Publ. Math. Debrecen
58 / 4 (2001), 751–778

Distribution functions of ratio sequences

By OTO STRAUCH (Bratislava) and JÁNOS T. TÓTH (Ostrava)

Dedicated to Prof. Kálmán Győry on his 60th birthday

Abstract. The paper deals with a block sequence defined by

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

where x_n is an increasing sequence of positive integers. We discuss the set of all distribution functions of such sequences.

1. Introduction

For every $n = 1, 2, \dots$, let

$$X_n = (x_{n,1}, \dots, x_{n,N_n})$$

be a finite sequence in $[0,1]$. The infinite sequence

$$\omega = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2} \dots),$$

abbreviated as $\omega = (X_n)_{n=1}^{\infty}$, will be called *a block sequence* associated with the sequence of single blocks X_n, $n = 1, 2, \dots$. We will distinguish

Mathematics Subject Classification: 11K31.
Key words and phrases: block sequences, ratio sequences, distribution functions, asymptotic density.
This research was supported by the Slovak Academy of Sciences Grant 5123 and the GAAV A 1187 101.

between block sequences and sequences of individual blocks. These sequences are the main tool for constructing sequences with given distribution properties. In the literature various types were published.

I. J. Schoenberg [14] introduced and studied the asymptotic distribution function (abbreviating a.d.f.) of X_n with $N_n = n$. For the definition see Section 2. He gave some criteria and mentioned a result of G. Pólya that

$$X_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1$$

has a.d.f. $g(x) = \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}\,t$. E. Hlawka in the monograph [4, p. 57–60], called sequences of single blocks X_n, for $N_n = n$, *double sequences* and, for general N_n, N_n*-double sequences*. As examples he included a proof of uniform distribution (abbreviating u.d.) for

$$X_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right), \quad \text{and} \quad X_n = \left(\frac{1}{n}, \frac{a_2}{n}, \dots, \frac{a_{\phi(n)}}{n}\right),$$

where $a_1 = 1 < a_2 < \dots < a_{\phi(n)}$, $\gcd(a_i, n) = 1$ and $\phi(n)$ denotes Euler's function. U.d. for related block sequences $\omega = (X_n)_{n=1}^{\infty}$ is given in the monograph of L. Kuipers and H. Niederreiter [6, Lemma 4.1, Example 4.1, p. 136]. G. Myerson [8, p. 172] called a sequence of blocks X_n (without any ordering in X_n) a sequence of sets. The same terminology is used by H. Niederreiter in his book [9]. Myerson called the associated block sequence ω (X_n with some order) an *underlying sequence* and established criteria for u.d. of X_n. The sequence of single blocks X_n with $N_n = n$ is also called a *triangular array*. R. F. Tichy [20] gave some examples of u.d. of such X_n.

Let x_n be an increasing sequence of positive integers. Extending a result of S. Knapowski [5], Š. Porubsky, T. Šalát and O. Strauch [10] have investigated a sequence of blocks X_n of the type

$$X_n = \left(\frac{1}{x_n}, \frac{2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

They obtained a complete theory for the uniform distribution of the related block sequence $\omega = (X_n)_{n=1}^{\infty}$.

In this paper we will consider a sequence of blocks X_n, $n = 1, 2, \dots$, with blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

The associated block sequence $\omega = (X_n)_{n=1}^{\infty}$ denoted as x_m/x_n, $m = 1, \dots, n$, $n = 1, 2, \dots$, is also called *the ratio sequence* of x_n. Everywhere density of x_m/x_n was first investigated by T. Šalát [12]. The set of limit points of x_m/x_n is characterized in J. Bukor and J. T. Tóth [2]. In [19] the authors prove that the lower asymptotic density of x_n greater than 1/2 implies everywhere density of x_m/x_n in $[0, \infty)$.

The main object of study is the set $G(X_n)$ of all distribution functions of the sequence of single blocks X_n. Occasionally, we also consider the set $G(\omega)$ of all distribution functions of the block sequence $\omega = (X_n)_{n=1}^{\infty}$. The study of the set of distribution functions of a given point-sequence was initiated by J. G. van der Corput [21]. He has given a characterization for this set which was formulated by R. Winkler [22] in the following general setting: It is nonempty, closed and connected in the weak topology, and consists either of one or of infinitely many functions. Further results can be found in [18].

In this paper we prove that if $g(x) \in G(X_n)$, then $\alpha g(x\beta) \in G(X_n)$ for some α and β. This shows that everywhere continuity on $(0, 1]$ for all $g \in G(X_n)$ follows from continuity at 1. We derive a limit which implies connectivity of $G(X_n)$, and also find a region that contains all graphs of $g \in G(X_n)$ assuming positive lower asymptotic density of x_n. We give the upper bound 1/2 for the minimum of the first moment of $g \in G(X_n)$ and we characterize a singleton $G(X_n) = \{g(x)\}$ as $g(x) = x^{\lambda}$, where $0 \le \lambda \le 1$. For a singleton $G(X_n)$ we have $G(\omega) = G(X_n)$. As an application, we solve a problem of S. Róka [11]. The paper concludes with some examples containing an explicit formula for $G(X_n)$.

2. Basic definitions

2.1. A function $g : [0, 1] \to [0, 1]$ will be called *a distribution function* (abbreviated d.f.) if the following conditions are satisfied:

(i) $g(0) = 0$, $g(1) = 1$, and

(ii) g is nondecreasing.

We will identify any two distribution functions $g, \tilde{g}$ coinciding at common points of continuity, or equivalently, if $g(x) = \tilde{g}(x)$ a.e. on $[0, 1]$.

2.2. A point $\beta \in [0, 1]$ is called a point of *increase* (or a point of the spectrum) of d.f. $g(x)$ if $g(x) > g(\beta)$ for every $x > \beta$ or $g(x) < g(\beta)$ for every $x < \beta$, $x \in [0, 1]$.

2.3. Let y_n, $n = 1, 2, \dots$, be an infinite sequence of points in $[0, 1]$. For an initial segment $y_1, y_2, \dots, y_N$ define a *step* d.f. $F_N(x)$ as

$$F_N(x) = \frac{\#\{n \le N; y_n < x\}}{N}$$

for $x \in [0, 1)$, and $F_N(1) = 1$.

2.4. A d.f. g is a d.f. of the sequence y_n if there exists an increasing sequence of positive integers $N_1, N_2, \dots$ such that

$$\lim_{k \to \infty} F_{N_k}(x) = g(x)$$

a.e. on $[0, 1]$.

2.5. Denote by $G(y_n)$ the set of all d.f. of the sequence y_n. We call $G(y_n)$ *the distribution of* y_n.

2.6. D.f. $\underline{g}(x)$ is the *lower* d.f. of y_n if $\underline{g}(x) = \inf_{g \in G(y_n)} g(x)$ and d.f. $\overline{g}(x)$ is the *upper* d.f. of y_n if $\overline{g}(x) = \sup_{g \in G(y_n)} g(x)$. These $\underline{g}(x)$ and $\overline{g}(x)$ need not be elements of $G(y_n)$.

2.7. If $G(y_n) = \{g\}$, then the sequence y_n is said to have the *asymptotic distribution function* (for short a.d.f.) $g(x)$. In this case

$$\lim_{N \to \infty} F_N(x) = g(x)$$

a.e. on $[0, 1]$. If $g(x) = x$, then the sequence y_n is said to be *uniformly distributed* (abbreviated u.d.) in $[0, 1]$.

2.8. Now, let y_n be a block sequence $\omega = (X_n)_{n=1}^{\infty}$ composed of blocks $X_n = (x_{n,1}, \dots, x_{n,N_n})$. The set $G(\omega)$ of all d.f. of ω is defined in 2.5.

2.9. For the block sequence ω we can use not only the step d.f. $F_N(x)$ defined in 2.3, but also a step d.f. $F(X_n, x)$ associated with the sequence of individual blocks X_n, defined as

$$F(X_n, x) = \frac{\#\{i \le N_n;\ x_{n,i} < x\}}{N_n}$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$.

2.10. A d.f. g is a d.f. of the sequence of single blocks X_n, if there exists an increasing sequence of positive integers $n_1, n_2, \dots$ such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

a.e. on $[0, 1]$.

