Publ. Math. Debrecen
58 / 4 (2001), 707–716

Homeomorphisms and monotone vector fields

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Abstract. A classical result of Minty [8] states that for a Hilbert space H and a continuous monotone map $A : H \to H$ the map $A + I$ is a homeomorphism of H. We extend this result to Hadamard manifolds.

1. Introduction

Let B be a Banach space and G a subset of B. The map $A : G \to B^*$ is called monotone with respect to duality (or in the sense of Minty–Browder) if $\langle Ay - Ax, y - x\rangle \geq 0$ for any x and y in G, where B^* is the dual of B and $\langle\, .\, , .\, \rangle$ is the natural pairing. If the strict inequality holds whenever $x \neq y$, then A is called strictly monotone. If B is a Hilbert space, then the pairing $\langle\, .\, , .\, \rangle$ can be identified with the scalar product of B. We extended the notion of monotonicity for vector fields of a Riemannian manifold. A classical result of Minty [8] states that for a Hilbert space H and a continuous monotone map $A : H \to H$ the map $A + I : H \to H$, where I is the identical map of H, is a homeomorphism. This result (and different variations of it) is widely used to prove existence and uniqueness theorems for operator equations, partial differential equations and variational inequalities (see [19]). Surprisingly, in the finite dimensional case this result boils down just to the continuity and expansivity of $A+I$, beeing a particular case (it is not trivial to show) of a classical homeomorphism theorem

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Mathematics Subject Classification: 47H05, 53C20, 53C21, 53C22, 58C07, 58C99.
Key words and phrases: monotone vector field, expansive map, homeomorphism, Hadamard manifold.
Supported by Bolyai János Research Fellowship and Hungarian Research Grant OTKA T029572.

of Browder [4, Theorem 4.10] (connected to this subject see also [1]–[3], [6], [13]–[15].) We shall generalize this result for a complete connected Riemannian manifold M. We shall prove that a continuous expansive map $A : M \to M$ is a homeomorphism. By an expansive map on a Riemannian manifold we mean a map which increases the distance between any two points. The distance function on a Riemannian manifold is given by [5, p. 146, Definition 2.4]. The expansivity of A can be greatly weakened. It is enough to suppose that A is reverse uniform continuous, which means that for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $d(Ax, Ay) < \delta$ implies $d(x, y) < \varepsilon$, where d denotes the distance function on M. Particularly if M is an Hadamard manifold (complete, simply connected Riemannian manifold, of nonpositive sectional curvature) and X is a monotone vector field on M we shall prove that $\exp X$ is expansive. Hence if X is continuous $\exp X$ is a homeomorphism of M, extending Minty's classical result. (We note that for a Hilbert space H we have $\exp X = X + I$, where X is identified with a map of H.)

The author expresses his gratitude to professor Tamás Rapcsák, professor János Szenthe and dr. Balázs Csikós to many helpful conversations.

2. Preliminary results

First we prove the following lemma:

Lemma 2.1. *Consider $\mathbb{R}^2$ endowed with the canonical scalar product $\langle\, .\, ,\, .\, \rangle$. Denote by $\|\, .\, \|$ the norm induced by $\langle\, .\, ,\, .\, \rangle$. Let abcd be a quadrilateral in $\mathbb{R}^2$ such that $\|c - d\| > \|a - b\|$. Denote by α, β, γ and δ the angles $\angle dab$, $\angle abc$, $\angle bcd$ and $\angle cda$, respectively. Then*

$$\|a - d\| \cos\delta + \|b - c\| \cos\gamma > 0. \tag{2.1}$$

(This holds even if abcd degenerates to a triangle.)

Proof. If $a = b$ the inequality follows from the relation

$$\|a - d\| \cos\delta + \|a - c\| \cos\gamma = \|c - d\|,$$

which can be easily obtained by projecting a to the straight line joining c and d. Suppose that $a \neq b$. From $\|c-d\| > \|a-b\|$ and the Schwarz inequality we have that

$$\langle d-c,\, a-b\rangle < \|d-c\|^2,$$

which is equivalent to

$$\langle c-d,\, a-d\rangle + \langle d-c,\, b-c\rangle > 0. \tag{2.2}$$

It is easy to see that (2.2) implies (2.1). □

In the following definition indices $i = 1, \dots, n$ are considered modulo n. A *geodesic n-sided poligon* in a Riemannian manifold M is a set formed by n segments of minimizing unit speed geodesics (called *sides* of the poligon)

$$\gamma_i : [0, l_i] \to M; \qquad i = 1, \dots, n,$$

in such a way that $\gamma_i(l_i) = \gamma_{i+1}(0)$; $i = l, \dots, n$. The endpoints of the geodesic segments are called *vertices* of the poligon. The angle

$$\angle(-\dot\gamma_i(l_i), \dot\gamma_{i+1}(0)); \qquad i = 1, \dots, n$$

is called the (interior) *angle* of the corresponding vertex.

