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Additive derivations and Jordan derivations on algebras of unbounded operators

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Abstract. Let $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$ be a standard algebra of unbounded operators on a dense domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$. We prove that for a large class of domains $\mathcal{D}$ every additive derivations (Jordan derivation resp.) D has the form $D(A) = TA - AT$ for some $T \in \mathcal{L}^+(\mathcal{D})$. A similar result is valid for Jordan $*$-derivations on every standard algebra of unbounded operators. These are generalizations of results valid for standard algebras of bounded operators.

1. Introduction and preliminaries

Several types of derivations play an important role in ring theory and in the theory of abstract as well as of operator algebras. Strange enough nowaday there seems to be only spurious interplay between ring theorists and operator algebraists. This is a pity because there are several results in abstract ring theory with interesting applications to certain classes of operator algebras. We are concerned here with derivations and corresponding applications to algebras of bounded operators on Hilbert or Banach spaces. The aim of the paper is to give some generalizations to algebras of unbounded operators. Looking for applications of abstract algebraic results concerning derivations to operator algebras one is faced with a variety of different notions related to derivations. For example, since an algebra can be considered as a ring one has to distinguish between ring and algebra derivations. Another class are the Jordan or Jordan $*$-derivations and so

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on. To fix our notation let us start with some definitions which are formulated only for algebras (or $*$-algebras). In this paper $\mathbb{F}$ stands for the scalar field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. Let $\mathcal{A}$ be an algebra over $\mathbb{F}$ and $\mathcal{A}_1 \subset \mathcal{A}$ a subalgebra. A mapping $D : \mathcal{A}_1 \to \mathcal{A}$ is called *multiplicative derivation* if D fulfils the Leibniz rule

$$D(ab) = D(a)b + aD(b) \quad \text{for all } a, b \in \mathcal{A}_1; \tag{1}$$

additive derivation or *ring derivation* if D is additive and fulfils (1); *derivation* if D is linear and fulfils (1); *inner derivation* if there is a $c \in \mathcal{A}_1$ such that $D(a) = [c, a] = ca - ac$ for all $a \in \mathcal{A}_1$. If we deal with $*$-subalgebras of $*$-algebras and $D(a^*) = D(a)^*$, then we call D a $*$-derivation ($*$-ring derivation and so on).

Now let us repeat the definition for several kinds of Jordan derivations.

Definition 1.2. Let $\mathcal{A}$ be an algebra over $\mathbb{F}$ and $\mathcal{A}_1 \subset \mathcal{A}$ a subalgebra. A mapping $J : \mathcal{A}_1 \to \mathcal{A}$ is called *additive Jordan derivation* if J is additive and fulfils

$$J(a^2) = J(a)a + aJ(a) \quad \text{for all } a \in \mathcal{A}_1; \tag{2}$$

(linear) Jordan derivation if J is linear and satifies (2); *inner Jordan derivation* if there is a $b \in \mathcal{A}_1$ such that $J(a) = ba - ab$ (i.e. J is already a derivation).

If $\mathcal{A}_1 \subset \mathcal{A}$ is a $*$-subalgebra of a $*$-algebra, $J : \mathcal{A}_1 \to \mathcal{A}$ is called *linear (additive) Jordan $*$-derivation* if J is real-linear (additive) and $J(a^2) = aJ(a) + J(a)a^*$; *inner Jordan $*$-derivation* if there is a $b \in \mathcal{A}_1$ such that $J(a) = ab - ba^*$ (i.e. J is automatically real-linear).

In an obvious manner there are defined local Jordan ($*$)-derivations and locally inner ($*$)-derivations.

Remarks 1.3. *i)* The so-called Jordan product in $\mathcal{A}$ is defined by $a \circ b = ab + ba$. J is a Jordan derivation (additive Jordan derivation) if and only if J is linear (additive) and $J(a \circ b) = J(a) \circ b + a \circ J(b)$.

ii) A Jordan $*$-derivation is *not* a Jordan derivation with an additional property. J is a Jordan derivation only on the symmetric elements.

iii) Every derivation is a Jordan derivation, but there are only few Jordan derivations which are not derivations.

Next let us fix our notation concerning operator algebras (for algebras of unbounded operators cf. also [10]). Let $\mathcal{X}$ be a Banach space, then $\mathcal{B}(\mathcal{X})$, $\mathcal{F}(\mathcal{X})$ denote the algebra of all bounded linear operators on $\mathcal{X}$, the two-sided ideal in $\mathcal{B}(X)$ consisting of all finite rank operators resp. A standard operator algebra on $\mathcal{X}$ is a subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ containing $\mathcal{F}(\mathcal{X})$ (it seems that this notion was first introduced by Chernoff [4]). If $\mathcal{X} = \mathcal{H}$ is a Hilbert space the notations $\mathcal{B}(\mathcal{H})$, $\mathcal{F}(\mathcal{H})$ have an obvious meaning. It should be noted that we have in this case operator $*$-algebras with respect to the involution $T \to T^*$.

Let $\mathcal{D}$ be a dense linear manifold in a Hilbert space $\mathcal{H}$ with scalar product $\langle\ ,\ \rangle$ (which is supposed to be antilinear in the first and linear in the second component). The set of all linear operators $\mathcal{L}^+(\mathcal{D}) = \{A : A\mathcal{D} \subset \mathcal{D}, A^*\mathcal{D} \subset \mathcal{D}\}$ is a $*$-algebra with respect to the natural operations and the involution $A \to A^+ = A^* \mid \mathcal{D}$. An O^*-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing the identity operator I.

By $\mathcal{F}(D)$ we denote the (two-sided $*$-) ideal in $\mathcal{L}^+(\mathcal{D})$ consisting of all finite rank operators in $\mathcal{L}^+(\mathcal{D})$. A standard operator algebra $\mathcal{A}$ on $\mathcal{D}$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing $\mathcal{F}(\mathcal{D})$ ($\mathcal{A}$ must not contain I).