2.11. Denote by $G(X_n)$ the set of all d.f. of the sequence of single blocks X_n.

2.12. If $G(X_n) = \{g\}$, we say that $g(x)$ is the a.d.f. of X_n. If $g(x) = x$, the sequence of blocks X_n is said to be u.d. in $[0, 1]$, i.e. X_n is u.d. if

$$\lim_{n\to\infty} F(X_n, x) = x$$

for all $x \in [0, 1]$.

2.13. In what follows, let x_n be an increasing sequence of positive integers and let

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

The block sequence $\omega = (X_n)_{n=1}^{\infty}$ composed of blocks X_n is denoted by x_m/x_n, $m = 1, \dots, n$, $n = 1, 2, \dots$ and is called, as above, the ratio sequence of x_n. In this case write $G(x_m/x_n)$ instead of $G(\omega)$.

2.14. For x_n we define a counting function $A(t)$ as

$$A(t) = \#\{n \in \mathbb{N}; x_n < t\},$$

further the lower asymptotic density $\underline{d}$ and the upper asymptotic density $\overline{d}$ are defined as

$$\underline{d} = \liminf_{t\to\infty} \frac{A(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n}, \quad \overline{d} = \limsup_{t\to\infty} \frac{A(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

2.15. We will specify the following type of d.f.:
one-jump d.f. $c_u : [0, 1] \to [0, 1]$, where $c_u(0) = 0$, $c_u(1) = 1$, and

$$c_u(x) = \begin{cases} 0, & \text{if } x \in [0, u), \\ 1, & \text{if } x \in (u, 1). \end{cases}$$

2.16. We will abbreviate the interval (xz, yz) as $(x, y)z$ and $(x/z, y/z)$ as $(x, y)/z$.

3. Basic properties of $G(X_n)$

Using

$$x_i < x x_m \Longleftrightarrow x_i < \left(x \frac{x_m}{x_n} \right) x_n$$

and that these inequalities imply $i < m$, it directly follows from Definition 2.9 of $F(X_n, x)$ that

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n} \right), \tag{1}$$

for every $m \le n$ and $x \in [0,1)$. Using the Helly selection principle from the sequence (m,n) we can select a subsequence (m_k, n_k) such that $F(X_{n_k}) \to g(x)$, $F(X_{m_k}) \to \tilde{g}(x)$ as $k \to \infty$, furthermore $x_{m_k}/x_{n_k} \to \beta$ and $m_k/n_k \to \alpha$, but α may be infinity. These limits have the following connection.

Proposition 3.1. *Let m_k and n_k be two increasing integer sequences satisfying $m_k \le n_k$, for $k = 1, 2, \dots$ and assume that*

(i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ *a.e.*,
(ii) $\lim_{k\to\infty} F(X_{m_k}, x) = \tilde{g}(x)$ *a.e.*,
(iii) $\lim_{k\to\infty} \frac{x_{m_k}}{x_{n_k}} = \beta > 0$,
(iv) $g(\beta - 0) > 0$.

Then there exists $\lim_{k\to\infty} \frac{n_k}{m_k} = \alpha < \infty$ such that

$$\tilde{g}(x) = \alpha g(x\beta) \text{ a.e. on } [0,1], \text{ and } \alpha = \frac{\tilde{g}(1-0)}{g(\beta-0)}. \tag{2}$$

Proof. Firstly we prove

$$\lim_{k\to\infty} F\left(X_{n_k}, x\frac{x_{m_k}}{x_{n_k}} \right) = g(x\beta). \tag{3}$$

Denoting $\beta_k = x_{m_k}/x_{n_k}$ and substituting $u = x\beta_k$, we find

$$0 \le \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2 \,\mathrm{d}\, x = \frac{1}{\beta_k}\int_0^{\beta_k} (F(X_{n_k}, u) - g(u))^2 \,\mathrm{d}\, u$$
$$\le \frac{1}{\beta_k}\int_0^1 (F(X_{n_k}, u) - g(u))^2 \,\mathrm{d}\, u \to 0,$$

which leads to $(F(X_{n_k}, x\beta_k) - g(x\beta_k)) \to 0$ a.e. as $k \to \infty$ (here necessarily $\beta > 0$). Furthermore,

$$\begin{aligned}\int_0^1 &(F(X_{n_k}, x\beta_k) - g(x\beta))^2 \,\mathrm{d}\, x \\ &= \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k) + g(x\beta_k) - g(x\beta))^2 \,\mathrm{d}\, x \\ &\le 2\left(\int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2 \,\mathrm{d}\, x + \int_0^1 (g(x\beta_k) - g(x\beta))^2 \,\mathrm{d}\, x\right).\end{aligned}$$

Since $g(x)$ is continuous a.e. on $[0,1]$ then $(g(x\beta_k) - g(x\beta)) \to 0$ a.e. and applying the Lebesgue theorem of dominant convergence we find $\int_0^1 (g(x\beta_k) - g(x\beta))^2 \,\mathrm{d}\, x \to 0$. This gives (3). The existence of the limit $\lim_{k\to\infty} \frac{n_k}{m_k} = \alpha < \infty$ follows from (1) and (iv). Now, let $t_n \in [0,1)$ increases to 1 and $\tilde{g}(x)$ be continuous in t_n. Then $g(x\beta)$ is also continuous in t_n and $\tilde{g}(t_n) = \alpha g(t_n\beta)$ for $n = 1, 2, \ldots$. The limit of this equation gives the desired form of α. □

The relation $\tilde{g}(x) = \alpha g(x\beta)$ will be written as $\alpha g(x\beta) \in G(X_n)$. Proposition 3.1 gives the following properties of $G(X_n)$ which are formulated without the knowledge of limit points of x_m/x_n.

Theorem 3.2. *For every $g \in G(X_n)$ and every $\beta \in (0,1]$ the following holds.*

(I) *If $g(x)$ increases and is continuous in β with $g(\beta) > 0$, then there exists $1 \le \alpha < \infty$ such that*

$$\alpha g(x\beta) \in G(X_n). \tag{4}$$

(II) *If β is a point of discontinuity of $g(x)$ with $g(\beta+0) - g(\beta-0) = h > 0$, then there exists a closed interval $I \subset [0,1]$, with length $|I| \ge h$ such that for every $\frac{1}{\alpha} \in I$ we have (4).*

Proof. (I) Let $F(X_{n_k}, x) \to g(x)$ a.e. as $k \to \infty$. If $g(x)$ increases in β, then β is an accumulation point of x_m/x_{n_k}, $m = 1, 2, \ldots, n_k$, $k = 1, 2, \ldots$ and thus we can find m_k such that $x_{m_k}/x_{n_k} \to \beta$. Helly's theorem gives that from (m_k, n_k) we can select (m'_k, n'_k) such that $F(X_{m'_k}, x) \to \tilde{g}(x)$ a.e. as $k \to \infty$. This and $g(\beta) > 0$ gives $n'_k/m'_k \to \alpha < \infty$.

(II) Let $F(X_{n_k}, x) \to g(x)$ a.e. and $\varepsilon_k \to 0$, $\varepsilon_k > 0$, as $k \to \infty$. Let $\beta_l < \beta < \beta'_l$, $(\beta'_l - \beta_l) \to 0$ as $l \to \infty$ and every β_l, β'_l are continuity points of $g(x)$. Denote by $\underline{m}(n_k) = \min i$ and $\overline{m}(n_k) = \max i$ for i satisfying $\frac{x_i}{x_{n_k}} \in (\beta_l, \beta_l')$. For a given ε_k we find sufficiently large l such that $|g(\beta_l') - g(\beta_l) - h| < \frac{\varepsilon_k}{2}$. From n_k we select n_k' such that $\left|\frac{\overline{m}(n_k') - \underline{m}(n_k')}{n_k'} - (g(\beta_l') - g(\beta_l))\right| < \frac{\varepsilon_k}{2}$. This gives $\left|\frac{\overline{m}(n_k') - \underline{m}(n_k')}{n_k'} - h\right| < \varepsilon_k$ for every k. Thus we see that the interval $\left[\frac{\underline{m}(n_k')}{n_k'}, \frac{\overline{m}(n_k')}{n_k'}\right] \subset [0,1]$ has length sufficiently close to h and may vary in $[0,1]$. We then select n_k'' from n_k' such that

$$\left[\frac{\underline{m}(n_k'')}{n_k''}, \frac{\overline{m}(n_k'')}{n_k''}\right] \to [\gamma, \delta].$$

