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Approximately multiplicative functions

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Abstract. We consider the inequality $|f(ts)-g_1(t)g_2(s)| \le \delta$, where f, g_1, g_2 are real-valued functions defined for nonzero reals. In the special case $g_1 = g_2$ our result is reduced to Šemrl's generalization of Baker's result on the superstability of multiplicative functions.

1. Introduction and statement of the result

The stability of functional equations was first studied by Ulam who posed the stability problem in 1940 (see [4, p. 63]). For Banach spaces and approximately additive mappings this problem was solved by Hyers in 1941 [2]. He showed that if $\delta > 0$ and $f : X \to Y$ is a mapping between real Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta, \quad x, y \in X,$$

then there exists a unique additive mapping $g : X \to Y$ with $\|f(x) - g(x)\| \le \delta$ for all $x \in X$. One can consider approximately multiplicative mappings instead of approximately additive ones. Baker [1] proved that the so called superstability phenomenon occurs in this case. More precisely, if $\delta > 0$ and $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfies

$$|f(ts) - f(t)f(s)| \le \delta, \quad t, s \in \mathbb{R} \setminus \{0\}, \tag{1}$$

then f is either multiplicative or bounded by $|f(t)| \le (1 + \sqrt{1+4\delta}\,)/2$ for every nonzero real t.

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Recently, Šemrl [3] proved that a bounded solution of (1) must be either close to the function $\operatorname{sign} t$ or to one of the constant functions $h(t) \equiv 0$ or $h(t) \equiv 1$. Moreover, Šemrl also considered inequality (1) with two functions involved instead of one and obtained the following result [3, Theorem 3].

Proposition. *Let $\delta > 0$ and let $f, g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfy*

$$|f(ts) - g(t)g(s)| \le \delta, \quad t, s \in \mathbb{R} \setminus \{0\}.$$

Then either there exist a real constant c and a multiplicative function $k : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$|f(t) - c^2 k(t)| \le \delta, \quad g(t) = ck(t), \quad t \in \mathbb{R} \setminus \{0\},$$

or f and g are bounded. More precisely:

$$|f(t)| \le 3\delta, \quad |g(t)| \le 2\sqrt{\delta}, \quad t \in \mathbb{R} \setminus \{0\},$$

or there exist $\eta > 0$ and $c \in \mathbb{R}$ satisfying $\eta(|c| + \eta) \le \delta$ such that either

$$|f(t) - c^2 + \eta^2| \le \delta, \quad |g(t) - c| \le \eta, \qquad t \in \mathbb{R} \setminus \{0\},$$

or

$$|f(t) - (c^2 - \eta^2) \operatorname{sign} t| \le \delta, \quad |g(t) - c \operatorname{sign} t| \le \eta, \quad t \in \mathbb{R} \setminus \{0\}.$$

It seems natural to go even one step further by considering Pexiderised version of (1).

Theorem. *Let $\delta > 0$ and let $f, g_1, g_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where g_1 and g_2 are nonzero, satisfy*

$$|f(ts) - g_1(t)g_2(s)| \le \delta, \qquad t, s \in \mathbb{R} \setminus \{0\}. \tag{2}$$

Then there are two possibilities: either there exist real constants c_1, c_2 and a multiplicative function $k : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$|f(t) - c_1 c_2 k(t)| \le \delta, \quad g_i(t) = c_i k(t), \quad t \in \mathbb{R} \setminus \{0\}, \ i = 1, 2,$$

or f, g_1 and g_2 are bounded functions. In the second case there exist constants $\eta_1, \eta_2 > 0$ satisfying $\eta_1 \eta_2 \le 4\delta$ and

$$|f(t)| \le 3\delta, \quad |g_i(t)| \le \eta_i, \quad t \in \mathbb{R} \setminus \{0\}, \ i = 1, 2, \tag{3}$$

or there exist constants $c_1, c_2 \in \mathbb{R}$, $\eta_1, \eta_2 > 0$ *satisfying*