The graph topology $t_{\mathcal{A}}$ on $\mathcal{D}$ induced by $\mathcal{A}(\mathcal{D})$ is generated by the directed family of seminorms $\varphi \to \|\varphi\|_A = \|A\varphi\|$, $\forall\ A \in \mathcal{A}(\mathcal{D})$, $\varphi \in \mathcal{D}$. In case $\mathcal{A}(D) = \mathcal{L}^+(\mathcal{D})$ this topology is simply denoted by t.

There are two important classes of domains $\mathcal{D}$: $(\mathcal{D}, t)$ is an (F)-space (Fréchet-space) or a (QF)-space (short for quasi-Fréchet space), that means for every bounded subset $\mathcal{M} \subset (\mathcal{D}, t)$ there is a subspace $\mathcal{E} \subset \mathcal{D}$ which is an (F)-space in the induced topology and which contains $\mathcal{M}$. Clearly, every (F)-space is a (QF)-space.

Let us collect some properties of $\mathcal{F}(\mathcal{D})$ which will be used in the sequel and which can be easily proved.

i) $\mathcal{F}(\mathcal{D})$ is a prime algebra in the sense that

$$(*) \quad \text{if for some } \ A, B \in \mathcal{F}(\mathcal{D}) : A\mathcal{F}(\mathcal{D})B = \{0\}, \text{ then } A = 0 \text{ or } B = 0.$$

ii) The implication $(*)$ is also valid for $A, B \in \mathcal{L}^+(\mathcal{D})$.

iii) If for some $X \in \mathcal{L}^+(\mathcal{D})$: $AX + XB = 0$ for all $A, B \in \mathcal{F}(\mathcal{D})$ or $CX + XC^+ = 0$ or $CX - XC^+ = 0$ for all $C \in \mathcal{F}(\mathcal{D})$ then $X = 0$.

iv) $\mathcal{F}(\mathcal{D})$ is a local matrix algebra, i.e. for every finite collection $F_1, \ldots, F_k \in \mathcal{F}(\mathcal{D})$ there is a subalgebra $\mathcal{B} \subset \mathcal{F}(\mathcal{D})$ such that all $F_i \in \mathcal{B}$ and $\mathcal{B}$ is isomorphic to some full matrix algebra $M_n(\mathbb{F})$.

2. Additive derivations and Jordan derivations

One of the main goals in the theory of derivations consists in answering the following question:

Given some kind of derivations on an algebra (or a ring) $\mathcal{A}$, which algebraic and/or topological properties of $\mathcal{A}$ imply additional properties of the derivation? For example, under which conditions:

– derivations are automatically continuous,

– multiplicative derivations are additive (cf. e.g. [5]),

– additive derivations are linear,

– additive (or linear) derivations are inner or spatial?

Many results from ring theory as well as from the theory of operator algebras are scattered in the literature. In this section we are dealing with the structure of additive derivations on operator algebras. Let us recall the following result of Šemrl [12].

Theorem 2.1. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ be a standard operator algebra on an infinite dimensional Banach space $\mathcal{X}$. Then every additive derivation $D : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ is spatial, i.e. there exists an operator $T \in \mathcal{B}(\mathcal{X})$ such that $D(A) = TA - AT$ for all $A \in \mathcal{A}$.*

If $\mathcal{X}$ is finite dimensional, additive derivations have a more complicated structure (cf. Lemma 2.2).

The main problem in showing Theorem 2.1 consists in the following. It is almost standard to prove the existence of an additive $T : \mathcal{X} \to \mathcal{X}$ such that $D(A) = TA - AT$. Next one shows that T fulfils

$$T(tA) = tT(A) + f(t)A \quad \text{for all } t \in \mathbb{F} \tag{3}$$

with some additive derivation $f : \mathbb{F} \to \mathbb{F}$. This f is the crucial point in all considerations, because it must be shown that f is zero. Let us recall some properties of such ring derivations f (cf. [1], [9]):

– $f(t) = 0$ for all algebraic t;

– any non-trivial f is (of course) discontinuous;

– any discontinuous additive $f : \mathbb{F} \to \mathbb{F}$ is unbounded on every neighbourhood of zero.

In discussing with P. Šemrl some extensions of Theorem 2.1 to algebras of unbounded operators, he remarked that the first step of the proof of Theorem 1 can be singled out in a more general context. Moreover the obtained result can be used to prove Theorem 2.6. To proceed let us introduce some terminology. Let $\mathcal{L}(\mathcal{V})$ denote the algebra of all linear operators on a vector space $\mathcal{V}$ over $\mathbb{F}$. We say that a subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{V})$ separates the points of $\mathcal{V}$ via rank one operators, if for every nonzero $\varphi \in \mathcal{V}$ there exists a linear functional $F : \mathcal{V} \to \mathbb{F}$ such that $F(\varphi) \neq 0$ and $\varphi \otimes F \in \mathcal{A}$. Here $\varphi \otimes F$ denotes the rank one operator $(\varphi \otimes F)(\psi) = F(\psi)\varphi$, $\forall \psi \in \mathcal{V}$. If $\mathcal{V} = \mathcal{D}$ is a pre-Hilbert space and $\varphi, \psi \in \mathcal{D}$ we denote by $\varphi \otimes \psi$ the rank one operator $(\varphi \otimes \psi)\chi = \langle \psi, \chi \rangle \varphi$.