We see that $\delta - \gamma = h$ and for every $\frac{1}{\alpha} \in [\gamma, \delta]$ there exists $m_k'' \in [\underline{m}(n_k''), \overline{m}(n_k'')]$ such that $\frac{m_k''}{n_k''} \to \frac{1}{\alpha}$ as $k \to \infty$. Putting $m = m_k''$ and $n = n_k''$ in (1) and by taking limits we find (4). $\square$

For some $G(X_n)$ all d.f. are of the form (4), more precisely:

Theorem 3.3. *Assume that all d.f. in $G(X_n)$ are continuous at 0. Then $c_1(x) \notin G(X_n)$ and for every $\tilde{g} \in G(X_n)$ and every $1 \le \alpha < \infty$ there exists $g \in G(X_n)$ and $0 < \beta \le 1$ such that*

$$\tilde{g}(x) = \alpha g(x\beta) \text{ a.e.}$$

Proof. Let $F(X_{m_k}, x) \to \tilde{g}(x)$ a.e. as $k \to \infty$. For any $\alpha \in [1, \infty)$ we can find a sequence n_k such that $m_k < n_k$ and $n_k/m_k \to \alpha$. From (m_k, n_k) we select a subsequence (m_k', n_k') such that $F(X_{n_k'}, x) \to g(x)$ a.e. and then a subsequence (m_k'', n_k'') of (m_k', n_k') such that $\frac{x_{m_k''}}{x_{n_k''}} \to \beta$. Since $F\big(X_{n_k''}, x\frac{x_{m_k''}}{x_{n_k''}}\big)$ $k = 1, 2, \dots$ is the sequence of d.f., then we can select (m_k''', n_k''') from (m_k'', n_k'') such that

$$F\left(X_{n_k'''}, x\frac{x_{m_k'''}}{x_{n_k'''}}\right) \to h(x).$$

The limit of (1) gives $\tilde{g}(x) = \alpha h(x)$. Furthermore, for every $0 < \beta' \le 1$ we have $F(X_{n_k'''}, x\beta') \to g(x\beta')$ and since

$$F\left(X_{n_k'''}, x\frac{x_{m_k'''}}{x_{n_k'''}}\right) \le F(X_{n_k'''}, x\beta')$$

(for sufficiently large k) we find $h(x) \leq g(x\beta')$. Hence, assuming that $\beta = 0$ and from continuity of $g(x)$ at 0 we have $h(x) = 0$ which gives $\tilde{g}(x) = c_1(x)$, a contradiction. Thus, $\beta > 0$ and we can use limit (3)

$$F\left(X_{n_k'''}, x\frac{x_{m_k'''}}{x_{n_k'''}}\right) \to g(x\beta)$$

which gives $\tilde{g}(x) = \alpha g(x\beta)$ a.e. on $[0,1]$. □

Using a mapping $[0,1]^2 \to [0,1]^2$ defined by $(x,y) \to (x/\beta, y\alpha)$ (as in [15, Figure 4), the graph of $\alpha g(x\beta)$ can viewed as a linear expansion of the graph of the restricted function $g \mid [0,\beta)$ by the following figure

Figure 1.

We shall write this mapping as $g \mid [0,\beta) \to \alpha g(x\beta)$.

4. Continuity of $g \in G(X_n)$

If all $g \in G(X_n)$ are everywhere continuous on $[0,1]$, then relation (4) is of the form

$$\frac{g(x\beta)}{g(\beta)} \in G(X_n). \tag{5}$$

As a criterion for continuity of all $g \in G(X_n)$ we can adapt the Wiener-Schoenberg theorem (cf. [6, p. 55]), but here we give the following simple sufficient condition.

Theorem 4.1. *Assume that all d.f. in $G(X_n)$ are continuous at 1. Then all d.f. in $G(X_n)$ are continuous on $(0,1]$, i.e. the only discontinuity point may be 0.*

Proof. Assume that $x_{m_k}/x_{n_k} \to \beta$ and $F(X_{n_k}, x) \to g(x)$ as $k \to \infty$. If from (m_k, n_k) we can select two sequences (m'_k, n'_k) and (m''_k, n''_k) such that $n'_k/m'_k \to \alpha_1$ and $n''_k/m''_k \to \alpha_2$ with a finite $\alpha_1 \neq \alpha_2$, then $\alpha_1 g(x\beta), \alpha_2 g(x\beta) \in G(X_n)$ and thus one of such d.f. $\tilde{g}(x)$ must be discontinuous at 1 (it holds also for g continuous at β). Thus, assuming that $G(X_n)$ has only continuous d.f. at 1, the limits $x_{m_k}/x_{n_k} \to \beta > 0$ and $F(X_{n_k}, x) \to g(x)$ imply the convergence of n_k/m_k. Thus, by Theorem 3.2, $g(x)$ cannot have a discontinuity point in $(0,1]$. □

There is a simple criterion for continuity at 1.

Proposition 4.2. *Denote*

$$\overline{d}(\varepsilon) = \limsup_{n\to\infty} \frac{\#\{i \le n; (1-\varepsilon)x_n < x_i < x_n\}}{n}.$$

Every $g \in G(X_n)$ is continuous at 1 if and only if

$$\lim_{\varepsilon\to 0} \overline{d}(\varepsilon) = 0.$$

Proof. By Definition 2.9, $\overline{d}(\varepsilon) = \limsup_{n\to\infty}(1 - F(X_n, 1-\varepsilon))$. Assume that $F(X_{n_k}, x) \to g(x)$ a.e. and $g(x)$ has a jump of size h in 1, i.e. $1 - g(1-\varepsilon) \ge h$ for every $\varepsilon > 0$. This implies

$$1 - F(X_{n_k}, 1-\varepsilon) > h - \varepsilon_0 \tag{6}$$

for arbitrary $\varepsilon_0 > 0$ and $k \ge k_0$ and thus $\overline{d}(\varepsilon) \ge h - \varepsilon_0$.

On the other hand, if $\limsup_{\varepsilon\to 0} \overline{d}(\varepsilon) = h > 0$, then for every $\varepsilon_k > 0$, $\varepsilon_k \to 0$, we have $\overline{d}(\varepsilon_k) \ge h$. We can select n_k such that $1 - F(X_{n_k}, 1-\varepsilon_k) > h - \varepsilon_0$ for every k which implies (6) for every $\varepsilon > 0$ and $k \ge k_0$. Then choose n'_k in n_k such that $F(X_{n'_k}, x) \to g(x)$, where $g(x)$ has a jump at 1. □

Similar condition holds for continuity at 0.

5. Connectivity of $G(X_n)$

As we have mentioned in the introduction, for a usual sequence y_n the set $G(y_n)$ of all d.f. of y_n is nonempty, closed and connected in the weak topology, and consists either of one or infinitely many functions. The closedness of $G(X_n)$ is clear, but connectivity of $G(X_n)$ is open. A general block sequence Y_n with non-connected $G(Y_n)$ can be found trivially. For our special X_n we have only the following sufficient condition.

Theorem 5.1. *If*

$$\lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{i=1}^{n+1}\sum_{j=1}^{n}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_n}\right| - \frac{1}{2(n+1)^2}\sum_{i,j=1}^{n+1}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_{n+1}}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|\right)=0, \tag{7}$$

then $G(X_n)$ is connected in the weak topology.

