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On the set for which 1 is univoque

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Abstract. For some integer $K \geq 2$ let $\mathcal{E}_K$ be the set of those $\Theta \in \left(\frac{1}{K+1}, \frac{1}{K}\right)$, for which 1 has only one expansion $1 = e_1\Theta + e_2\Theta^2 + \dots$ with digits $e_j \in \{0, \dots, K\}$, $(j = 1, 2, \dots)$. In this paper we prove, that the Lebesgue measure of the set $\mathcal{E}_K$ is 0.

1. Introduction

Let $K \in \mathbb{N}$, $q \in [K, K+1)$, $\mathcal{A}_K = \{0, 1, \dots, K\}$, $\Theta = \frac{1}{q}$, $L = \frac{K\Theta}{1-\Theta}$ $\left(= \frac{K}{q-1}\right)$. Since

$$[0, L] \subseteq \bigcup_{j=0}^{K} (j\Theta + \Theta[0, L]),$$

therefore each $x \in [0, L]$ has at least one expansion of the form

$$x = a_1\Theta + a_2\Theta^2 + \dots, \qquad a_j \in \mathcal{A}_K \quad (j = 1, 2, \dots). \tag{1}$$

The structure of the so called univoque numbers, i.e. those for which no more than one expansion (1) exist was investigated in the case $K = 1$ by Z. Daróczy and I. Kátai [1], [2], and for $K \geq 2$ by G. Kallós [3], [4]. In [2] it was demonstrated that this set is quite simple for $K = 1$ if 1 has at least two representations of form (1), and that the set of those $\Theta \in \left(\frac{1}{2}, 1\right)$ for which 1 is univoque has Lebesgue measure 0 (and Hausdorff dimension 1).

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Our purpose in this short paper is to prove

Theorem 1. *For each $K \in \mathbb{N}$, the set of those $\Theta \in \left(\frac{1}{K+1}, \frac{1}{K}\right)$, for which 1 is univoque is of Lebesgue measure 0.*

2. Some remarks and definitions

The regular (or Rényi–Parry [5]) expansion of some $x \in [0,1)$ is generated by the iteration of the rule:

$$x = \varepsilon_1(x)\Theta + \Theta x_1, \quad \varepsilon_1(x) = [qx], \quad x_1 = \{qx\}.$$

The quasiregular expansion of some $x \in (0,1]$, $x = \delta_1(x)\Theta + \Theta x_1 = \delta_1(x)\Theta + \delta_2(x)\Theta^2 + \dots$ is defined as follows: $\delta_1(x)$ is the largest integer s for which $x - s\Theta > 0$, $x_1 = qx - s$.

Let $\mathcal{A}_K^{\mathbb{N}}$ be the set of infinite sequences over $\mathcal{A}_K$, and for some $\alpha = a_1a_2\dots$ let $\overline{\alpha} = \overline{a}_1\overline{a}_2\dots$, where $\overline{n} = K - n$ for $n \in \mathcal{A}_K$. For some integer $h > 0$ let $\mathcal{A}_K^h$ be the set of sequences of length h over $\mathcal{A}_K$. The shift operator $\sigma : \mathcal{A}_K^{\mathbb{N}} \to \mathcal{A}_K^{\mathbb{N}}$ is defined as usual by $\sigma(a_1a_2\dots) = a_2a_3\dots$. Let $<$ denote the lexicographic ordering in $\mathcal{A}_K^{\mathbb{N}}$. Furthermore let $\underline{\Theta} = (\Theta, \Theta^2, \dots)$, and for $\underline{\ell} = (\ell_1, \ell_2, \dots) \in \mathcal{A}_K^{\mathbb{N}}$ let $\langle \underline{\ell}, \underline{\Theta} \rangle = \ell_1\Theta + \ell_2\Theta^2 + \dots$.