Lemma 2.2 (Šemrl, private comm.). *Let $\mathcal{V}$ be a vector space over $\mathbb{F}$ and assume that $\mathcal{A} \subset \mathcal{L}(\mathcal{V})$ is a subalgebra which separates the points of $\mathcal{V}$ via rank one operators. Assume further that there exists a nonzero linear functional $F_0 : \mathcal{V} \to \mathbb{F}$ such that $\varphi \otimes F_0 \in \mathcal{A}$ for all $\varphi \in \mathcal{V}$. Let $D : \mathcal{A} \to \mathcal{L}(\mathcal{V})$ be a ring derivation. Then there exists a ring derivation $f : \mathbb{F} \to \mathbb{F}$ and an additive mapping $T : \mathcal{V} \to \mathcal{V}$ satisfying $T(t\varphi) = tT(\varphi) + f(t)\varphi$ for all $\varphi \in \mathcal{V}, t \in \mathbb{F}$ such that $D(A) = TA - AT$ for all $A \in \mathcal{A}$.*

Proof. The assertion is trivially true if dim $\mathcal{V} = 1$. So, assume that dim $\mathcal{V} > 1$. Choose $\psi \in \mathcal{V}$ such that $F_0(\psi) = 1$ and define a map $T : \mathcal{V} \to \mathcal{V}$ by $T\varphi = D(\varphi \otimes F_0)\psi$, $\varphi \in \mathcal{V}$. Then T is additive and for every $X \in \mathcal{A}$ we have

$$D(X\varphi \otimes F_0) = XD(\varphi \otimes F_0) + D(X)(\varphi \otimes F_0).$$

Applying both sides of this equation to ψ it follows that

$$D(X)\varphi = TX\varphi - XT\varphi,$$

hence $D(X) = TX - XT$. Now let $\varphi \in \mathcal{V}$ be an arbitrary nonzero vector. According to our assumption we can find a rank one idempotent $P \in \mathcal{A}$ such that $P\varphi = \varphi$. Since $D(P)$ is linear it follows that

$$TP(t\varphi) - PT(t\varphi) = t(TP\varphi - PT\varphi)$$

for every $t \in \mathbb{F}$. This implies $P(T(t\varphi) - tT\varphi) = T(t\varphi) - tT\varphi$. Applying the fact that P is an idempotent of rank one whose range is spanned by the vector φ we conclude that there exists an additive map $f_\varphi : \mathbb{F} \to \mathbb{F}$ such that $T(t\varphi) = tT\varphi + f_\varphi(t)\varphi$ for all $t \in \mathbb{F}$.

Let $\varphi, \chi \in \mathcal{V}$ be linearly independent. Comparing $T(t(\varphi + \chi))$ with $T(t\varphi) + T(t\chi)$ one can see that $f_\varphi = f_{\varphi+\chi} = f_\chi$. It follows easily that $f_\varphi = f$ is independent of φ. Moreover, it follows from $tsT\varphi + f(ts)\varphi = T((ts)\varphi) = T(t(s\varphi)) = tT(s\varphi) + f(t)s\varphi = tsT\varphi + tf(s)\varphi + f(t)s\varphi$ that f is a ring derivation. This completes the proof. □

Remarks 2.3. i) The converse is trivially true, namely if $f : \mathbb{F} \to \mathbb{F}$ is a ring derivation, $T : \mathcal{V} \to \mathcal{V}$ is additive and so that $T(t\varphi) = tT(\varphi) + f(t)\varphi$, then $TX - XT$ is linear for all $X \in \mathcal{L}(\mathcal{V})$ and $D(X) = TX - XT$ is a ring derivation.

ii) There are many algebras fulfilling these assumptions, e.g. all standard operator algebras (bounded as well as unbounded ones), but also non-standard operator algebras. An example can be obtained as follows: identify the elements of $\mathcal{B}(\mathcal{H})$ (or of $\mathcal{L}^+(\mathcal{D})$) with infinite matrices, take the subalgebra $\mathcal{A}$ consisting of all matrices with nonzero elements only in a finite number of columns. Of course $\mathcal{A}$ is not a $*$-algebra.

Our aim is to prove a generalization of Theorem 2.1. The crucial point is the proof that f in the representation (3) is identical zero. This can be done along the same line as in [12]. To do this, we suppose that $\mathcal{D}$ has the following property:

(B) There exists an infinite orthonormal system (φ_n) in $\mathcal{D}$ with the following two properties:

i) there is a sequence (t_n), $t_n \neq 0$, $t_n \in \mathbb{F}$ such that $\sum t_n \varphi_n \in \mathcal{D}$,

ii) for all (s_n), $s_n \in \mathbb{F}$ and $|s_n| \leq |t_n|$, $\sum s_n \varphi_n$ belongs also to $\mathcal{D}$.

Let us remark that this property is clearly fulfilled in the case that $\mathcal{D} = \mathcal{H}$ is an infinite dimensional Hilbert space. In a Banach space (φ_n) must be replaced by an appropriate basic sequence. In our context of algebras of unbounded operators this property holds if $(\mathcal{D}, t)$ is an (F)-space or a (QF)-space which contains at least one bounded set $\mathcal{M}$ which spans an infinite dimensional (F)-subspace of $(\mathcal{D}, t)$.

Theorem 2.4. *Let $\mathcal{D} \subset \mathcal{H}$ be a dense domain with property* (B). *Further let $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$ be a standard operator algebra and $D : \mathcal{A} \to \mathcal{L}^+(\mathcal{D})$ an additive derivation. Then there is a $T \in \mathcal{L}^+(\mathcal{D})$ such that $D(X) = TX - XT$ for all $X \in \mathcal{A}$, i.e. D is a spatial derivation.*

Proof. By Lemma 2.2 there are an additive mapping $T : \mathcal{D} \to \mathcal{D}$ and a ring derivation $f : \mathbb{F} \to \mathbb{F}$ so that

$$(4) \qquad D(A) = TA - AT, \quad T(t\psi) = tT\psi + f(t)\psi, \qquad \forall \psi \in \mathcal{D},\ t \in \mathbb{F}.$$

Let (φ_n) be the orthonormal system from (B) and extend it to a Hamel basis $\{\varphi_\alpha, \alpha \in J\}$ of $\mathcal{D}$. Define an additive mapping $T_1 : \mathcal{D} \to \mathcal{D}$ by

$$T_1\Big(\sum_\alpha t_\alpha \varphi_\alpha\Big) = \sum_\alpha f(t_\alpha)\varphi_\alpha.$$