Recall that on Hadamard manifolds every two points can be uniquely joined by a geodesic arc [11]. Hence the distance between two points of an Hadamard manifold is the length of the geodesic joining these points.

Let M be an Hadamard manifold. If a, b, c are three arbitrary points of M then ab will denote the distance of a from b and $abc_\triangle$ the geodesic triangle of vertices a, b, c (which is uniquely defined). In general a geodesic poligon in M, of consecutive vertices $a_1, \dots, a_n$ will be denoted by $a_1 \dots a_n$.

Lemma 2.2. *Let $abcd$ be a quadrilateral in a Hadamard manifold M and α, β, γ, δ the angles of the vertices a, b, c, d, respectively. Then*

$$\alpha + \beta + \gamma + \delta \leq 2\pi.$$

Proof. Let α_1, α_2 be the angles of the vertex a in $adc_\triangle$ and $abc_\triangle$, respectively. Similarly, let γ_1 and γ_2 be the angles of the vertex c in $adc_\triangle$ and $abc_\triangle$, respectively. It is known that an angle formed by two edges of a trieder is bounded by the sum of the other two angles formed by edges. Hence

$$\alpha_1 + \alpha_2 \geq \alpha \tag{2.3}$$

and

$$\gamma_1 + \gamma_2 \geq \gamma. \tag{2.4}$$

On the other hand by [5, p. 259, Lemma 3.1 (ii)] we have that

$$\alpha_1 + \gamma_1 + \delta \leq \pi, \tag{2.5}$$

$$\alpha_2 + \gamma_2 + \beta \leq \pi. \tag{2.6}$$

Summing inequalities (2.5), (2.6) and using (2.3), (2.4) we obtain

$$\alpha + \beta + \gamma + \delta \leq 2\pi. \qquad \square$$

The next lemma follows from [18, Lemma 1].

Lemma 2.3. *Let $(M, \langle\,.\,,\,.\,\rangle)$ be an Hadamard manifold and $abcd$ be a quadrilateral in M such that α is nonacute and β is obtuse (nonacute), where α, β, γ, δ are the angles of the vertices a, b, c, d, respectively. Then $cd > ab$ ($cd \geq ab$).*

The following lemma is a generalization of Lemma 2.1.

Lemma 2.4. *Let $(M, \langle\,.\,,\,.\,\rangle)$ be an Hadamard manifold and $abcd$ be a quadrilateral in M such that $cd > ab$. Denote by α, β, γ, δ the angles of the vertices a, b, c, d, respectively. Then*

$$ad\cos\delta + bc\cos\gamma > 0.$$

(This holds even if $abcd$ degenerates to a triangle.)

Proof. We identify T_aM with $\mathbb{R}^n$, where $n = \dim M$. Denote by $\|\,.\,\|$ the norm generated by the canonical scalar product of $\mathbb{R}^n$.

If δ, $\gamma \geq \pi/2$ then Lemma 2.3 implies $ab \geq cd$ which contradicts $cd > ab$. Hence we have either $\delta < \pi/2$ or $\gamma < \pi/2$. We can suppose without loss of generality that

$$\gamma < \pi/2. \tag{2.7}$$

The lengths of the sides of a geodesic triangle satisfy the triangle inequalities. Hence there exist the points b', c', d' of $T_a(M)$ such that $\|a-d'\| = ad$, $\|a-c'\| = ac$, $\|d'-c'\| = dc$, $\|a-b'\| = ab$, $\|b'-c'\| = bc$ and b' is contained in the plane of $ad'c'_\triangle$, such that b' and d' are contained in different half planes defined by the straight line in $T_a(M)$ joining a and c'. Let $\alpha' = \angle d'ab'$, $\beta' = \angle ab'c'$, $\gamma' = \angle b'c'd'$, $\delta' = \angle c'd'a$, $\gamma_1' = \angle ac'd'$ and $\gamma_2' = \angle ac'b'$. Using Lemma 2.1 to the quadrilateral $ab'c'd'$ we obtain

$$\|a-d'\| \cos \delta' + \|b'-c'\| \cos \gamma' > 0. \tag{2.8}$$

Denote by γ_1, γ_2 the angles of the vertex c in the triangles $adc_\triangle$, $abc_\triangle$, respectively. Then we have, by [5, p. 259, Lemma 3.1 (i)] that

$$\delta' \geq \delta. \tag{2.9}$$

and

$$\gamma_1' + \gamma_2' \geq \gamma_1 + \gamma_2 \geq \gamma. \tag{2.10}$$

We consider two cases:

1) $\gamma_1' + \gamma_2' \leq \pi$.