For some $\Theta \in \left(\frac{1}{K+1}, \frac{1}{K}\right)$ let $\underline{t} = \underline{t}(\Theta) = t_1t_2\dots$ be the sequence of the digits in the quasiregular expansion of 1. Let

$$\mathcal{F}_K = \left\{ \underline{t}(\Theta) \;\middle|\; \Theta \in \left(\frac{1}{K+1}, \frac{1}{K}\right) \right\}.$$

According to a theorem due to W. Parry [5], $\underline{t} \in \mathcal{F}_K$ if and only if

$$\sigma^j(\underline{t}) < \underline{t} \qquad (j = 1, 2, \dots), \quad t_1 = K$$

holds. Furthermore, for a fixed Θ the sequence $\underline{e} \in \mathcal{A}_K^{\mathbb{N}}$ is the regular expansion of some $x \in [0,1)$, if and only if

$$\sigma^j(\underline{e}) < \underline{t} \qquad (j = 0, 1, 2, \dots).$$

Let $\mathcal{F}_K^{(u)} \subseteq \mathcal{F}_K$ be the set of those sequences $\underline{t}$, for which 1 is univoque with respect to Θ, $1 = \langle \underline{t}, \underline{\Theta} \rangle$, and let $\mathcal{E}_K$ be the set of those $\Theta \in \left(\frac{1}{K+1}, \frac{1}{K}\right)$, for which 1 is univoque, i.e.

$$\mathcal{E}_K = \left\{ \Theta \;\middle|\; \langle \underline{t}, \underline{\Theta} \rangle = 1,\ \underline{t} \in \mathcal{F}_K^{(u)} \right\}.$$

We should prove that $\lambda(\mathcal{E}_K) = 0$ (λ is the Lebesgue measure). Since the case $K = 1$ was treated in [2], from now on we assume that $K \geq 2$.

A trivial but important observation is that for some fixed Θ the number x is univoque, if and only if $L - x$ is univoque. Since $L < 2$, therefore $0 < L - 1 < 1$. If $1 = t_1\Theta + t_2\Theta^2 + \dots$, then $L - 1 = \bar{t}_1\Theta + \bar{t}_2\Theta^2 + \dots$. 1 is univoque with respect to Θ if the expansion $\bar{t}_1\bar{t}_2\dots$ of $L - 1$ is the quasiregular expansion of it. Thus $\underline{t} \in \mathcal{F}_K^{(u)}$ if and only if

$$\bar{\underline{t}} < \sigma^j(\underline{t}) < \underline{t} \quad (j = 1, 2, \dots),\ t_1 = K$$

holds (clearly $\bar{\underline{t}} < \underline{t}$).

W. Parry proved that, if $\underline{t}^1, \underline{t}^2 \in \mathcal{F}_K$, $\langle \underline{t}^1, \underline{\Theta_1} \rangle = 1$, $\langle \underline{t}^2, \underline{\Theta_2} \rangle = 1$, then $\underline{t}^1 < \underline{t}^2$ implies that $\Theta_1 > \Theta_2$.

3. A useful lemma

Lemma 1. *Let $u \geq 1$, $t_1, t_2, \dots, t_u \in \mathcal{A}_K$, and $B_K(t_1, t_2, \dots, t_u)$ be the set of those Θ-s, for which 1 is univoque with respect to Θ, with first u digits $t_1, t_2, \dots, t_u$. Then $B_K(t_1, t_2, \dots, t_u) \subseteq [\alpha_u, \beta_u]$, where α_u is the positive solution of*

$$1 = (t_1 y + \dots + t_u y^u)\frac{1}{1 - y^u}$$

and β_u is the positive solution of the polynomial $1 = t_1 x + \dots + t_u x^u$. Consequently, $\beta_u - \alpha_u \leq \frac{\alpha_u^u}{K}$.

Proof of Lemma 1. Let $\Theta \in B_K(t_1, t_2, \dots, t_u)$, $1 = t_1\Theta + \dots + t_u\Theta^u + \dots$. Then

$$t_1 \dots t_u 0^\infty \leq \underline{t} \leq (t_1 \dots t_u)^\infty,$$

and for α_u we get

$$1 = (t_1 y + \dots + t_u y^u)(1 + y^u + y^{2u} + \dots) = (t_1 y + \dots + t_u y^u)\frac{1}{1 - y^u},$$

thus the first assertion holds. Let

$$\phi_1(x) = t_1 x + \dots + t_u x^u - 1$$

$$\phi_2(x) = t_1 x + \dots + t_{u-1} x^{u-1} + (t_u + 1)x^u - 1,$$

$\Delta = \beta_u - \alpha_u$. We have $0 = \phi_1(\beta_u) = \phi_2(\alpha_u)$. From the Taylor expansion

$$0 = \phi_1(\alpha_u + \Delta) = \phi_1(\alpha_u) + \Delta\phi_1'(\alpha_u) + \frac{\Delta^2}{2}\phi_1''(\alpha_u) + \dots,$$

with $\phi_1(\alpha_u) = -\alpha_u^u$ we obtain that

$$\alpha_u^u = \Delta\phi_1'(\alpha_u) + \frac{\Delta^2}{2}\phi_1''(\alpha_u) + \dots.$$