Put $T_2\varphi = T\varphi - T_1\varphi$ for all $\varphi \in \mathcal{D}$. Since f is a ring derivation, T_2 is linear and consequently D is of the form $D(A) = T_1A - AT_1 + T_2A - AT_2$ with a linear mapping $T_2 : \mathcal{D} \to \mathcal{D}$. T_1 fulfils $T_1(t\varphi_n) = f(t)\varphi_n$ for all $t \in \mathbb{F}$, $n \in \mathbb{N}$. In particular:

$$T_1(s\varphi_n) = 0 \quad \text{for all } n \text{ and all algebraic numbers } s \in \mathbb{F}. \tag{5}$$

Define the one-dimensional orthoprojections $P_n = \varphi_n \otimes \varphi_n$. Since $P_n \in \mathcal{A}$ it follows $D(P_n) \in \mathcal{L}^+(\mathcal{D})$ and $D(P_n)$ is a bounded operator because it is at most two-dimensional. Let $(t_n) \subset \mathbb{F}$ so that $t_n \neq 0$ and $\sum_n t_n\varphi_n \in \mathcal{D}$ (which is possible due to (B)). Put

$$A = \Big(\sum_n t_n\varphi_n\Big) \otimes \varphi_1,$$

which defines an operator from $\mathcal{A}$. The same way as in [12] we obtain:

$$D(P_nA)\varphi_1 = (f(t_n) - t_n\langle\varphi_1, T_2\varphi_1\rangle)\varphi_n + t_nT_2\varphi_n \tag{6}$$

and

$$P_nD(A)\varphi_1 = P_nT_1\Big(\sum_i t_i\varphi_i\Big) + P_nT_2\Big(\sum_i t_i\varphi_i\Big) - t_n\langle\varphi_1, T_2\varphi_1\rangle\varphi_n. \tag{7}$$

Therefore

$$D(P_n)A\varphi_1 = D(P_n)\sum_i t_i\varphi_i = \sum_i t_iD(P_n)\varphi_i. \tag{8}$$

Put $\varepsilon_n = \max\{1, \|T_2\varphi_n\|\}$, $n \in \mathbb{N}$. Choose a sequence (s_n) of algebraic numbers so that $0 < |s_n| < \min\{|t_n|, \varepsilon_n^{-1}2^{-n}\}$. Put

$$B = \Big(\sum_{i=1}^k s_i\varphi_i\Big) \otimes \varphi_1.$$

Then (5)–(8) and $D(P_nB) = P_nD(B) + D(P_n)B$ imply

$$\sum_{i=1}^k s_iD(P_n)\varphi_i = D(P_n)B\varphi_1 = D(P_nB)\varphi_1 - P_nD(B)\varphi_1$$

$$
\begin{aligned}
&= \{f(s_n) - s_n\langle\varphi_1, T_2\varphi_1\rangle\}\varphi_n + \alpha(k,n)s_nT_2\varphi_n \\
&\quad - P_nT_1\left(\sum_{i=1}^k s_i\varphi_i\right) - P_nT_2\left(\sum_{i=1}^k s_i\varphi_i\right) + s_n\langle\varphi_1, T_2\varphi_1\rangle\varphi_n \\
&= \alpha(k,n)s_nT_2\varphi_n - P_nT_2\left(\sum_{i=1}^k s_i\varphi_i\right)
\end{aligned}
$$

where $\alpha(k,n) = 1$ if $n \leq k$, $\alpha(n,k) = 0$ otherwise. This implies the estimation

$$
\begin{aligned}
\left\|\sum_{i=1}^k s_iD(P_n)\varphi_i\right\| &\leq |s_n|\,\|T_2\varphi_n\| + \sum_{i=1}^k |s_i|\,\|T_2\varphi_i\| \\
&\leq 2^{-n} + \sum_{i=1}^k 2^{-i} < 2.
\end{aligned} \tag{9}
$$

Consequently we have for all pairs k, n:

$$
\left\|\sum_{i=1}^k s_iD(P_n)\varphi_i\right\| < 2 \tag{10}
$$

and

$$
\|s_kD(P_n)\varphi_k\| = \left\|\sum_{i=1}^k s_iD(P_n)\varphi_i - \sum_{i=1}^{k-1} s_iD(P_n)\varphi_i\right\| \leq 2 + 2 = 4.
$$

Hence

$$
\|D(P_n)\varphi_k\| \leq \frac{4}{|s_k|}\,. \tag{11}
$$

Assume now that f is not identical zero. Then f must be unbounded on every neighborhood of $s_n, n \in \mathbb{N}$. Therefore we can find a sequence of numbers $(p_n) \subset \mathbb{F}$ with the following properties:

$$
|p_n| \leq |t_n|, \quad |p_n - s_n| < 2^{-n-1}|s_n| \tag{12}
$$

and

$$
|f(p_n)| > |\langle\varphi_1, T_2\varphi_1\rangle| + \|T_2\varphi_n\| + 4 + n. \tag{13}
$$

Now we define the following one-dimensional operator (which leads us to the desired contradiction): $C = \left(\sum_{i=1}^{\infty} p_i \varphi_i\right) \otimes \varphi_1$. For all $n \in \mathbb{N}$ we have trivially

$$\|P_n D(C)\varphi_1\| \leq \|D(C)\varphi_1\|. \tag{14}$$

On the other hand, $D(P_n C) = P_n D(C) + D(P_n)C$ implies

$$\|P_n D(C)\varphi_1\| \geq \|D(P_n C)\varphi_1\| - \|D(P_n)C\varphi_1\|.$$

Using (6) and (8) we get

$$\begin{aligned}\|P_n D(C)\varphi_1\| &\geq |f(p_n)| - |p_n|\,|\langle \varphi_1, T_2\varphi_1\rangle| - |p_n|\,\|T_2\varphi_n\| \\ &\quad - \left\|\sum_{i=1}^{\infty} p_i D(P_n)\varphi_i\right\| \geq |f(p_n)| - |\langle \varphi_1, T_2\varphi_1\rangle| - \|T_2\varphi_n\| \\ &\quad - \left\|\sum_{i=1}^{\infty} (p_i - s_i) D(P_n)\varphi_i\right\| - \left\|\sum_{i=1}^{\infty} s_i D(P_n)\varphi_i\right\|.\end{aligned}$$

Applying (10)–(13) we get

$$\|P_n D(C)\varphi_1\| \geq 4 + n - \sum_{i=1}^{\infty} 2^{-i-1} \cdot 4 - 2 = n, \qquad \forall n \in \mathbb{N}.$$

But this clearly contradicts (14). Hence f is a trivial ring derivation on $\mathbb{F}$, and consequently T_1 is equal to zero. This implies that D is of the form

$$D(A) = TX - XT, \qquad X \in \mathcal{A}$$

with a linear mapping $T : \mathcal{D} \to \mathcal{D}$.