$$2\eta_1\eta_2 + |c_1|\eta_2 + |c_2|\eta_1 \le 2\delta, \tag{4}$$

such that either

$$|f(t) - c_1c_2 + \eta_1\eta_2 \operatorname{sign}(g_1(1)g_2(1))| \le \delta, \quad |g_i(t) - c_i| \le \eta_i, \tag{5}$$

or

$$\begin{gathered} |f(t) - (c_1c_2 - \eta_1\eta_2 \operatorname{sign}(g_1(1)g_2(1))) \operatorname{sign} t| \le \delta, \\ |g_i(t) - c_i \operatorname{sign} t| \le \eta_i \end{gathered} \tag{6}$$

for every $t \in \mathbb{R} \setminus \{0\}$, $i = 1, 2$. *If* $g_1 = g_2$, *then we can choose constants in such a way that* $\eta_1 = \eta_2$ *and* $c_1 = c_2$.

Remarks. By (4) it follows that $\eta_1\eta_2 \le \delta$, so (5) implies that $f(t)$ is near c_1c_2 and (6) implies that $f(t)$ is near $c_1c_2 \operatorname{sign} t$. Our theorem is formulated in such a way to draw the analogy with Šemrl's Proposition, which is obtained as a special case of our theorem when $g_1 = g_2$.

Also note that η_1 and η_2 are not both necessarily small. For example, functions

$$g_1(t) = \begin{cases} c_1 - \eta_1 & \text{if } t < 0 \\ c_1 + \eta_1 & \text{if } t > 0 \end{cases}, \qquad g_2(t) = \frac{\delta}{\eta_1} \quad \text{and} \quad f(t) = \frac{\delta c_1}{\eta_1}$$

satisfy the equation (2) for arbitrary $c_1 > \eta_1 > 0$ and we obtain (5) for $c_2 = \delta/\eta_1$ and $\eta_2 = 0$.

2. Proof of the theorem

We begin by showing that if any of the functions f, g_1, g_2 is bounded then all of them must be bounded. Since g_1 and g_2 are nonzero, there exist $u_1, u_2 \in \mathbb{R} \setminus \{0\}$ such that $g_i(u_i) \ne 0$, $i = 1, 2$. Suppose f is bounded by a constant M. Then

$$|g_i(t)| \le \frac{|f(tu_j)| + \delta}{|g_j(u_j)|} \le \frac{M + \delta}{|g_j(u_j)|}, \quad t \in \mathbb{R} \setminus \{0\}, \quad i = 1, 2, \quad j = 3 - i,$$

meaning that all three functions are bounded. If any of g_i, $i \in \{1, 2\}$, is bounded, then so is f, since

$$|f(t)| \le |g_i(t)g_j(1)| + \delta, \quad t \in \mathbb{R} \setminus \{0\}, \quad i \in \{1, 2\}, \quad j = 3 - i,$$

and consequently also g_j is bounded.

To prove the theorem we must therefore consider two cases: f, g_1, g_2 unbounded and f, g_1, g_2 bounded.

Let f, g_1 and g_2 be unbounded. Assume that there exists $u_0 \in \mathbb{R}\setminus\{0\}$ with $g_i(u_0) = 0$. Then

$$|f(t)| = \left|f(t) - g_i(u_0)g_j\left(\frac{t}{u_0}\right)\right| \leq \delta, \quad t \in \mathbb{R}\setminus\{0\}, \quad i \in \{1,2\}, \quad j = 3-i,$$

which contradicts the fact that f is unbounded, so $g_i(t) \neq 0$ for all $t \in \mathbb{R}\setminus\{0\}$, $i = 1,2$. We define three auxiliary functions and a constant

$$f_1(t) = \frac{f(t)}{g_1(1)g_2(1)}, \quad h_i(t) = \frac{g_i(t)}{g_i(1)}, \quad i = 1,2, \quad \delta_1 = \frac{\delta}{|g_1(1)g_2(1)|}.$$