Since the derivates $\phi_1^{(\mu)}(\alpha_u)$ are nonnegative ($\mu = 1, 2, \dots$), we get

$$\Delta \le \frac{\alpha_u^u}{t_u u \alpha_u^{u-1} + \dots + t_1} \le \frac{\alpha_u^u}{K}.$$

The proof is completed. □

4. Proof of Theorem 1

Let $r \ge 1$, $0 \le s < K$, $T = K^r s$ $(\in \mathcal{A}_K^{r+1})$. Let $\mathcal{B}_N(T)$ be the set of those sequences $s_1 s_2 \dots s_N \in \mathcal{A}_K^N$, for which

$$\overline{T} \le s_{i+1} \dots s_{i+(r+1)} \le T \qquad (i = 0, 1, \dots, N-r-1), \tag{2}$$

$$0^{p+1} \le s_{N-p} \dots s_N \le K^{p+1} \quad (p = 0, 1, \dots, r) \tag{3}$$

holds. For some α of length $M < N$ let $\mathcal{B}_N(T \mid \alpha)$ be the subset of those elements of $\mathcal{B}_N(T)$ for which additionally $s_1 s_2 \dots s_M = \alpha$ holds. Let $\mathcal{D}_N = \mathcal{D}_N(T)$, $M_N(\alpha)$ be the size of $\mathcal{B}_N(T)$, $\mathcal{B}_N(T \mid \alpha)$, respectively.

We shall give an upper estimate for $\mathcal{D}_N$.

Assume that $N \ge r + 2$. Then

$$\mathcal{D}_N = M_N(0) + M_N(K) + \sum_{j=1}^{K-1} M_N(j).$$

It is clear that $M_N(j) = \mathcal{D}_{N-1}$, $(j = 1, \dots, K-1)$. If the conditions (2), (3) hold then they remain valid for $\overline{s_1 s_2} \dots \overline{s_N}$ as well. Thus $M_N(0) = M_N(K)$, and so

$$\mathcal{D}_N - (K-1)\mathcal{D}_{N-1} = 2M_N(K). \tag{4}$$

Similarly, if $1 \le h \le r$, then

$$\begin{aligned} M_N(K^h) &= M_N(K^h0) + M_N(K^{h+1}) + \sum_{j=1}^{K-1} M_N(K^h j) \\ &= M_{N-h}(K) + M_N(K^{h+1}) + (K-1)\mathcal{D}_{N-(h+1)}, \end{aligned}$$

and so

$$\sum_{h=1}^{r-1} \big(M_N(K^h) - M_N(K^{h+1})\big) = \sum_{h=1}^{r-1} M_{N-h}(K) + (K-1)\sum_{\nu=2}^{r} \mathcal{D}_{N-\nu}.$$

The left hand side is $M_N(K) - M_N(K^r)$.

Other hand, if $s > 0$ then

$$M_N(K^r) = M_N(K^r0) + \sum_{j=1}^{s} M_N(K^r j) = M_{N-r}(K) + s\mathcal{D}_{N-(r+1)},$$

and so

$$M_N(K) - M_{N-r}(K) - s\mathcal{D}_{N-(r+1)} = \sum_{h=1}^{r-1} M_{N-h}(K) + (K-1)\sum_{\nu=2}^{r} \mathcal{D}_{N-\nu}.$$

Hence, by (4), substituting $2M_N(K)$ and $2M_{N-h}(K)$

$$\begin{aligned} \mathcal{D}_N - (K-1)\mathcal{D}_{N-1} - \sum_{h=1}^{r} \Big(\mathcal{D}_{N-h} - (K-1)\mathcal{D}_{N-(h+1)}\Big) \\ = 2(K-1)\sum_{\nu=2}^{r} \mathcal{D}_{N-\nu} + 2s\mathcal{D}_{N-(r+1)}, \end{aligned}$$

Consequently

$$\mathcal{D}_N - \left(\sum_{\mu=1}^{r} K\mathcal{D}_{N-\mu} + s\mathcal{D}_{N-(r+1)}\right) + (K-1-s)\mathcal{D}_{N-(r+1)} = 0,$$

and so

$$\mathcal{D}_N \le \sum_{\mu=1}^{r} K\mathcal{D}_{N-\mu} + s\mathcal{D}_{N-(r+1)}.$$