To complete the proof it remains to show $T \in \mathcal{L}^+(\mathcal{D})$. This is quite standard, cf. e.g. [10], but let us repeat the proof for the sake of completeness. Recall that T was defined by $T\varphi = D(\varphi \otimes \psi)\psi$ for some fixed $\psi \in \mathcal{D}$, cf. the proof of Lemma 2.2. Moreover we know that T is linear. Put $X = \psi \otimes \psi$, then:

$$\begin{aligned}\langle T\psi, \psi\rangle &= \langle D(\psi \otimes \psi)\psi, \psi\rangle \\ &= \langle T(\psi \otimes \psi)\psi, \psi\rangle - \langle (\psi \otimes \psi)T\psi, \psi\rangle = 0.\end{aligned} \tag{15}$$

If we define S by $S\varphi = (D(\psi \otimes \varphi)^+)\psi$, $\varphi \in \mathcal{D}$ we get a linear operator which maps $\mathcal{D}$ into $\mathcal{D}$ and fulfils (taking into account (15)):

$$\begin{aligned}\langle S\varphi, \chi\rangle &= \langle (D(\psi \otimes \varphi)^+)\psi, \chi\rangle = \langle \psi, D(\psi \otimes \varphi)\chi\rangle \\ &= \langle \psi, T(\psi \otimes \varphi)\chi\rangle - \langle \psi, (\psi \otimes \varphi)T\chi\rangle = -\langle \varphi, T\chi\rangle\end{aligned}$$

This implies $T \in \mathcal{L}^+(\mathcal{D})$ and $T^+ = -S$. □

Remark that it is not necessary to refer explicitly to Lemma 2.2. Theorem 2.4 can be applied to get a result about the structure of additive Jordan derivations. This is a generalization of [13] to unbounded operator algebras. The proof is the same as in [13].

Corollary 2.5. *Let the assumptions of Theorem 2.4 be fulfilled and let $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$ be a standard operator algebra. Suppose $J : \mathcal{A} \to \mathcal{L}^+(\mathcal{D})$ is an additive Jordan derivation. Then there is a $T \in \mathcal{L}^+(\mathcal{D})$ such that $J(A) = TA - AT$, $A \in \mathcal{A}$, i.e. J is a (linear) spatial derivation.*

Proof. Remark that $\mathcal{F}(\mathcal{D})$ is a prime ring, i.e. $X\mathcal{F}(\mathcal{D})Y = \{0\}$, $X, Y \in \mathcal{F}(\mathcal{D})$ implies $X = 0$ or $Y = 0$. Moreover, J maps $\mathcal{F}(\mathcal{D})$ into itself. This can be seen as follows. Every $F \in \mathcal{F}(\mathcal{D})$ is a linear combination of idempotent operators of rank one. Let P be such an operator, $\lambda = \mu^2 \in \mathbb{F}$, then $J(\lambda P) = J(\mu^2 P^2) = \mu[PJ(P) + J(\mu P)P]$ has rank at most two. Hence $J : \mathcal{F}(\mathcal{D}) \to \mathcal{F}(\mathcal{D})$. A classical result of Herstein [6] implies that J restricted to $\mathcal{F}(\mathcal{D})$ is an additive derivation, i.e. $J(AB) = J(A)B + AJ(B)$ for all $A, B \in \mathcal{F}(\mathcal{D})$. By Theorem 2.4 there is a $T \in \mathcal{L}^+(\mathcal{D})$ such that

$$J(A) = TA - AT, \quad A \in \mathcal{F}(\mathcal{D}). \tag{16}$$

Now let $A \in \mathcal{A}$, $B \in \mathcal{F}(\mathcal{D})$. From the definition of a Jordan derivation it follows that $J((A+B)^2) = (A+B)J(A+B) + J(A+B)(A+B)$, hence $J(AB + BA) = AJ(B) + BJ(A) + J(A)B + J(B)A$. Since $(AB + BA) \in \mathcal{F}(\mathcal{D})$ we can apply (16) to get

$$B[TA - AT - J(A)] + [TA - AT - J(A)]B = 0, \qquad B \in \mathcal{F}(\mathcal{D}).$$

But this implies $J(A) = TA - AT$, $A \in \mathcal{A}$. □

Let $\boldsymbol{d} \subset l^2$ denote the vector space of all sequences $x = (x_i)$ with only finitely many nonzero elements. Clearly, $\boldsymbol{d}$ does not have property (B). The starting point in the discussion with P. Šemrl was the question whether or not the conclusion of Theorem 2.4 is valid for standard operator algebras $\mathcal{A} \subset \mathcal{L}^+(\boldsymbol{d})$. The following Theorem is included here with kind permission of P. Šemrl (private communication). Remark that every $X \in \mathcal{L}^+(\boldsymbol{d})$ can be identified with a matrix (x_{ij}) having only finitely many elements in each row and column, while the elements of $\mathcal{F}(\boldsymbol{d})$ can be identified with matrices having nonzero elements only in a block of finite size in the upper left corner.