Then

$$|f_1(ts) - h_1(t)h_2(s)| \leq \delta_1, \quad t,s \in \mathbb{R}\setminus\{0\}$$

and it follows from $h_i(1) = 1$, $i = 1,2$, that

$$(7) \quad |f_1(t) - h_i(t)| \leq \delta_1,\ i = 1,2, \text{ and } |h_1(t) - h_2(t)| \leq 2\delta_1,\ t \in \mathbb{R}\setminus\{0\}.$$

So,

$$(8) \quad |h_1(ts) - h_1(t)h_2(s)| \leq |h_1(ts) - f_1(ts)| + |f_1(ts) - h_1(t)h_2(s)| \leq 2\delta_1$$

for all $t,s \in \mathbb{R}\setminus\{0\}$. Let us show that $h_1(t) = h_2(t)$ for all $t \in \mathbb{R}\setminus\{0\}$. Suppose, on the contrary, that there exists $s_0 \in \mathbb{R}\setminus\{0\}$ such that $h_2(s_0) = h_1(s_0) + \varepsilon$, $\varepsilon \neq 0$. Then by (8)

$$|h_1(ts_0) - h_1(t)(h_1(s_0) + \varepsilon)| \leq 2\delta_1, \qquad t \in \mathbb{R}\setminus\{0\},$$

hence

$$|h_1(ts_0) - h_1(t)h_1(s_0)| \geq |h_1(t)\varepsilon| - 2\delta_1, \quad t \in \mathbb{R}\setminus\{0\}.$$

Since h_1 is unbounded, there exists $t_0 \in \mathbb{R}\setminus\{0\}$ satisfying

$$|h_1(t_0)| > \frac{|h_1(s_0)|2\delta_1}{|\varepsilon|} + \frac{4\delta_1}{|\varepsilon|}.$$

If we write $h_2(t_0) = h_1(t_0) + \omega$, then, by (7), $|\omega| \leq 2\delta_1$. Since by (8)

$$|h_1(s_0t_0) - h_1(s_0)(h_1(t_0) + \omega)| \leq 2\delta_1,$$

it follows that

$$\begin{aligned}2\delta_1 + |h_1(s_0)\omega| &\geq |h_1(t_0 s_0) - h_1(t_0)h_1(s_0)| \\ &\geq |h_1(t_0)\varepsilon| - 2\delta_1 > |h_1(s_0)|2\delta_1 + 2\delta_1,\end{aligned}$$

which contradicts the fact that $|\omega| \leq 2\delta_1$. So, $h_1(t) = h_2(t)$ for all $t \in \mathbb{R} \setminus \{0\}$ and

$$|f_1(ts) - h_1(t)h_1(s)| \leq \delta_1.$$

By [3, Theorem 3] there exist a constant c and a multiplicative function $k : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ satisfying $h_1(t) = ck(t)$ and $|f_1(t) - c^2 k(t)| \leq \delta_1$. Therefore

$$g_i(t) = h_i(t)g_i(1) = cg_i(1)k(t) = c_i k(t), \quad i = 1, 2,$$

and

$$|f(t) - c_1 c_2 k(t)| \leq \delta.$$

It remains to examine the case when all three functions g_1, g_2 and f are bounded. Following the notation used in [3] we denote by

$$\begin{gathered}a_i = \inf\{|g_i(t)| : t \in \mathbb{R} \setminus \{0\}\}, \quad b_i = \sup\{|g_i(t)| : t \in \mathbb{R} \setminus \{0\}\}, \\ e_i = \frac{a_i + b_i}{2}, \quad \mu_i = \frac{b_i - a_i}{2}, \quad i = 1, 2.\end{gathered} \tag{9}$$

Let $t \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$ and let $s \in \mathbb{R} \setminus \{0\}$ be such that $|g_2(s)| < a_2 + \varepsilon$, so $|f(t)| \leq |g_1(t/s)g_2(s)| + \delta < b_1(a_2 + \varepsilon) + \delta$. Therefore $|f(t)| \leq b_1 a_2 + \delta$ and similarly we show that $|f(t)| \leq a_1 b_2 + \delta$, $|f(t)| \geq a_1 b_2 - \delta$ and $|f(t)| \geq b_1 a_2 - \delta$. Hence $|f(t)| \leq (a_1 b_2 + b_1 a_2)/2 + \delta = e_1 e_2 - \mu_1 \mu_2 + \delta$, $|f(t)| \geq (a_1 b_2 + b_1 a_2)/2 - \delta = e_1 e_2 - \mu_1 \mu_2 - \delta$, and consequently

$$\big| |f(t)| - e_1 e_2 + \mu_1 \mu_2 \big| \leq \delta, \quad t \in \mathbb{R} \setminus \{0\}. \tag{10}$$