We have

$$\gamma' = \gamma_1' + \gamma_2'. \tag{2.11}$$

Relations (2.10) and (2.11) implies

$$\gamma' \geq \gamma. \tag{2.12}$$

Since $\|a-d'\| = ad$, $\|b'-c'\| = bc$ and the cosine function is strictly decreasing on $]0,\pi]$ (2.8), (2.9) and (2.12) imply

$$ad \cos \delta + bc \cos \gamma > 0.$$

2) $\gamma_1' + \gamma_2' > \pi$.

If $\delta < \pi/2$ then $ad \cos \delta + bc \cos \gamma > 0$ holds trivially, since $\gamma < \pi/2$. We suppose that $\delta \geq \pi/2$. By (2.9) we have that $\delta' \geq \pi/2$. [5, p. 259, Lemma 3.1 (ii)] implies that

$$\gamma_1' \leq \pi/2. \tag{2.13}$$

We also have

$$\gamma_2' \leq \pi. \tag{2.14}$$

Hence (2.13) and (2.14) implies

$$2\pi - \gamma' = \gamma_1' + \gamma_2' \leq \frac{3\pi}{2}. \tag{2.15}$$

By (2.7) and (2.15) we have $0 \leq \gamma < \gamma' \leq \pi$. Since the cosine function is strictly decreasing on $[0, \pi]$ we have

$$\cos\gamma > \cos\gamma'. \tag{2.16}$$

Similarly (2.9) implies

$$\cos\delta \geq \cos\delta'. \tag{2.17}$$

By $\|a - d'\| = ad$, $\|b' - c'\| = bc$, (2.8), (2.16) and (2.17) we have

$$ad\cos\delta + bc\cos\gamma > 0. \qquad \square$$

3. Monotone vector fields on Riemannian manifolds

Let M be a Riemannian manifold. We recall that a subset K of M is called (*geodesic*) *convex* [12] if for every two points of M there is a geodesic arc joining these points contained in K.

If N is an arbitrary manifold, we shall denote by $\mathrm{Sec}(TN)$ the family of sections of the tangent bundle TN of N. Using this notation, we have the following definition:

Definition 3.1. Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold, $K \subset M$ a convex open set and $X \in \mathrm{Sec}(TK)$ a vector field on K. X is called *monotone* [9] if for every $x, y \in K$ and every unit speed geodesic arc $\gamma : [0, l] \to M$ joining x and y ($\gamma(0) = x$, $\gamma(l) = y$) contained in K, we have that

$$\langle X_x, \dot\gamma(0)\rangle \leq \langle X_y, \dot\gamma(l)\rangle,$$

where $\dot\gamma$ denotes the tangent vector of γ with respect to the arclength.

Let X be monotone. With the previous notations X is called *strictly monotone* [9] if for every distinct x and y

$$\langle X_x, \dot\gamma(0)\rangle < \langle X_y, \dot\gamma(l)\rangle.$$

Since the length of the tangent vector of an arbitrary parametrized geodesic is constant, the relations of Definition 3.1 can be given for any parametrization of γ. It is also easy to see that X is monotone (strictly monotone), if and only if for every geodesic γ (arbitrarily parametrized) the $\upsilon : \tau \mapsto \langle X_{\gamma(\tau)}, \gamma'(\tau)\rangle$ is monotone (strictly monotone), where $\gamma'(\tau)$ is the tangent vector of γ with respect to its parameter τ.

The following example makes connection between monotone vector fields and monotone operators of a Euclidean space, showing that with few modifications the formers are generalizations of the latters:

Example 3.2. Let E be a Euclidean space, $G \subset E$ an open and convex set and $h : G \to E$ a monotone (strictly monotone) operator. Then, the vector field $X \in \mathrm{Sec}(TG)$; $x \mapsto h(x)_x$, where $h(x)_x$ is the tangent vector in 0 of the curve $t \mapsto x + th(x)$, is monotone (strictly monotone).

The next remark follows easily from Definition 3.1.