Theorem 2.6. *Let $\mathcal{A}$ be a standard operator algebra on $\boldsymbol{d}$ and $D : \mathcal{A} \to \mathcal{L}^+(\boldsymbol{d})$ a multiplicative derivation. Then there exist a ring derivation $f : \mathbb{F} \to \mathbb{F}$ and a $T \in \mathcal{L}^+(\boldsymbol{d})$ such that*

$$D(X) = D((x_{ij})) = (f(x_{ij})) + TX - XT, \quad X \in \mathcal{A}. \tag{17}$$

Proof. Denote the restriction of D to $\mathcal{F}(\boldsymbol{d})$ by E. Since any finite rank operator X can be written as $X = YZ$ with finite rank operators Y, Z, we conclude that $D(X)$ is also of finite rank, i.e. $E : \mathcal{F}(\boldsymbol{d}) \to \mathcal{F}(\boldsymbol{d})$ is a multiplicative derivation. Therefore we can apply a result of Daif [5] to conclude that E is a ring derivation. By Lemma 2.2 there are a ring derivation $f : \mathbb{F} \to \mathbb{F}$ and an additive mapping $T_0 : \boldsymbol{d} \to \boldsymbol{d}$ with $T_0(tx) = tT_0(x) + f(t)x$, $x \in \boldsymbol{d}$, $t \in \mathbb{F}$ such that $E(X) = T_0X - XT_0$ for all $X \in \mathcal{F}(\boldsymbol{d})$. Let us define an additive mapping $T_1 : \boldsymbol{d} \to \boldsymbol{d}$ by $T_1(x_1, x_2, \dots, x_n, 0, 0, \dots) = (f(x_1), f(x_2), \dots, f(x_n), 0, 0, \dots)$. Then $T = T_0 - T_1$ is linear and $E(X) = T_1X - XT_1 + TX - XT$. A direct computation shows that $T_1X - XT_1 = T_1(x_{ij}) - (x_{ij})T_1 = (f(x_{ij}))$. It follows that $TX - XT \in \mathcal{F}(\boldsymbol{d})$ for every $X \in \mathcal{F}(\boldsymbol{d})$. But this implies that $T \in \mathcal{L}^+(\boldsymbol{d})$. Hence E has the desired form.

Now let $X = (x_{ij}) \in \mathcal{A}$ arbitrary. Then for every $Y = (y_{ij}) \in \mathcal{F}(\boldsymbol{d})$ we have $YD(X) = D(YX) - D(Y)X$. Applying the fact that YX and Y are of finite rank one can easily verify that

$$YD(X) = Y((f(x_{ij})) + TX - XT).$$

Accordingly, $D(X) - (f(x_{ij})) - TX + XT$ annihilates $\mathcal{F}(\boldsymbol{d})$ and therefore is identical zero, i.e. (17) is valid. □

3. Jordan $*$-derivations

The structure of Jordan $*$-derivations was considered by several authors and with increasing generality. Let us mention some of those results which are relevant in our context.

Brešar and Vukman proved in [2] among other things the following theorem:

Every Jordan $*$-derivation on a complex algebra with unit is inner. If one tries to drop the assumption about the existence of the unit one must restrict the class of algebras. Moreover, real algebras are more difficult to handle than complex algebras. As a further step Šemrl proved in [11] that every Jordan $*$-derivation on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a real Hilbert space with dimension greater than 1 is inner. Brešar and Zalar gave in [3] another proof of this result. Moreover, they pointed out the connection with double centralizers (see below).

Finally, Šemrl has shown the following general result in [13]:

Theorem 3.1. *Let $\mathcal{H}$ be a real or complex Hilbert space,* $\dim \mathcal{H} > 1$, *and let $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$. Suppose that $J : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an additive Jordan $*$-derivation. Then there exists a unique linear operator $T \in \mathcal{B}(\mathcal{H})$ such that $J(A) = AT - TA^*$ for all $A \in \mathcal{A}$.*

The aim of this section is to generalize this theorem to unbounded standard operator algebras. We include the proof for sake of completeness but emphasize that it is with minor modifications the original proof of Šemrl. In the second part of this section we discuss the relationship with double centralizers.

Theorem 3.2. *Let $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$ be a standard operator algebra and $J : \mathcal{A} \to \mathcal{L}^+(\mathcal{D})$ an additive Jordan $*$-derivation. Then there is a unique $T \in \mathcal{L}^+(\mathcal{D})$ such that $J(A) = AT - TA^+$.*

Proof. We repeat the proof given in [13] for bounded standard operator algebras and indicate the necessary changes. Denote the restriction of J to $\mathcal{F}(\mathcal{D})$ by J_1 and define a map $\Phi : \mathcal{F}(\mathcal{D}) \to \mathcal{L}^+(\mathcal{D} \oplus \mathcal{D})$ by

$$\Phi(A) = \begin{pmatrix} A & J_1(A) \\ 0 & A^+ \end{pmatrix}. \tag{18}$$

Using $J_1(A^2) = AJ_1(A) + J_1(A)A^+$ it is easily to see that Φ is a Jordan homomorphism, i.e. Φ is additive and $\Phi(A^2) = \Phi(A)^2$. Since $\mathcal{F}(\mathcal{D})$ is a local matrix algebra we can apply a result of Jacobson and Rickart [7] stating that $\Phi = F + G$ where $F : \mathcal{F}(\mathcal{D}) \to \mathcal{L}^+(\mathcal{D} \oplus \mathcal{D})$ is a ring homomorphism and $G : \mathcal{F}(\mathcal{D}) \to \mathcal{L}^+(\mathcal{D} \oplus \mathcal{D})$ is a ring antihomomorphism (i.e. G is additive and $G(AB) = G(B)\ G(A)$). Since

$$\operatorname{Im}\Phi \subset \left\{ \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix} \in \mathcal{L}^+(\mathcal{D} \oplus \mathcal{D}) : X, Y, W \in \mathcal{L}^+(\mathcal{D}) \right\}$$

the maps F, G have the following form:

$$F(A) = \begin{pmatrix} F_1(A) & F_2(A) \\ 0 & F_3(A) \end{pmatrix}, \quad G(A) = \begin{pmatrix} G_1(A) & G_2(A) \\ 0 & G_3(A) \end{pmatrix} \tag{19}$$