Proof. The connection follows from the limit

$$\lim_{n\to\infty}\int_0^1 (F(X_{n+1},x)-F(X_n,x))^2\,\mathrm{d}\,x=0,$$

since by a theorem of H. G. Barone [1] if t_n is a sequence in a metric space (X,ρ) satisfying
(i) any subsequence of t_n contains a convergent subsequence and
(ii) $\lim_{n\to\infty}\rho(t_n,t_{n+1})=0$,
then the set of all limit points of t_n is connected. Next we use the expression

$$\int_0^1 (g(x)-\tilde{g}(x))^2\,\mathrm{d}\,x=\int_0^1\int_0^1|x-y|\,\mathrm{d}\,g(x)\,\mathrm{d}\,\tilde{g}(y) -\frac{1}{2}\int_0^1\int_0^1|x-y|\,\mathrm{d}\,g(x)\,\mathrm{d}\,g(y)-\frac{1}{2}\int_0^1\int_0^1|x-y|\,\mathrm{d}\,\tilde{g}(x)\,\mathrm{d}\,\tilde{g}(y). \tag{8}$$

Putting $g(x)=F(X_{n+1},x)$ and $\tilde{g}(x)=F(X_n,x)$ we get the desired limit. □

As a consequence we have:

Theorem 5.2. *If* $\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=1$, *then* $G(X_n)$ *is connected.*

Proof. After some manipulation, (7) follows from

$$\lim_{n\to\infty}\left(\frac{1}{nx_n}\sum_{i=1}^{n}x_i\right)\left(1-\frac{x_n}{x_{n+1}}\right)=0. \qquad \square$$

6. $G(X_n)$ and an asymptotic density of x_n

Since

$$\#\{i\le n; x_i/x_n<x\}=\#\{i\in\mathbb{N}; x_i<xx_n\}$$

comes directly from the definitions $F(X_n,x)$ and $A(t)$ we have

$$\frac{F(X_n,x)n}{xx_n}=\frac{A(xx_n)}{xx_n} \tag{9}$$

for every $x\in[0,1]$. This gives

Proposition 6.1. *Assume for a sequence* n_k, $k=1,2,\dots$ *that*

(i) $\lim_{k\to\infty}F(X_{n_k},x)=g(x)$ *a.e.,*

(ii) $\lim_{k\to\infty}\frac{n_k}{x_{n_k}}=d_1$.

Then there exists a $d_2(x)$ *such that* $\lim_{k\to\infty}\frac{A(xx_{n_k})}{xx_{n_k}}=d_2(x)$ *and*

$$\frac{g(x)}{x}d_1=d_2(x) \tag{10}$$

a.e. on $[0,1]$.

As a consequence we have:

Theorem 6.2.

(i) *If* $\overline{d}>0$, *then there exits* $g\in G(X_n)$ *such that* $g(x)\le x$ *for every* $x\in[0,1]$.

(ii) *If* $\underline{d}>0$, *then there exits* $g\in G(X_n)$ *such that* $g(x)\ge x$ *for every* $x\in[0,1]$.

(iii) *If* $\underline{d}>0$, *then for every* $g\in G(X_n)$ *we have*

$$(\underline{d}/\overline{d})x\le g(x)\le(\overline{d}/\underline{d})x \tag{11}$$

for every $x \in [0,1]$.

(iv) *If* $\underline{d} > 0$*, then every* $g \in G(X_n)$ *is everywhere continuous in* $[0,1]$.

(v) *If* $\underline{d} > 0$*, then for every limit point* $\beta > 0$ *of* x_m/x_n *there exist* $g \in G(X_n)$ *and* $0 \le \alpha < \infty$ *such that* $\alpha g(x\beta) \in G(X_n)$.

Proof. (i) Assume that $n_k/x_{n_k} \to \overline{d}$ as $k \to \infty$. Select a subsequence n'_k of n_k such that $F(X_{n'_k}, x) \to g(x)$ a.e. on $[0,1]$. Since $d_2(x) \le \overline{d}$ a.e., (10) gives $(g(x)/x)\overline{d} \le \overline{d}$ a.e., which leads to $g(x) \le x$ a.e. and implies $g(x) \le x$ for every $x \in [0,1]$.

(ii) Similarly to (i), let $n_k/x_{n_k} \to \underline{d}$ as $k \to \infty$. Select a subsequence n'_k of n_k such that $F(X_{n'_k}, x) \to g(x)$ a.e. on $[0,1]$. Since $d_2(x) \ge \underline{d}$ a.e., (10) implies $(g(x)/x)\underline{d} \ge \underline{d}$ a.e. again, which gives $g(x) \ge x$ a.e., whence, $g(x) \ge x$ everywhere on $x \in [0,1]$.

(iii) For any $g \in G(X_n)$ there exists n_k such that $F(X_{n_k}, x) \to g(x)$ a.e. From n_k we can choose a subsequence n'_k such that $n'_k/x_{n'_k} \to d_1$. Using (10) and the fact that $\underline{d} \le d_1 \le \overline{d}$ and $\underline{d} \le d_2 \le \overline{d}$ we have $(g(x)/x)\underline{d} \le \overline{d}$ and $(g(x)/x)\overline{d} \ge \underline{d}$ a.e. If $\underline{d} > 0$, these inequalities are valid for every $x \in (0,1]$.

(iv) Continuity of $g \in G(X_n)$ at 1 follows from Proposition 4.2, since

$$\overline{d}(\varepsilon) \le \limsup_{n\to\infty} (1-\varepsilon)\frac{x_n}{n} = \frac{1-\varepsilon}{\underline{d}}.$$

Applying Theorem 4.1, we have continuity of g in $(0,1]$. Continuity at 0 follows from (11).

(v) It follows from the fact that if $\underline{d} > 0$ and $\lim_{k\to\infty} x_{m_k}/x_{n_k} = \beta > 0$ for $m_k < n_k$, then $\limsup_{k\to\infty} n_k/m_k < \infty$. More precisely, if we pick (m'_k, n'_k) from (m_k, n_k) such that $n'_k/m'_k \to \alpha$, then

$$\frac{\underline{d}}{\overline{d}\beta} \le \alpha \le \frac{\overline{d}}{\underline{d}\beta}. \tag{12}$$

This is so because if we select (m''_k, n''_k) from (m'_k, n'_k) such that $n''_k/x_{n''_k} \to d_1$ and $m''_k/x_{m''_k} \to d_2$, then, by

$$\frac{n''_k}{m''_k} = \frac{\frac{n''_k}{x_{n''_k}} x_{n''_k}}{\frac{m''_k}{x_{m''_k}} x_{m''_k}},$$

we see $\alpha = d_1/(d_2\beta)$. $\square$

Thus, in the case $\underline{d} > 0$, the graph $g(x)$ of any distribution function of X_n lies in the following region.

Figure 2.

From Figure 2 we see

Theorem 6.3.

(i) *If* $\underline{d} = \overline{d} > 0$ *then* $G(X_n) = \{x\}$.

(ii) *If* $\underline{d} > 0$, *then*

a) $g(x) > 0$ *for every* $x \in (0, 1]$;

b) $g(x)$ *has positive Dini derivatives at* 0.

c) $c_\alpha(x) \notin G(X_n)$ *for every* $\alpha \in [0, 1]$.

7. First moment of $g \in G(X_n)$

For a computation of the first moment we use the following expressions:

$$\int_0^1 x \,\mathrm{d}\, g(x) = 1 - \int_0^1 g(x) \,\mathrm{d}\, x, \quad \int_0^1 x \,\mathrm{d}\, F(X_n, x) = \frac{1}{n x_n} \sum_{i=1}^{n} x_i,$$

$$\int_0^1 F(X_n, x) \,\mathrm{d}\, x = \frac{1}{n x_n} \sum_{i=1}^{n-1} i(x_{i+1} - x_i).$$

Theorem 7.1. *For every strictly increasing sequence x_n of positive integers we have*

$$\max_{g\in G(X_n)}\int_0^1 g(x)\,\mathrm{d}\,x \ge \frac{1}{2}, \tag{13}$$

or in equivalent forms

$$\limsup_{n\to\infty}\frac{1}{nx_n}\sum_{i=1}^{n-1} i(x_{i+1}-x_i) \ge \frac{1}{2}, \tag{14}$$

$$\min_{g\in G(X_n)}\int_0^1 x\,\mathrm{d}\,g(x) \le \frac{1}{2}, \qquad \liminf_{n\to\infty}\frac{1}{nx_n}\sum_{i=1}^{n} x_i \le \frac{1}{2}. \tag{15}$$

Proof. We prove (14). In what follows, let $t_i = x_{i+1}-x_i$ for $i = 1,2,\dots$. Putting $t'_i = t_i+1$ and $t'_j = t_j-1$ for some $i<j$ and $t'_k = t_k$ for $k = 1,2,\dots,n-1$, we have

$$\sum_{i=1}^{n-1} t'_i = \sum_{i=1}^{n-1} t_i = x_n - x_1, \quad \frac{1}{nx_n}\sum_{i=1}^{n-1} it'_i < \frac{1}{nx_n}\sum_{i=1}^{n-1} it_i.$$

Define the following algorithm which, for every $n\ge 2$, produces a sequence $t_i^{(n)}$, $i = 1,2,\dots,n-1$.