Remark 3.3. If M is an Hadamard manifold, $K \subset M$ a convex open set and $X \in \mathrm{Sec}(TK)$ is a vector field on K then X is monotone if and only if for every $x, y \in K$

$$\langle X_x, \exp_x^{-1} y\rangle + \langle X_y, \exp_y^{-1} x\rangle \le 0, \tag{3.1}$$

where $\exp : TM \to M$ is the exponential map of M.

Examples for monotone vector fields on Riemannian manifolds can be found in [9], [10]. We also remark that the gradient of every (geodesic) convex function [12] on a Riemannian manifold is monotone (see [16], [17]).

4. Homeomorphisms of Hadamard manifolds

The following proposition is a consequence of Lemma 2.4.

Proposition 4.1. *Let M be an Hadamard manifold and $X \in \mathrm{Sec}(TM)$ a monotone vector field on M. Then the map $A = \exp X : M \to M$ defined by $Ax = \exp_x X_x$ is expansive.*

Proof. Suppose that A is not expansive. Hence there exist x and y in M such that $x'y' < xy$, where $x' = Ax$ and $y' = Ay$. Consider the quadrilateral $xyy'x'$. Denote by the same letters the angles corresponding to the vertices x and y, respectively. Then by Lemma 2.4 we have

$$xx' \cos x + yy' \cos y > 0. \tag{4.1}$$

It is easy to see that (4.1) is equivalent to

$$\langle X_x, \exp_x^{-1} y\rangle + \langle X_y, \exp_y^{-1} x\rangle > 0. \tag{4.2}$$

But by (3.1) inequality (4.2) contradicts the monotonicity of X. Hence A is expansive. □

Definition 4.2. Let M be a Riemannian manifold and d its distance function, which is a metric on M (see [5, p. 146, Proposition 2.5]). $A : M \to M$ is called *reverse uniform continuous* if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $d(Ax, Ay) < \delta$ implies $d(x, y) < \varepsilon$.

Let $\alpha \geq 1$ and $L > 0$ be two arbitrary positive constants and $A : M \to M$ such that for any x and y in M to have $d(Ax, Ay) \geq Ld(x, y)^\alpha$. Then A is reverse uniform continuous. If $\alpha = L = 1$ we obtain the set of expansive maps.

Theorem 4.3. *Let M be a complete connected Riemannian manifold and $A : M \to M$ a continuous and reverse uniform continuous map. Then A is a homeomorphism. Particularly this is true for A continuous and expansive.*

Proof. Let $n = \dim M$. It is easy to see that the reverse uniform continuity of A implies that it is injective and $A^{-1} : AM \to M$ is continuous, where $AM = \{Ax : x \in M\}$. Hence $A : M \to AM$ is a homeomorphism. It remains to show that $AM = M$. Suppose that we have already proved that AM is closed. Since $A : M \to AM$ is a homeomorphism, by Brouwer's domain invariance theorem, [7, p. 65] AM is open. Since M is connected and AM is an open and closed subset of M we have $AM = M$. Hence if we prove that AM is closed we are done. For this let us consider a sequence $x'_n = Ax_n$ in M convergent to $x' \in M$ and prove that $x' \in AM$ i.e. there is an $x \in M$ such that $x' = Ax$. Since x'_n is convergent it is a Cauchy sequence. It is easy to see that the reverse continuity of A implies that x_n is also a Cauchy sequence. Since M is complete, by Hopf–Rinow theorem for Riemannian manifolds it is complete as a metric space (see

[5, p. 146]). Hence x_n is convergent. Denote by x its limit. Since A is continuous taking the limit in the relation $x'_n = Ax_n$ as $n \to \infty$ we obtain $x' = Ax$. □

By Proposition 4.1 we have the following extension to Hadamard manifolds of Minty's classical homeomorphism theorem for monotone maps [8, Corollary of Theorem 4].

Corollary 4.4. *Let M be an Hadamard manifold and X be a continuous monotone vector field. Then $\exp X : M \to M$ is a homeomorphism.*

In [10] we proved that if $p_1, p_2, \ldots, p_n$ are projection maps onto closed convex sets of an Hadamard manifold [18] then the vector field

$$X = -\exp^{-1}(p_1 \circ \ldots \circ p_n)$$

defined by

$$X_x = -\exp_x^{-1}[(p_1 \circ \ldots \circ p_n)(x)]$$

is continuous and monotone. Hence we have the following corollary:

Corollary 4.5. *Let M be an Hadamard manifold and $p_1, p_2, \ldots, p_n$ projection maps onto closed convex sets of M. Then* $\exp[-\exp^{-1}(p_1 \circ \ldots \circ p_n)]$ *is a homeomorphism of M onto M.*

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(Received February 24, 2000)