Let the sequence X_j be defined by the equation $X_j = \mathcal{D}_j$, $(j = 1, \dots, r{+}1)$, and let

$$X_n = \sum_{\mu=1}^{r} K X_{n-\mu} + s X_{n-(r+1)}, \qquad (n = r+2, \dots, N). \tag{5}$$

Then $\mathcal{D}_j \le X_j$, $(j = r+2, \dots, N)$. Let

$$\phi(x) = x^{r+1} - K(x^r + \dots + x) - s$$

be the characteristic polynomial of the difference equation (5), and

$$\lambda(x) := x^{r+1} \phi\left(\frac{1}{x}\right) = 1 - K(x + \dots + x^r) - s x^{r+1}. \tag{6}$$

Let η be the positive root of $\lambda(x)$. This is a simple root, and no other roots do exist in the disc $|x| \le \eta$. Thus

$$\mathcal{D}_N \le X_N < c \left(\frac{1}{\eta}\right)^N, \tag{7}$$

with a suitable positive constant c, which may depend only on r and s.

In the case $s = 0$, similarly as above, we deduce the recursion

$$\mathcal{D}_N - K \sum_{\nu=1}^{r} \mathcal{D}_{N-\nu} + (K-1)\mathcal{D}_{N-(r+1)} = 0,$$

whence

$$\mathcal{D}_N \le K \sum_{\nu=1}^{r} \mathcal{D}_{N-\nu}$$

follows. Let $X_j = \mathcal{D}_j$ $(j = 1, \dots, r)$, and

$$X_n := K \sum_{\nu=1}^{r} X_{n-\nu}, \qquad (n = r+1, \dots, N). \tag{8}$$

As earlier, we have $\mathcal{D}_j \le X_j$, $(j = r+1, \dots, N)$. Let

$$\psi(x) = x^r - K(x^{r-1} + \dots + 1)$$

be the characteristic polynomial of (8), and

$$\beta(x) := x^r \psi\left(\frac{1}{x}\right) = 1 - K(x + \cdots + x^r).$$

Let η^* be its positive root. Then, as above, we obtain that

$$\mathcal{D}_N \le X_N \le c\left(\frac{1}{\eta^*}\right)^N,$$

with some positive constant $c = c(r)$.

To finish the proof we shall prove that for every $r \ge 1$ and $0 \le s < K$, $\lambda(B_K(K^r s)) = 0$. Here

$$B_K(K^r s) = B_K(\underbrace{K \dots K}_{r} s)$$

is defined in Lemma 1. If $K^r s s_1 s_2 \dots \in \mathcal{F}_K^{(u)}$ then (2) and (3) hold.

Assume that $s \ge 1$. We have

$$B_K(K^r s) = \sum_{s_1,\dots,s_N} B_K(K^r s s_1 s_2 \dots s_N),$$

where on the right hand sum we sum only over those $s_1, \dots, s_N$, for which (2), (3) hold. From Lemma 1 it follows that each summand on the right hand side can be covered by an interval $[\alpha_{N+(r+1)}, \beta_{N+(r+1)}]$, the length of which is $\beta_{N+(r+1)} - \alpha_{N+(r+1)} \le \alpha_{N+(r+1)}^{N+(r+1)}$. If (2) holds, then among $s_1 \dots s_{r+1}$ there exists a nonzero element, consequently $\alpha_{N+(r+1)}$ is smaller than ω, where ω is the positive root of the equation

$$1 = K(x + \cdots + x^r) + s x^{r+1} + 1 \cdot x^{2r+1}. \tag{9}$$

It is obvious that $\omega < \eta$ (see (6)). Thus, from (7),

$$\lambda(B_K(K^r s)) \le c\left(\frac{1}{\eta}\right)^N \omega^{N+r+1},$$

which by $N \to \infty$ implies that $\lambda(B_K(K^r s)) = 0$.

In the case $s = 0$ the argument is similar. We can observe only that the positive root ω of the equation (9) with $s = 0$ is less than η^*. Hence we obtain that $\lambda(B_K(K^r s)) = 0$. Since $\mathcal{E}_K = \sum_{r=1}^{\infty} \sum_{s=0}^{K-1} B_K(K^r s)$, therefore $\lambda(\mathcal{E}_K) = 0$.

The proof of the theorem is completed. □

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