where F_1, F_3 are additive homomorphisms, G_1, G_3 are additive antihomomorphisms on $\mathcal{F}(\mathcal{D})$ and the equations $F_1(A) + G_1(A) = A$, $F_3(A) + G_3(A) = A^+$ hold for all $A \in \mathcal{F}(\mathcal{D})$. Now choose an idempotent $P \in \mathcal{F}(\mathcal{D})$ of rank one. The equation $F_1(P) + G_1(P) = P$ implies either $F_1(P) = 0$ or $G_1(P) = 0$, i.e. at least one of F_1 and G_1 has a nonzero kernel. But the kernels of homomorphisms or antihomomorphisms are ideals and the only nonzero ideal of $\mathcal{F}(\mathcal{D})$ is $\mathcal{F}(\mathcal{D})$ itself, we have $F_1 = 0$ or $G_1 = 0$. This implies $G_1 = 0$ and $F_1(A) = A$ for all $A \in \mathcal{F}(\mathcal{D})$. Remark that $F_1 = 0$ would imply $G_1(A) = A$, a contradiction since G_1 is an antihomomorphism. Similarly it can be shown that $F_3 = 0$ and $G_3(A) = A^+$ for all $A \in \mathcal{F}(\mathcal{D})$. Thus

$$F(A) = \begin{pmatrix} A & F_2(A) \\ 0 & 0 \end{pmatrix}, \quad G(A) = \begin{pmatrix} 0 & G_2(A) \\ 0 & A^+ \end{pmatrix}$$

and F_2, G_2 are additive mappings satisfying

$$F_2(AB) = AF_2(B) \tag{20}$$

and

$$G_2(AB) = G_2(B)A^+ \tag{21}$$

for all $A, B \in \mathcal{F}(\mathcal{D})$. This follows immediately from the properties of F, G. From $F_2 + G_2 = J_1$, $J_1(A^2) = AJ_1(A) + J_1(A)A^+$, (20) and (21) we obtain

$$\begin{aligned} F_2(A^2) + G_2(A^2) &= A(F_2(A) + G_2(A)) + (F_2(A) + G_2(A))A^+ \\ &= AF_2(A) + G_2(A)A^+, \end{aligned}$$

hence

$$AG_2(A) + F_2(A)A^+ = 0.$$

Replacing A by $A + B$ this implies

$$F_2(A)B^+ + F_2(B)A^+ + AG_2(B) + BG_2(A) = 0 \quad \text{for all } A, B \in \mathcal{F}(\mathcal{D}). \tag{22}$$

Now we replace B by CB, $C \in \mathcal{F}(\mathcal{D})$ arbitrary and obtain

$$C\big[F_2(B)A^+ + BG_2(A)\big] + \big[F_2(A)B^+ + AG_2(B)\big]C^+ = 0.$$

This together with (22) implies

$$F_2(A)B^+ + AG_2(B) = 0 \quad \text{for all } A, B \in \mathcal{F}(\mathcal{D}). \tag{23}$$

From (20) we conclude that F_2 is a linear map on $\mathcal{F}(\mathcal{D})$. This can be seen as follows. Let $P = P^2 \in \mathcal{F}(\mathcal{D})$. Then $F_2(P) = F_2(P^2) = PF_2(P)$, thus $F_2(\lambda P) = F_2(\lambda PP) = \lambda PF_2(P) = \lambda F_2(P)$. The assertion follows now from the additivity of F_2 and the fact that every $A \in \mathcal{F}(\mathcal{D})$ is a linear combination of idempotents from $\mathcal{F}(\mathcal{D})$. For every $\varphi \in \mathcal{D}$ we put $L_\varphi = \{\varphi \otimes \psi = \langle \psi, \cdot \rangle \varphi : \psi \in \mathcal{D}\} \subset \mathcal{F}(\mathcal{D})$. Then (20) implies $F_2(L_\varphi) \subset L_\varphi$. Thus for every nonzero $\varphi \in \mathcal{D}$ there is a linear map $S_\varphi : \mathcal{D} \to \mathcal{D}$ such that $F_2(\varphi \otimes \psi) = \varphi \otimes (S_\varphi \psi)$. It is rather standard (cf. [13] or [8]) to show that S_φ is independent of φ. Hence there is a linear operator $S : \mathcal{D} \to \mathcal{D}$ such that

$$F_2(\varphi \otimes \psi) = \varphi \otimes (S\psi) = \langle S\psi, \cdot \rangle \varphi. \tag{24}$$

Let G_2' be the mapping defined by $G_2'(A) = (G_2(A))^+$. Equation (21) implies $G_2'(AB) = AG_2'(B)$. Thus there is a linear operator $T : \mathcal{D} \to \mathcal{D}$ such that

$$G_2(\varphi \otimes \psi) = -(T\psi) \otimes \varphi = -\langle \varphi, \cdot \rangle T\psi. \tag{25}$$

Substituting in (23) $A = \langle \psi, \cdot \rangle \varphi$, $B = \langle \chi, \cdot \rangle \varrho$ we obtain

$$F_2(\langle \psi, \cdot \rangle \varphi) \cdot \langle \varrho, \cdot \rangle \chi = -\langle \psi, \cdot \rangle \varphi \cdot G_2(\langle \chi, \cdot \rangle \varrho).$$

Using (24) and (25) this implies

$$\langle S\psi, \chi \rangle \cdot \langle \varrho, \cdot \rangle \varphi = \langle \psi, T\chi \rangle \cdot \langle \varrho, \cdot \rangle \varphi, \quad \text{for all } \varphi, \psi, \chi, \varrho \in \mathcal{D}.$$