1) For $n=2$, $t_1^{(2)} = t_1$;
2) For $n=3$, $t_1^{(3)} = t_1+t_2-1$ and $t_2^{(3)} = 1$;
3) Assume that for $n-1$ we have $t_i^{(n-1)}$, $i = 1,2,\dots,n-2$, for n we put $t'_i = t_i^{(n-1)}$, $i = 1,2,\dots,n-2$, and $t'_{n-1} = x_n - x_{n-1}$. The following steps produce new $t'_1,\dots,t'_{n-1}$.
   (a) If there exists k, $1\le k<n-1$, such that $t'_1 = t'_2 = \dots = t'_{k-1} > t'_k$ and $t'_{n-1} > 1$, then we put $t'_k := t'_k+1$, $t'_{n-1} := t'_{n-1}-1$ and $t'_i := t'_i$ in all other cases.
   (b) If such k does not exists and $t'_{n-1} > 1$, then we put $t'_1 := t'_1+1$, $t'_{n-1} := t'_{n-1}-1$ and $t'_i := t'_i$ in all other cases.

In the nth step we will repeat (a) and (b) and the algorithm ended if $t'_{n-1} = 1$ and which gives the resulting $t_1^{(n)} := t'_1,\dots,\ t_{n-1}^{(n)} := t'_{n-1}$. Thus, the algorithm produces $t_i^{(n)}$ satisfying

$$\sum_{i=1}^{n-1} t_i^{(n)} = x_n - x_1, \qquad \frac{1}{nx_n}\sum_{i=1}^{n-1} it_i^{(n)} \le \frac{1}{nx_n}\sum_{i=1}^{n-1} it_i.$$

and having the form

$$t_1^{(n)} = t_2^{(n)} = \cdots = t_m^{(n)} = D_n > t_{m+1}^{(n)} \ge t_{m+2}^{(n)} = t_{m+3}^{(n)} = \dots t_{n-1}^{(n)} = 1,$$

or

$$\begin{aligned} t_1^{(n)} &= \cdots = t_m^{(n)} = D_n > t_{m+1}^{(n)} = \cdots = t_{m+k}^{(n)} = D_n - 1 \\ &\ge t_{m+k+1}^{(n)} = \cdots = t_{n-1}^{(n)} = 1. \end{aligned}$$

There are two possibilities:
(I) D_n is bounded;
(II) $D_n \to \infty$.

(I) Evidently, boudedness of D_n implies $\underline{d} > 0$ and Theorem 6.2 (ii) implies the existence of $g \in G(X_n)$ such that $g(x) \ge x$ for $x \in [0,1]$. This gives (13), since $\int_0^1 g(x)\,\mathrm{d}\,x \ge \int_0^1 x\,\mathrm{d}\,x = \frac{1}{2}$.

(II) In this case there exists infinitely many n for which $D_n > D_{n-1}$. For such n the sequence $t_i^{(n)}$, $i = 1, 2, \dots, n-1$ has the form

$$\begin{aligned} t_1^{(n)} &= t_2^{(n)} = \cdots = t_m^{(n)} = D_n > t_{m+1}^{(n)} = t_{m+2}^{(n)} = \cdots \\ &= t_{n-2}^{(n)} = D_n - 1 > t_{n-1}^{(n)} = 1. \end{aligned}$$

Summing it up

$$\text{(16)} \qquad \frac{1}{nx_n}\sum_{i=1}^{n-1} it_i^{(n)} = \frac{1}{2}\,\frac{\frac{m^2}{n^2} + O\left(\frac{m}{n^2}\right) + \left(\frac{D_n - 1}{D_n}\right)\left(1 - \frac{m^2}{n^2} + O\left(\frac{1}{n}\right)\right)}{\frac{m}{n} + \left(1 - \frac{2+m}{n}\right)\left(\frac{D_n-1}{D_n}\right) + \frac{1}{nD_n}}.$$

and selecting a subsequence n_k of such n such that $m_k/n_k \to \alpha$, the limit of (16) gives $1/2$ as $k \to \infty$ and for any $0 \le \alpha \le 1$. □

Note that for $x_n = [\log n]$ we have $G(X_n) = \{c_1(x)\}$ and $\int_0^1 c_1(x)\,\mathrm{d}\,x = 0$, but in this case x_n is not strictly increasing.

In some cases the first moment characterizes $G(X_n)$.

Theorem 7.2. *Assume that all d.f. in $G(X_n)$ are continuous at 1 and that, for every $g \in G(X_n)$, we have $\int_0^1 g(x)\,\mathrm{d}\,x = c$, where $c > 0$ is a constant. Then*

(i) *If $g \in G(X_n)$ increases at every point $\beta \in (0,1)$, then $g(x) = x^{\frac{1-c}{c}}$ for every $x \in [0,1]$.*

(ii) *If $g \in G(X_n)$ is constant on $(\alpha,\beta) \subset (0,1]$, then $G(X_n)$ is a singleton and $g(x) = c_0(x)$ a.e. on $[0,1]$.*

Proof. (i) By Theorem 4.1, such $g \in G(X_n)$ is continuous on $(0,1]$ and by (5) for every $\beta \in (0,1)$ we have $\frac{1}{g(\beta)} g(x\beta) \in G(X_n)$. Thus

$$c = \int_0^1 \frac{1}{g(\beta)} g(x\beta)\,\mathrm{d}\,x = \frac{1}{g(\beta)\beta} \int_0^\beta g(x)\,\mathrm{d}\,x.$$

Using the equation $cg(\beta)\beta = \int_0^\beta g(x)\,\mathrm{d}\,x$ and the fact that the right-hand side is differentiable, we get $c(g(\beta) + g'(\beta)\beta) = g(\beta)$, so

$$\frac{g'(\beta)}{g(\beta)} = \left(\frac{1-c}{c}\right)\frac{1}{\beta}$$

and integration gives $g(x) = x^{(1-c)/c}$.

(ii) Assume that $0 < \alpha < \beta < 1$ and $g(x)$ increases in α and β. Then as in (i) we have

$$\frac{1}{g(\alpha)\alpha} \int_0^\alpha g(x)\,\mathrm{d}\,x = \frac{1}{g(\beta)\beta} \int_0^\beta g(x)\,\mathrm{d}\,x.$$

Expressing the right-hand side as $\frac{1}{g(\alpha)\beta} \left(\int_0^\alpha g(x)\,\mathrm{d}\,x + g(\alpha)(\beta - \alpha)\right)$ we get

$$\frac{1}{g(\alpha)\alpha} \int_0^\alpha g(x)\,\mathrm{d}\,x = 1.$$

Consequently, $c = 1$ and thus $\int_0^1 g(x)\,\mathrm{d}\,x = 1$, which implies $g(x) = c_0(x)$, moreover, $G(X_n) = \{c_0(x)\}$. □

8. Singleton $G(X_n) = \{g\}$

For general $G(X_n)$, the connection between $G(X_n)$ and $G(x_m/x_n)$ is open, but for singleton $G(X_n)$ we have

Theorem 8.1. *If $G(X_n) = \{g\}$, then $G(x_m/x_n) = \{g\}$.*

Proof. A proof of the theorem is the same as the proof of [10, Proposition 1, (ii)], since

$$\lim_{n\to\infty} \frac{|X_n|}{|X_1| + \cdots + |X_n|} = \lim_{n\to\infty} \frac{n}{n(n+1)/2} = 0. \qquad \square$$

Theorem 7.2 indicates the following possibilities for a singleton $G(X_n)$. We give an alternative proof.

Theorem 8.2. *Assume that $G(X_n) = \{g\}$. Then either*

(i) $g(x) = c_0(x)$ *for* $x \in [0, 1]$ *or*

(ii) $g(x) = x^\lambda$ *for some* $0 < \lambda \le 1$ *and* $x \in [0, 1]$. *Moreover,*

(iii) *if* $\overline{d} > 0$ *then* $g(x) = x$.

Proof. Let $G(X_n) = \{g\}$. We divide the proof into the following six steps.

