(1) \quad \bar{h}_{hi} R^h{}_{0j} - (i/j) = 0, \qquad (2) \quad \bar{h}_{hi} R^h{}_{jk} + (i, j, k) = 0.$$

Now we shall return to the metric $\bar{L} = e^{c(x,y)} L$. Then we have

$$\bar{l}_{ij} = e^c \{ l_{ij} + l_i c_j + l_j c_i + L(c_{ij} + c_i c_j) \}.$$

Since we have $l_h R^h{}_{ij} = 0$ and $h_{hi} R^h{}_{jk} = R_{0ijk}$, the equations (1) and (2) above are written respectively in the form

$$\{ L c_{hi} + c_h (L c_i + l_i) \} R^h{}_{0j} - (i/j) = 0,$$
$$\{ L c_{hi} + c_h (L c_i + l_i) \} R^h{}_{jk} + (i, j, k) = 0.$$

Putting $S_{ij} = c_h R^h{}_{ij}$, as in Example 2, we have

$$S_{ij \cdot k} = c_{hk} R^h{}_{ij} + c_h H_k{}^h{}_{ij},$$
$$(S_{0j})_{\cdot i} = c_{hi} R^h{}_{0j} + c_h R^h{}_{ij} + c_h H_i{}^h{}_{0j}.$$

Therefore, paying attention to (3.3), we can conclude as follows:

Theorem 3. *If a Finsler space $F^n = (M^n, L)$ is projective to a Finsler space $\bar{F}^n = (M^n, e^{c(x,y)}L)$, then $S_{ij} = c_h R^h{}_{ij}$ must satisfy one of the equations*

$$(1) \qquad (Lc_i + l_i)S_{0j} + L(S_{0j})._i - (i/j) = 3S_{ij} \,,$$

$$(2) \qquad (Lc_i + l_i)S_{jk} + LS_{jk\cdot i} + (i, j, k) = 0 \,.$$

Consequently the integrability condition (1.5), that is, $S_{ij} = 0$ is eased into (1) or (2) above, though they are not the integrability condition.

Example 3. We again consider the *Schwarzschild space–time* which was dealt with in Example 2. If we put

$$S_{ijk} = S_{ij\cdot k} + (i, j, k), \quad T_{ijk} = (Lc_i + l_i)S_{jk} + (i, j, k)\,,$$

and further

$$(3.6) \qquad K_{12} = c_1 y^2/R + Rc_2 y^1, \quad K_{34} = c_4 y^3 - \sin^2\theta c_3 y^4,$$

then we have from (1.11)

$$S_{123} = -(3m/r^3)K_{12\cdot 3}, \quad S_{124} = -(3m/r^3)K_{12\cdot 4},$$
$$S_{134} = (3m/r)K_{34\cdot 1}, \qquad S_{234} = (3m/r)K_{34\cdot 2}.$$

Next, if we pay attention to

$$l_1 = (R/L)y^1, \quad l_2 = -y^2/LR, \quad l_3 = -(r^2/L)y^3,$$
$$l_4 = -(r^2\sin^2\theta/L)y^4,$$

we have from (1.11)

$$T_{123} = -(3m/r^3)(Lc_3 + l_3)K_{12}, \quad T_{124} = -(3m/r^3)(Lc_4 + l_4)K_{12},$$
$$T_{134} = (3m/r)(Lc_1 + l_1)K_{34}, \qquad T_{234} = (3m/r)(Lc_2 + l_2)K_{34}.$$

Therefore the equation (2) of Theorem 3 states

$$(3.7) \qquad \begin{aligned} (c + \log L)._i K_{12} + K_{12\cdot i} &= 0, \quad i = 3, 4, \\ (c + \log L)._j K_{34\cdot j} + K_{34\cdot j} &= 0, \quad j = 1, 2. \end{aligned}$$

Consequently we can conclude as follows:

Proposition 2. *If a Finsler space $F^4 = (M^4, e^{c(x,y)}L)$ is projective to the Schwarzschild space–time (M^4, L), then the function $c(x, y)$ must be such that*
(1) $e^c L(c_1\dot{r}/R + Rc_2\dot{t})$ *is independent of* $(\dot\theta, \dot\varphi)$,
(2) $e^c L(c_4\dot\theta - \sin^2\theta c_3\dot\varphi)$ *is independent of* $(\dot{t}, \dot{r})$,

(3) $c_1\dot{t} + c_2\dot{r} + c_3\dot{\theta} + c_4\dot{\varphi} = 0,$

where $R = 1 - 2m/r$.