If we choose $0 < \varepsilon < \max\{b_1, b_2\}$, then there exist $t, s \in \mathbb{R} \setminus \{0\}$ with $|g_1(t)| > b_1 - \varepsilon$ and $|g_2(s)| > b_2 - \varepsilon$, so $e_1 e_2 - \mu_1 \mu_2 + \delta \geq |f(ts)| \geq |g_1(t)g_2(s)| - \delta > (b_1 - \varepsilon)(b_2 - \varepsilon) - \delta$. We obtain $e_1 e_2 - \mu_1 \mu_2 + \delta \geq b_1 b_2 - \delta = (e_1 + \mu_1)(e_2 + \mu_2) - \delta$, which gives

$$2\mu_1 \mu_2 + e_1 \mu_2 + e_2 \mu_1 \leq 2\delta.$$

We will distinguish two cases. First, let $e_1e_2 - \mu_1\mu_2 \leq 2\delta$. From (10) we obtain $|f(t)| \leq 3\delta$. Thus $|g_1(t)g_2(s)| \leq 4\delta$ for all $t, s \in \mathbb{R} \setminus \{0\}$ and if we denote $\eta_i = b_i$, $i = 1, 2$, we see that $\eta_1\eta_2 \leq 4\delta$. So, in this case we have (3).

Second, suppose $e_1e_2 - \mu_1\mu_2 > 2\delta$. Then, by (10), $|f(t)| \geq e_1e_2 - \mu_1\mu_2 - \delta > \delta$ for all $t \in \mathbb{R}\setminus\{0\}$. If $f(ts) > \delta$, then $g_1(t)g_2(s) \geq f(ts) - \delta > 0$. And if $g_1(t)g_2(s) > 0$, then $f(ts) \geq g_1(t)g_2(s) - \delta > -\delta$, so $f(ts) > \delta$. In the same way we see that $f(ts) < -\delta$ if and only if $g_1(t)g_2(s) < 0$. These two equivalences are used repeatedly in what follows. Let

$$P_1 = \{t \in \mathbb{R} \setminus \{0\} : g_1(t) > 0\} \text{ and } P_2 = \{t \in \mathbb{R} \setminus \{0\} : g_2(t) > 0\}$$

and let us denote for an arbitrary set P the set $\{-t : t \in P\}$ with $-P$ and the complement of P with P^C. We claim that either

$$P_1 = -P_1 \text{ and } P_2 = -P_2 \quad \text{or} \quad P_1 = -P_1^C \text{ and } P_2 = -P_2^C.$$

If $P_1 = P_2 = \emptyset$, then clearly $P_1 = -P_1$ and $P_2 = -P_2$. Suppose $P_i = \emptyset$ and $P_{3-i} \neq \emptyset$, $i \in \{1, 2\}$. If $p \in \mathbb{R} \setminus \{0\} = P_i^C$, then $r \in P_{3-i}$ is equivalent to $g_i(p)g_{3-i}(r) < 0$, to $f(pr) < -\delta$ and to $g_i(-p)g_{3-i}(-r) < 0$, which is equivalent to $-r \in P_{3-i}$ because $-p \in P_i^C$. So, $P_1 = -P_1$ and $P_2 = -P_2$. We now turn to the case when $P_1 \neq \emptyset$ and $P_2 \neq \emptyset$. If, on one hand, there exists $p \in P_1 \cap -P_1$, then $r \in P_2$ is equivalent to $g_1(-p)g_2(r) > 0$, to $f(-pr) > \delta$, to $g_1(p)g_2(-r) > 0$ and to $-r \in P_2$, so $P_2 = -P_2$. We can apply the same arguments again to obtain $P_1 = -P_1$ because $P_2 \cap -P_2$ is not empty. If, on the other hand, $P_1 \cap -P_1 = \emptyset$, then for any $p \in P_1$ and $r \in P_2$ we have $g_1(p)g_2(r) > 0$, hence $f(pr) > \delta$ and $g_1(-p)g_2(-r) > 0$. Since $-p \in P_1^C$, it follows that $-r \in P_2^C$. Similarly, for $p \in P_1$ and $r \in P_2^C$ we see that $-r \in P_2$, so $P_2 = -P_2^C$ and since $P_2 \cap -P_2 = \emptyset$, we obtain in the same way that $P_1 = -P_1^C$.