(I) Applying Theorem 7.1, we get $\int_0^1 g(x)\,\mathrm{d}\,x \ge \frac{1}{2}$ which implies $g(x) \ne c_1(x)$.

(II) g must be continuous on $(0, 1)$, since otherwise Theorem 3.2, for a discontinuity point $\beta \in (0, 1)$, guarantees the existence of $\alpha_1 \ne \alpha_2$ such that $\alpha_1 g(x\beta) = \alpha_2 g(x\beta) = g(x)$ a.e. which is a contradiction.

(III) Assume that $g(x)$ increases in every point $\beta \in (0, 1)$. In this case relation (5) gives the well-known Cauchy equation $g(x)g(\beta) = g(x\beta)$ for a.e. $x, \beta \in [0, 1]$ For a monotonic $g(x)$ the Cauchy equation has solutions only of the type $g(x) = x^\lambda$.

(IV) Assume that $g(x)$ has a constant value on the interval $(\gamma, \delta) \subset [0, 1]$. For $\beta \in (0, 1]$ satisfying (i) $g(x)$ increases in β and (ii) $g(\beta) > 0$ the basic relation (4) gives $g(x) = \alpha g(x\beta)$ which implies that $g(x)$ has a constant value also on $\beta(\gamma, \delta)$ and if $\delta \le \beta$ then also on $\beta^{-1}(\gamma, \delta)$. Thus, if (γ_i, δ_i), $i \in \mathcal{I}$ is a system of all intervals (maximal under inclusion) in which $g(x)$ possesses constant values, then for every $i \in \mathcal{I}$ there exists $j \in \mathcal{I}$ such that $\beta(\gamma_i, \delta_i) = (\gamma_j, \delta_j)$ and vice-versa for every $j \in \mathcal{I}$, $\delta_j \le \beta$, there exists $i \in \mathcal{I}$ such that $\beta^{-1}(\gamma_j, \delta_j) = (\gamma_i, \delta_i)$. This is true also for $\beta = \beta_1^{n_1}\beta_2^{n_2}\ldots$, where $\beta_1, \beta_2, \ldots$ satisfy (i) and (ii) and $n_1, n_2, \ldots \in \mathbb{Z}$. Thus, there exists $0 < \theta < 1$ such that every such β has the form θ^n, $n \in \mathbb{N}$. The end points γ_i, δ_i (without $\gamma_i = 0$) satisfy (i) and (ii) and thus the intervals (γ_i, δ_i) is of the form (θ^n, θ^{n-1}), $n = 1, 2, \dots$ and all discontinuity points of $g(x)$ are θ^n, $n = 1, 2, \dots$, a contradiction with (II). For $g(x) = c_0(x)$ there exists no $\beta \in (0, 1]$ satisfying (i) and (ii).

(V) We have the possibilities $g(x) = c_0(x)$ and $g(x) = x^\lambda$ for some $\lambda > 0$. Applying Theorem 7.1 we have $\int_0^1 g(x)\,\mathrm{d}\,x \geq 1/2$ which reduces λ to $\lambda \leq 1$.

(VI) If $\overline{d} > 0$, then by Theorem 6.2, (i) must be $g(x) \leq x$ which is contrary to $x^\lambda > x$ for $\lambda < 1$. □

The possibilities (i), (ii) are achievable. Trivially, for $x_n = [n^\lambda]$, $G(X_n) = \{x^{1/\lambda}\}$ and for x_n satisfying $\lim_{n\to\infty} x_n/x_{n+1} = 0$ we have $G(X_n) = \{c_0(x)\}$. Less trivially, every lacunary x_n, i.e. $x_n/x_{n+1} \leq \lambda < 1$, gives $G(X_n) = \{c_0(x)\}$.

The following limit covers all of $G(X_n) = \{g\}$.

Theorem 8.3. *The set $G(X_n)$ is a singleton if and only if*

(17)
$$\lim_{m,n\to\infty} \left(\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \frac{x_i}{x_m} - \frac{x_j}{x_n} \right| - \frac{1}{2m^2} \sum_{i,j=1}^{m} \left| \frac{x_i}{x_m} - \frac{x_j}{x_m} \right| - \frac{1}{2n^2} \sum_{i,j=1}^{n} \left| \frac{x_i}{x_n} - \frac{x_j}{x_n} \right| \right) = 0.$$

Proof. It follows directly from the limit

$$\lim_{m,n\to\infty} \int_0^1 (F(X_m, x) - F(X_n, x))^2 \,\mathrm{d}\,x = 0,$$

after applying (8) for $g(x) = F(X_m, x)$ and $\tilde{g}(x) = F(X_n, x)$. □

Using the theory of statistically convergent sequences, we can characterize x_n having $G(X_n) = \{c_0(x)\}$.

Theorem 8.4. $G(X_n) = \{c_0(x)\}$ *if and only if any of the following limit relations holds.*

(i)
$$\lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0;$$

(ii)
$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \frac{x_i}{x_m} - \frac{x_j}{x_n} \right| = 0.$$

Furthermore, any of (i) and (ii) implies $\overline{d} = 0$.

PROOF. (i) $\int_0^1 x \,\mathrm{d}\, g(x) = 1 - \int_0^1 g(x) \,\mathrm{d}\, x = 0$ only if $g(x) = c_0(x)$.

(ii) Here we adapt [17, Sect. 22]. Assume that $F(X_{m_k}, x) \to \tilde{g}(x)$ and $F(X_{n_k}, x) \to g(x)$ a.e. as $k \to \infty$. Riemann–Stieltjes integration yields

$$\frac{1}{m_k n_k} \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} \left| \frac{x_i}{x_{m_k}} - \frac{x_j}{x_{n_k}} \right| = \int_0^1 \int_0^1 |x-y| \,\mathrm{d}\, F(X_{m_k}, x) \,\mathrm{d}\, F(X_{n_k}, y)$$

which, after using Helly's theorem, tends to

$$\int_0^1 \int_0^1 |x-y| \,\mathrm{d}\, \tilde{g}(x) \,\mathrm{d}\, g(y)$$

as $k \to \infty$. The final integral is equal to 0 if and only if $\tilde{g}(x) = g(x) = c_\alpha(x)$ for some fixed $\alpha \in [0,1]$. By Theorem 8.2, α must be 0.

$\overline{d} = 0$ follows from Theorem 6.2, part (i). □

Theorem 8.5. $G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\}$ *if and only if*

$$\lim_{n\to\infty} \frac{1}{n^2 x_n} \sum_{i,j=1}^{n} |x_i - x_j| = 0,$$

and this equation implies $c_0(x) \in G(X_n)$.

PROOF. As in (ii), the limit of the selected subsequence tends to

$$\int_0^1 \int_0^1 |x-y| \,\mathrm{d}\, g(x) \,\mathrm{d}\, g(y)$$

and is equal to 0 if and only if $g(x) = c_\alpha(x)$ for some $\alpha \in [0,1]$.

If $c_0(x) \notin G(X_n)$, then all d.f. in $G(X_n)$ are continuous and thus by Theorem 3.3, for any $c_{\alpha'}(x) \in G(X_n)$, $\alpha' < 1$, and for any $\alpha > 1$ there exists $c_{\alpha''}(x) \in G(X_n)$ and $\beta \in (0,1]$ such that

$$c_{\alpha'}(x) = \alpha c_{\alpha''}(x\beta)$$

which is a contradiction. The case $G(X_n) = \{c_1(x)\}$ is impossible by Theorem 7.1. □

9. U.d. of X_n

By Theorem 8.1, u.d. of the single block sequence X_n implies the u.d. of the ratio sequence x_m/x_n. Applying Theorem 6.3 part (i) we have

Theorem 9.1. *If the increasing sequence x_n of positive integers has a positive asymptotic density, i.e. $\underline{d} = \overline{d} > 0$, then the associated ratio sequence x_m/x_n, $m = 1, 2, \dots, n$, $n = 1, 2, \dots$ is u.d. in $[0, 1]$.*

Positive asymptotic density is not necessary. According to T. Šalát [12] we can use also a sequence x_n with $\underline{d} = 0$.