Hence $\langle S\psi, \chi\rangle = \langle \psi, T\chi\rangle$ for all $\psi, \chi \in \mathcal{D}$. This means $S = T^+$, $T = S^+$ and $S, T \in \mathcal{L}^+(\mathcal{D})$. Moreover the equations

$$F_2(\langle \psi, \cdot\rangle \varphi) = \langle \psi, \cdot\rangle \varphi \cdot T \quad \text{and} \quad G_2(\langle \psi, \cdot\rangle \varphi) = -T(\langle \psi, \cdot\rangle \varphi)^+$$

yield

$$F_2(A) = AT,\ G_2(A) = -TA^+ \quad \text{for all } A \in \mathcal{F}(\mathcal{D}).$$

Using $J_1 = F_2 + G_2$ we obtain

$$J_1(A) = AT - TA^+, \quad \text{for all } A \in \mathcal{F}(\mathcal{D}). \tag{26}$$

The extension of (26) to $A \in \mathcal{A}$ is standard. First replace A by $A + B$ in $J(A^2) = AJ(A) + J(A)A^+$ to get

$$J(AB) + J(BA) = AJ(B) + BJ(A) + J(A)B^+ + J(B)A^+ \quad \text{for all } A, B \in \mathcal{A}.$$

Applying (26) to this equation, it follows that

$$B(J(A) - AT + TA^+) + (J(A) - AT + TA^+)B^+ = 0,$$
$$\text{for all } A \in \mathcal{A}, B \in \mathcal{F}(\mathcal{D}).$$

This means (26) is valid for all $A \in \mathcal{A}$. □

Now we comment on another approach to additive Jordan $*$-derivations used by Brešar and Zalar [3]. Let us recall some definitions. Let $\mathcal{A}$ be an algebra. A linear (additive) mapping $L : \mathcal{A} \to \mathcal{A}$ is called an (additive) left centralizer of $\mathcal{A}$ if $L(xy) = L(x)y$ for all $x, y \in \mathcal{A}$. Analogously, an (additive) right centralizer of $\mathcal{A}$ is a linear (additive) mapping $R : \mathcal{A} \to \mathcal{A}$ such that $R(xy) = xR(y)$. An (additive) double centralizer of $\mathcal{A}$ is a pair (L, R), where L is a left and R is a right centralizer such that

$$xL(y) = R(x)y. \tag{27}$$

In [3] there was proved the following theorem.

Theorem 3.3. *Let $\mathcal{A}$ be a complex $*$-algebra such that $\mathcal{A}x = 0$ or $x\mathcal{A} = 0$, $x \in \mathcal{A}$ implies $x = 0$. If J is a Jordan $*$-derivation on $\mathcal{A}$ then there exists a unique double centralizer (L, R) such that $J(x) = L(x^*) - S(x)$ for all $x \in \mathcal{A}$.*

Let us remark the following. If under the assumptions of the theorem above L, R are additive left resp. right centralizers and (L, R) is a double centralizer, than L, R are automatically linear. This implies that J is real linear.

Since a standard operator algebra satisfies the assumptions of Theorem 3.3, this theorem gives a representation of Jordan $*$-derivations on such algebras via double centralizers. The next proposition describes the structure of double centralizers on unbounded standard operator algebras (in the same manner as in the bounded case).

Proposition 3.4. *Let (L, R) be a double centralizer on a standard operator algebra $\mathcal{A}$ on $\mathcal{D}$. Then there is a unique $T \in \mathcal{L}^+(\mathcal{D})$ such that $L(A) = TA$, $R(A) = AT$.*

Proof. Let $\varphi, \psi \in \mathcal{D}$ be arbitrary. Then

$$L(A \cdot \langle \varphi, \cdot \rangle \psi) = \langle \varphi, \cdot \rangle L(A)\psi. \tag{28}$$

Define T by $T(A\psi) = L(A)\psi$, $\psi \in \mathcal{D}$. Since $\mathcal{F}(\mathcal{D}) \subset \mathcal{A}$ this is a linear operator $T : \mathcal{D} \to \mathcal{D}$. To see that the definition of T is correct suppose $\psi_1, \psi_2 \in \mathcal{D}$, $A_1, A_2 \in \mathcal{A}$ so that $A_1\psi_1 = A_2\psi_2$. For arbitrary $\varphi \in \mathcal{D}$ we have $\langle \varphi, \cdot \rangle A_1\psi_1 = \langle \varphi, \cdot \rangle A_2\psi_2$ and $L(A_1 \cdot \langle \varphi, \cdot \rangle \psi_1) = L(A_2 \cdot \langle \varphi, \cdot \rangle \psi_2)$. Thus (28) implies $L(A_1)\psi_1 = L(A_2)\psi_2$, i.e. T is correctly defined. Next define a linear operator $S : \mathcal{D} \to \mathcal{D}$ by $S(A^+\varphi) = R(A)^+\varphi$, $\varphi \in \mathcal{D}$. The definition of T implies $AL(B)\varphi = ATB\varphi$, hence $\langle AL(B)\varphi, \psi \rangle = \langle ATB\varphi, \psi \rangle = \langle R(A)B\varphi, \psi \rangle = \langle B\varphi, R(A)^+\psi \rangle = \langle B\varphi, SA^+\psi \rangle = \langle TB\varphi, A^+\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}, A, B \in \mathcal{A}$. Using $\mathcal{F}(\mathcal{D}) \subset \mathcal{A}$ this means:

$$\langle T\chi, \varrho \rangle = \langle \chi, S\varrho \rangle \quad \text{for all } \chi, \varrho \in \mathcal{D}.$$

Hence $S, T \in \mathcal{L}^+(\mathcal{D})y$, $S = T^+$ and $L(A) = TA$, $R(A) = AS^+ = AT$ for all $A \in \mathcal{A}$. We omit the easy proof of the uniqueness of T. $\square$

Together with Theorem 3.3 we get a shorter proof of Theorem 3.2 in the complex case, namely $J(A) = L(A^+) - R(A) = TA^+ - AT$ (or setting $T_1 = -T$:$J(A) = AT_1 - T_1A^+$). Remark that Proposition 3.4 becomes trivial if $\mathcal{A}$ contains the unit operator, because then we put $T = L(I)$.

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