Next, we show that

$$P_1 = P_2 \quad \text{or} \quad P_1 = P_2^C. \tag{11}$$

Suppose $1 \in P_1 \cap P_2$. If $r \in P_1$, then $g_1(r)g_2(1) > 0$ and $f(r) > \delta$. It follows that $g_1(1)g_2(r) > 0$ and $r \in P_2$. So, $P_1 \subset P_2$ and, by symmetry, $P_2 \subset P_1$, which implies $P_1 = P_2$. Similarly, if $1 \in P_1^C \cap P_2^C$, then $P_1 = P_2$

and in the same way we prove also the remaining cases where as well as $1 \in P_1 \cap P_2^C$ also $1 \in P_1^C \cap P_2$ implies $P_1 = P_2^C$.

Further, let us prove that

$$P_1 \in \{\mathbb{R} \setminus \{0\},\ \emptyset,\ \mathbb{R}^+,\ \mathbb{R}^-\}. \tag{12}$$

In order to see this, we examine eight possibilities.

1. $P_1 = -P_1$, $1 \in P_1$, $P_1 = P_2$. If $t \in P_1$, then $g_1(t)g_2(t) > 0$, so $f(t^2) > \delta$, again, $g_1(1)g_2(t^2) > 0$ and therefore $t^2 \in P_2 = P_1$ because $1 \in P_1$. And if $t \in P_1^C$, then $f(t^2) > \delta$ and $t^2 \in P_1$. Every positive s is then in P_1 and since $P_1 = -P_1$, it follows that $P_1 = \mathbb{R} \setminus \{0\}$.

2. $P_1 = -P_1$, $1 \in P_1$, $P_1 = P_2^C$. If $t \in P_1$, then $g_1(t)g_2(t) < 0$, so $f(t^2) < -\delta$ and $t^2 \in P_2^C = P_1$. And if $t \in P_1^C$, then again $f(t^2) < -\delta$ and $t^2 \in P_1$. We conclude once more that $P_1 = \mathbb{R} \setminus \{0\}$.

3. and *4.* $P_1 = -P_1$, $1 \in P_1^C$ and $P_1 = P_2$ or $P_1 = P_2^C$. In both cases we see as before that $t \in \mathbb{R} \setminus \{0\}$ implies $t^2 \in P_1^C$. So, $P_1^C = \mathbb{R} \setminus \{0\}$ and $P_1 = \emptyset$.

5. $P_1 = -P_1^C$, $1 \in P_1$, $P_1 = P_2$. Since $t \in \mathbb{R} \setminus \{0\}$ implies $t^2 \in P_1$, every positive s is in P_1 and since $P_1 = -P_1^C$, every negative s is in P_1^C, so $P_1 = \mathbb{R}^+$.

The remaining three cases can be treated in a similar way. Finally, it remains to combine (10), (11), (12) and the facts that $f(st) > \delta$ if and only if $g_1(s)g_2(t) > 0$ and that $f(st) < -\delta$ if and only if $g_1(s)g_2(t) < 0$ in order to obtain (5) or (6) where $\eta_i = \mu_i$, $|c_i| = e_i$ and $\operatorname{sign} c_i = \operatorname{sign} g_i(1)$, $i = 1, 2$. It is also evident from (9) that $\eta_1 = \eta_2$ and $c_1 = c_2$ if $g_1 = g_2$.

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