Theorem 9.2. *Let x_n be an increasing sequence of positive integers and $h : [0, \infty) \to [0, \infty)$ be a function satisfying*

(i) *$A(x) \sim h(x)$ as $x \to \infty$, where*

(ii) *$h(xy) \sim xh(y)$ as $y \to \infty$ and for every $x \in [0, 1]$, and*

(iii) $\lim_{n\to\infty} \frac{n}{h(x_n)} = 1$.

Then X_n (and consequently x_m/x_n) is u.d. in $[0, 1]$.

Proof. Starting with $F(X_n, x)n = A(xx_n)$ it follows from (i) that $\frac{F(X_n,x)n}{h(xx_n)} \to 1$ as $n \to \infty$, then by (ii) $\frac{F(X_n,x)n}{xh(x_n)} \to 1$, which gives the limit $F(X_n, x)\frac{n}{h(x_n)} \to x$ as $n \to \infty$. □

Assuming (i) and (ii), we have $\liminf_{n\to\infty} n/h(x_n) \geq 1$, since otherwise $n_k/h(x_{n_k}) \to \alpha < 1$ implies $F(X_{n_k}, x) \to x/\alpha$ for every $x \in [0, 1]$ which is a contradiction. Also, $G(X_n) \subset \{x\lambda; \lambda \in [0, 1]\}$.

Another criterion can be found by using the so called L^2 discrepancy of the block X_n defined by

$$D^{(2)}(X_n) = \int_0^1 (F(X_n, x) - x)^2 \,\mathrm{d}\, x,$$

which can be expressed (cf. [16, IV. Appl.]) as

$$D^{(2)}(X_n) = \frac{1}{n^2} \sum_{i,j=1}^{n} F\left(\frac{x_i}{x_n}, \frac{x_j}{x_n}\right),$$

where

$$F(x, y) = \frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2} - \frac{|x - y|}{2}.$$

Thus

$$D^{(2)}(X_n) = \frac{1}{3} + \frac{1}{nx_n^2}\sum_{i=1}^n x_i^2 - \frac{1}{nx_n}\sum_{i=1}^n x_i - \frac{1}{2n^2x_n}\sum_{i,j=1}^n |x_i - x_j|, \tag{18}$$

which gives (cf. [16])

Theorem 9.3. *For every increasing sequence x_n of positive integers we have*

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \Longleftrightarrow \lim_{n\to\infty} F(X_n, x) = x.$$

The left hand-side can be divided into three limits (cf. [15, Theorem 1])

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \Longleftrightarrow \begin{cases} \text{(i)} & \lim_{n\to\infty} \frac{1}{nx_n}\sum_{i=1}^n x_i = \frac{1}{2}, \\ \text{(ii)} & \lim_{n\to\infty} \frac{1}{nx_n^2}\sum_{i=1}^n x_i^2 = \frac{1}{3}, \\ \text{(iii)} & \lim_{n\to\infty} \frac{1}{n^2x_n}\sum_{i,j=1}^n |x_i - x_j| = \frac{1}{3}. \end{cases} \tag{19}$$

Weyl's criterion for u.d. of X_n is not well applicable in our case. It says (cf. [14, (7)]).

Theorem 9.4. *X_n is u.d. if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n e^{2\pi i h \frac{x_k}{x_n}} = 0$$

for all non-zero integers h.

10. Applications

S. Róka [11] formulated the following open question:

Question 10.1. *Let $a_1, a_2, \dots$ be an infinite sequence of positive real numbers. Is it true that*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty \Longleftrightarrow \lim_{n\to\infty} \frac{na_n}{a_1 + a_2 + \cdots + a_n} > 2? \tag{20}$$

We disprove this equivalence in the following five steps.

1. Put $a_n = x_n$, where x_n is an increasing sequence of positive integers. Using the associated sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

and rewriting

$$\frac{nx_n}{x_1 + x_2 + \dots + x_n} = \frac{1}{\frac{1}{n}\left(\frac{x_1}{x_n} + \frac{x_2}{x_n} + \dots + \frac{x_n}{x_n}\right)},$$

we obtain by Helly's theorem, that every limit point of the sequence $\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$, $n = 1, 2, \dots$ is of the form $\int_0^1 x \,\mathrm{d}\, g(x)$ for some $g \in G(X_n)$. Thus, by $\int_0^1 x \,\mathrm{d}\, g(x) = 1 - \int_0^1 g(x)\,\mathrm{d}\, x$, (20) has the form

$$\text{(21)} \qquad \sum_{n=1}^{\infty}\frac{1}{x_n} < \infty \Longleftrightarrow \forall (g \in G(X_n)) \int_0^1 g(x)\,\mathrm{d}\, x = c > \frac{1}{2},$$

where c is a constant.

2. By Theorem 7.1, independently of the divergence or convergence of the series $\sum_{n=1}^{\infty}\frac{1}{x_n}$, we have $\max_{g\in G(X_n)}\int_0^1 g(x)\,\mathrm{d}\, x \ge \frac{1}{2}$.

3. Since

$$\frac{x_i^2}{x_n^2} < x \Longleftrightarrow \frac{x_i}{x_n} < \sqrt{x},$$

for the blocks sequence

$$X_n^2 = \left(\frac{x_1^2}{x_n^2}, \frac{x_2^2}{x_n^2}, \dots, \frac{x_n^2}{x_n^2}\right),$$

we have

$$G(X_n^2) = \{g(\sqrt{x}); g \in G(X_n)\},$$

and in any case, $\sum_{n=1}^{\infty}\frac{1}{x_n^2}$ converges.

4. Example 11.2 below gives $G(X_n)$ such that, if we choose $\frac{\gamma}{\delta}$ sufficiently close to 1 and a is sufficiently large, then d.f. $g_1(x)$ is sufficiently near to $c_1(x)$ and $g_0(x)$ to $c_0(x)$. The same holds for $G(X_n^2)$.

5. Thus, we can find an increasing sequence x_n of positive integers such that the series $\sum_{n=1}^{\infty}\frac{1}{x_n}$ converges, $\max_{g\in G(X_n)}\int_0^1 g(x)\,\mathrm{d}\, x$ is sufficiently close to 1 and $\min_{g\in G(X_n)}\int_0^1 g(x)\,\mathrm{d}\, x$ is sufficiently close to 0.

11. Examples

Example 11.1. Put $x_n = p_n$, the nth prime and denote

$$X_n = \left(\frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n}\right).$$

By the prime number theorem, $h(x) = x/\log x$ satisfies assumptions (i) and (ii) of Theorem 9.2. Applying [7, p. 247] $p_n/(n\log n) \to 1$ we also find that (iii) $p_n/(n\log p_n) \to 1$ by this theorem. Thus the sequence of blocks X_n is u.d. and therefore the ratio sequence p_m/p_n, $m = 1, 2, \dots, n$, $n = 1, 2, \dots$ is u.d. in $[0, 1]$. This generalizes a result of A. Schinzel (cf. [13, p. 155]). Note that from u.d. of X_n applying the equivalent limits (19) for the L^2 discrepancy of X_n we get the following interesting limit

$$\lim_{n\to\infty} \frac{1}{n^2 p_n} \sum_{i,j=1}^{n} |p_i - p_j| = \frac{1}{3}.$$

Example 11.2. Let γ, δ, and a be given real numbers satisfying $1 \le \gamma < \delta \le a$. Let x_n be an increasing sequence of all integer points lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), \dots, (\gamma a^k, \delta a^k), \dots.$$

The set $G(X_n)$ has the following properties:

1. Every $g \in G(X_n)$ is continuous.

2. Every $g \in G(X_n)$ has infinitely many intervals with constant values, i.e. with $g'(x) = 0$, and in the infinitely many complement intervals it has a constant derivative $g'(x) = c$, where

$$\frac{1}{\overline{d}} \le c \le \frac{1}{\underline{d}}$$

and for lower $\underline{d}$ and upper $\overline{d}$ asymptotic density we have

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a-1)}, \quad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a-1)}.$$

3. More precisely, the set of all d.f. can be expressed parametrically as $G(X_n) = \{g_t(x); t \in [0, 1]\}$, where $g_t(x)$ has constant values

$$g_t(x) = \frac{1}{a^i(1 + t(a-1))} \text{ for } x \in \frac{(\delta, a\gamma)}{a^{i+1}(t\delta + (1-t)\gamma)}, \quad i = 0, 1, 2, \dots$$

and on the component intervals it has a constant derivative

$$g_t'(x) = \frac{t\delta + (1-t)\gamma}{(\delta-\gamma)(\frac{1}{a-1}+t)}$$

$$\text{for } x \in \frac{(\gamma,\delta)}{a^{i+1}(t\delta+(1-t)\gamma)},\ i = 0,1,2,\dots \text{ and } x \in \left(\frac{\gamma}{t\delta+(1-t)\gamma}, 1\right).$$

Here, as above, we write $(xz, yz) = (x, y)z$ and $(x/z, y/z) = (x, y)/z$.