Thus the condition $c_i = 0$, obtained from (1.5) and (1.6), is eased into (1), (2) and (3) above, if, instead of (a2), we consider the projectivity.

§4. Projective β–change

We shall apply Rapcsák's fundamental Theorem D to a β–change $L \to \bar{L} = f(L, \beta)$, where β is a differential 1–form $\beta = b_i(x)y^i$ and $f(u^1, u^2)$ is a positively homogeneous function of degree one in (u^1, u^2).

Denoting $f_a = \partial f/\partial u^a$ and $f_{ab} = \partial f_a/\partial u^b$, $a, b = 1, 2$, we have

$$\bar{L}_{;i} = f_2\beta_{;i}, \quad \bar{L}_{;i\cdot j} = (f_{21}l_j + f_{22}b_j)\beta_{;i} + f_2 b_{j;i}.$$

Since the homogeneity implies $f_{21}L + f_{22}\beta = 0$, we have

$$\bar{L}_{;i\cdot j} = f_{22}\beta_{;i}(b_j - \beta l_j/L) + f_2 b_{j;i}.$$

Thus (3) of Theorem D leads us to

Theorem E [3]. *A β–change $L \to \bar{L} = f(L, \beta)$, $\beta = b_i(x)y^i$, is projective, if and only if $f(L, \beta)$ satisfies*

(4.1) $2f_2 F_{ij} = f_{22}(\beta_{;i}B_j - \beta_{;j}B_i),$

where we put $F_{ij} = (b_{i;j} - b_{j;i})/2$ and $B_i = b_i - \beta l_i/L$.

We shall apply Theorem E to the projective change $L = \alpha \to \bar{L} = e^{c(\alpha, \beta)}\alpha$, treated in Theorem 2, where L is a Riemannian metric $\alpha = (a_{ij}(x)y^iy^j)^{1/2}$ and $c(\alpha, \beta)$ is assumed to be a positively homogeneous function of degree zero in α and β. In this case we have $l_i = \partial\alpha/\partial y^i = a_{ir}y^r/\alpha$. Putting $y_i = a_{ir}y^r$ and $c_\beta = \partial c/\partial\beta$, (4.1) is written in the form

(4.2) $2c_\beta f_{ij} = (c_{\beta\beta} + c_\beta^2)\{\beta_{;i}(b_j - \beta y_j/\alpha^2) - (i/j)\}.$

Transvecting (4.2) by y^i, we have

(4.3) $2c_\beta F_{0j} = \beta_{;0}(c_{\beta\beta} + c_\beta^2)(b_j - \beta y_j/\alpha^2),$

which corresponds to (1) of Theorem D. Then assuming $\beta_{;0} \neq 0$ and $c_\beta \neq 0$, (4.2) may be written in the form

$$\beta_{;0}F_{ij} = \beta_{;i}F_{0j} - \beta_{;j}F_{0i}.$$

This is a quadratic polynomial in y^i, because $\beta_{;i} = b_{h;i}y^h$ and $F_{0i} = F_{hi}y^h$. Thus it is equivalent to

(4.4) $(b_{h;k} + b_{k;h})F_{ij} = b_{h;i}F_{kj} + b_{k;i}F_{hj} - (i/j).$

Consequently we obtain

Proposition 4. *The change* $\alpha \to e^{c(\alpha,\beta)}\alpha$ *is projective, if and only if* $\beta = b_i(x)y^i$ *and* $c(\alpha,\beta)$ *satisfy* (4.2) *or* (4.3). *In this case we have* (4.4), *provided that* $\beta_{;0} \neq 0$ *and* $c_\beta \neq 0$.