4. The graph of every $g \in G(X_n)$ lies in the intervals

$$\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right] \cup \left[\frac{1}{a^2}, \frac{1}{a}\right] \times \left[\frac{1}{a^2}, \frac{1}{a}\right] \cup \dots$$

Figure 3.

(Here $a = 5$.) Moreover, the graph g in $\left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right] \times \left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right]$ is similar to the graph of g in $\left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right] \times \left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right]$ with coefficient $\frac{1}{a}$. Using the parametric expression, it can be written as

$$\forall x \in \left(\frac{1}{a^{i+1}}, \frac{1}{a^i}\right) g_t(x) = \frac{g_t(a^i x)}{a^i}, \quad i = 0,1,2,\dots.$$

5. Thus, we need to known only the graphs $g \in G(X_n)$ in $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$. They are of the form displayed in the following figure.

(Here $\gamma = 1$, $\delta = 2$ and $a = 5$.) The curves $y = y(x_1)$, $x_1 \in \left(\frac{1}{a}, \frac{\delta}{a\gamma}\right)$, and $y = y(x_2)$, $x_2 \in \left(\frac{\gamma}{\delta}, 1\right)$ are of the form

$$y(x_1) = \left(1 + \frac{1}{\overline{d}}\left(\frac{1}{x_1} - \frac{a\gamma}{\delta}\right)\right)^{-1}, \quad y(x_2) = \left(1 + \frac{1}{\underline{d}}\left(\frac{1}{x_2} - 1\right)\right)^{-1}.$$

Figure 4.

6. From the graphs of $g \in G(X_n)$ it follows that $G(X_n)$ is connected and the upper distribution function $\overline{g}(x) = g_0(x) \in G(X_n)$ and the lower distribution function $\underline{g}(x) \notin G(X_n)$. The graph of $\underline{g}(x)$ on $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$ coincides with the graph of $y(x_2)$ on $\left[\frac{\gamma}{\delta}, 1\right]$, further, on $\left[\frac{1}{a}, \frac{\gamma}{\delta}\right]$ we have $\underline{g}(x) = \frac{1}{a}$.

Figure 5.

7. We have $G(X_n)$ in the form (5):

$$G(X_n) = \left\{ \frac{g_0(x\beta)}{g_0(\beta)}; \beta \in \left[\frac{1}{a}, \frac{\delta}{a\gamma}\right] \right\}.$$

For the proofs of 1–7 we only note: Assume that $x_n \in a^k(\gamma, \delta)$, $i, i+1, i+2, \ldots \in a^j(\gamma, \delta)$ for some $j < k$, and let $F(X_n, x) \to g(x)$ for some sequence of n. Then $g(x)$ has a constant derivative in the intervals containing $\frac{i}{x_n}, \frac{i+1}{x_n}, \frac{i+2}{x_n}, \ldots$, since

$$\frac{\frac{1}{n}}{\frac{i+1}{x_n} - \frac{i}{x_n}} = \frac{x_n}{n},$$

and thus $\frac{x_n}{n}$ must be convergent to $g'(x)$, so $\frac{1}{d} \leq g'(x) \leq \frac{1}{\underline{d}}$. For $x_n = [ta^k\delta + (1-t)a^k\gamma]$ we can find

$$g'(x) = \lim_{n\to\infty} \frac{x_n}{n} = \lim_{k\to\infty} \frac{a^k(t\delta + (1-t)\gamma)}{\sum_{j=0}^{k-1} a^j(\delta-\gamma) + a^k(t\delta + (1-t)\gamma) - a^k\gamma}$$
$$= \frac{t\delta + (1-t)\gamma}{(\delta-\gamma)\left(\frac{1}{a-1} + t\right)}.$$

References

[1] H. G. Barone, Limit points of sequences and their transforms by methods of summability, *Duke Math. J.* **5** (1939), 740–752, MR **1**, 218 a.

[2] J. Bukor and J. T. Tóth, On accumulation points of ratio sets of positive integers, *Amer. Math. Monthly* **103** (1996), 502–504.

[3] M. Drmota and R. F. Tichy, Sequences, Discrepancies and Applications, Lecture Notes in Mathematics 1651, *Springer-Verlag*, *Berlin, Heidelberg*, 1997.

[4] E. Hlawka, The theory of uniform distribution, *A B Academic Publishers*, *London*, 1984, (English translation of: Theorie der Gleichverteilung. Bibl. Inst., Mannheim–Wien–Zürich, 1979).

[5] S. Knapowski, Über ein Problem der Gleichverteilung, *Colloq. Math.* **5** (1958), 8–10.

[6] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, *John Wiley & Sons*, *New York*, 1974.

[7] D. S. Mitrinović, J. Sándor and B. Crstici, Handbook of Number Theory, *Kluwer Academic Publishers*, *Dordrecht, Boston, London*, 1996.

[8] G. Myerson, A sampler of recent developments in the distribution of sequences, Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991), vol. 147, *Marcel Dekker*, *New York*, 1993, 163–190.

[9] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, *SIAM Conf. Ser. Appl. Math.,* Vol. 63, *Society for Industrial and Applied Mathematics*, *Philadelphia*, 1992.

[10] Š. Porubský, T. Šalát and O. Strauch, On a class of uniform distributed sequences, *Math. Slovaca* **40** (1990), 143–170, (MR 92d:11076).

[11] S. Róka, *Octogon Math. Mag.* **5** (1997), no. 1, 98.

[12] T. Šalát, On ratio sets of sets of natural numbers, *Acta Arith.* **XV** (1969), 273–278.

[13] W. Sierpiński, Elementary Theory of Numbers, *PWN*, *Warszawa*, 1964.

[14] I. J. Schoenberg, Über die asymptotische Vertaeilung reeller Zahlen mod 1, *Math. Z.* **28** (1928), 171–199.

[15] O. Strauch, A new moment problem of distribution functions in the unit interval, *Math. Slovaca* **44** (1994), 171–211, (MR 95i:11082).

[16] O. Strauch, L^2 discrepancy, *Math. Slovaca* **44** (1994), 601–632.

[17] O. Strauch, Uniformly maldistributed sequences in a strict sense, *Monatsh. Math.* **120** (1995), 153–164.

[18] O. Strauch, On set of distribution functions of a sequence, Proc. Conf. Analytic and Elementary Number Theory (W. G. Nowak and J. Schoißengeier, eds.), In Honor of E. Hlawka's 80th Birthday, Vienna, July 18–20, 1996, *Universität Vien and Universität für Bodenkultur*, *Vienn*, 1997, 214–229.

[19] O. Strauch and J. T. TÓTH, Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$, *Acta Arith.* **LXXXVII** no. 1 (1998), 67–78.

[20] R. F. Tichy, Three examples of triangular arrays with optimal discrepancy and linear recurrences, Applications of Fibonacci numbers, vol. 7, 1998, 415–423.

[21] J. G. van der Corput, Verteilungsfunktionen I–VIII, *Proc. Akad. Amsterdam* **38** (1935), 813–821, 1058–1066 **39** (1936), 10–19, 19–26, 149–153, 339–344, 489–494, 579–590.

[22] R. Winkler, On the distribution behaviour of sequences, *Math. Nachr.* **186** (1997), 303–312.

OTO STRAUCH
MATHEMATICAL INSTITUTE OF THE SLOVAK ACADEMY OF SCIENCES
ŠTEFÁNIKOVA 49
SK–814 73 BRATISLAVA
SLOVAKIA

E-mail: strauch@mat.savba.sk

JÁNOS T. TÓTH
DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCES
UNIVERSITY OF OSTRAVA
30. DUBNA, 22
70103 OSTRAVA
CZECH REPUBLIK

E-mail: janos.toth@albert.osu.cz

(Received May 15, 2000; revised July 29, 2000)