Example 4. If β and $c(\alpha,\beta)$ satisfy

$$(1) \qquad F_{ij} = 0, \qquad (2) \qquad c_{\beta\beta} + c_\beta^2 = 0,$$

then (4.2) obviously holds and the Finsler metric $e^{c(\alpha,\beta)}\alpha$ is projective to the Riemannian metric α. (1), a condition for β, shows $\partial_j b_i - \partial_i b_j = 0$, so that $b_i(x)$ *is locally a gradient vector field*. (2), a condition for $c(\alpha,\beta)$, is written as $c_{\beta\beta}/c_\beta + c_\beta = 0$, which is integrated to obtain $\log(c_\beta) + c = \log A(\alpha)$ and further $e^{c(\alpha,\beta)} = A(\alpha)\beta + B(\alpha)$, where $A(\alpha)$ and $B(\alpha)$ are functions of α alone, positively homogeneous of degree-1 and 0 respectively, so we have $A(\alpha) = k_2/\alpha$ and $B(\alpha) = k_1$ with constants k_1 and k_2. Consequently we obtain a special (α,β)–metric [2]:

$$e^{c(\alpha,\beta)}\alpha = k_1\alpha + k_2\beta,$$

which is *of the Randers type*.

Example 5. If we assume $c(\alpha,\beta) = \alpha^{-r}\beta^r$, $r \neq 0$, then (4.3) is written in the form

$$(4.5) \qquad 2\alpha^2\beta F_{0j} = \beta_{;0}(r - 1 + r\alpha^{-r}\beta^r)(\alpha^2 b_j - \beta y_j).$$

(1) Assume that r is an odd number. Then α^r is irrational in y^i. Hence (4.5) must imply $\beta_{;0} = 0$ and we get $F_{0j} = 0$. $\beta_{;0} = b_{i;j}y^i y^j = 0$ shows $b_{i;j} + b_{j;i} = 0$. $F_{0j} = (b_{i;j} - b_{j;i})y^i/2 = 0$ shows $b_{i;j} - b_{j;i} = 0$. Therefore we get $b_{i;j} = 0$ and then (4.5) holds.

(2) Assume that r is a positive even number. Then (4.5) is rewritten in the form

$$(4.6) \qquad 2\alpha^{r+2}\beta F_{0j} = \beta_{;0}\{(r-1)\alpha^r + r\beta^r\}(\alpha^2 b_j - \beta y_j).$$

In this polynomial in y^i, we observe that only the term $\beta_{;0}(r\beta^r)(-\beta y_j)$ does not contain α^2, so that we must have $z_j = z_{jr}(x)y^r$ satisfying (i) $\beta_{;0}y_j = \alpha^2 z_j$. Further only the term $\beta_{;0}(r - 1)\alpha^r(\alpha^2 b_j)$ does not contain β, so that we must have $u_j = u_{jr}(x)y^r$ satisfying (ii) $\beta_{;0}b_j = \beta u_j$. Then, substituting from (i) and (ii), (4.6) is written as

$$2\alpha^r F_{0j} = \{(r-1)\alpha^r + r\beta^r\}(u_j - z_j),$$

which obviously implies $u_j - z_j = 0$ and $F_{0j} = 0$. The former shows $\beta_{;0}(\alpha^2 b_j - \beta y_j) = 0$, so that we have $\beta_{;0} = 0$.

(3) Assume that r is a negative even number $-s$. Then (4.5) is written in the form

$$(4.7) \qquad 2\alpha^2\beta^{s+1}F_{0j} = -\beta_{;0}\{(s+1)\beta^s + s\alpha^s\}(\alpha^2 b_j - \beta y_j).$$

Similarly to (2), we easily get $\beta_{;0} = 0$ and $F_{0j} = 0$.

As a conclusion, *if we assume $c(\alpha, \beta) = \alpha^{-r}\beta^r$ with a non–zero integer r, then we must have $b_{i;j} = 0$.* Compare with Roxburgh's paper [8].

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