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Power integral bases in a parametric family of sextic fields

By PÉTER OLAJOS (Debrecen)

Abstract. The purpose of the paper is to investigate power integral bases in a parametric family of totally complex sextic fields. These fields are composits of a complex quadratic subfield and a totally real cubic subfield. GAÁL [2] studied a similar family of fields. Using his method we show that the fields of the family in question do not admit power integral bases if the parameters are not very small. Moreover, using direct computations we also deal with those fields in the family which correspond to small parameters and are not covered by the main theorem.

1. Introduction

Let K be an algebraic number field of degree n with ring of integers $\mathbb{Z}_K$, integral basis $\{1, \omega_2, \dots, \omega_n\}$ and discriminant D_K. The element $\alpha = x_1 + \omega_2 x_2 + \dots + \omega_n x_n$ generates a power integral basis in K if $\{1, \alpha, \dots, \alpha^{n-1}\}$ is an integral basis. We have

$$D_{K/\mathbb{Q}}(\omega_2 x_2 + \dots + \omega_n x_n) = I(x_2, \dots, x_n)^2 \cdot D_K$$

where $I(x_2, \dots, x_n)$ is the index form corresponding to the above integral basis. As it is well known (cf. [5]) α generates a power integral basis if and only if

$$I(x_2, \dots, x_n) = \pm 1. \tag{1}$$

It is a classical problem in algebraic number theory to decide if a number field admits power integral bases. This question is satisfactory solved for

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lower degree number fields, cf. GAÁL and SCHULTE [11], GAÁL, PETHŐ and POHST [7], [8] for cubic and quartic fields. There is a general algorithm for quintic fields by GAÁL and GYŐRY [6] which already requires several hours of CPU time.

For higher degree number fields this problem is very complicated because of the high degree the large number of variables of equation (1), and the resolution of this equation is only hopeful if K has proper subfields, because in this case the index form is reducible.

Hence for sextic fields till now only some special classes were investigated by GAÁL [3], GAÁL and POHST [9], [10]. A very interesting case was studied in [2], when the field K is the composite of its complex quadratic and real cubic subfields. In this case the index form equation can be reduced to some cubic Thue inequalities, which enables one to describe power integral bases even in an infinite parametric family of such fields.

We are going to apply the method of [2] in a new parametric family of fields. For sufficiently large values of the parameters we shall show that there exist no power integral bases. Moreover, by extending the algorithm of [2] we shall use direct computations to study the existence of power integral bases also for the small values of the parameters, not covered by the main theorem.

2. Auxiliary results

Let ϑ be a totally real cubic algebraic integer and let m be a square-free positive integer. Let us consider the sextic field $K = \mathbb{Q}(\vartheta, i\sqrt{m})$, with discriminant D_K and ring of integers $\mathbb{Z}_K$. Let $M = \mathbb{Q}(i\sqrt{m})$ and $L = \mathbb{Q}(\vartheta)$ be the subfields of K. Set

$$\omega = \begin{cases} (1 + i\sqrt{m})/2, & \text{if } -m \equiv 1 \pmod 4, \\ i\sqrt{m}, & \text{if } -m \equiv 2, 3 \pmod 4. \end{cases} \tag{2}$$

We represent any $\alpha \in \mathbb{Z}_K$ in the form

$$\alpha = \frac{x_0 + x_1\vartheta + x_2\vartheta^2 + y_0\omega + y_1\omega\vartheta + y_2\omega\vartheta^2}{g} \tag{3}$$

with $x_0, x_1, x_2, y_0, y_1, y_2 \in \mathbb{Z}$ and with a fixed common denominator $g \in \mathbb{Z}$. Set $\mathcal{O} = \mathbb{Z}[1, \vartheta, \vartheta^2, \omega, \omega\vartheta, \omega\vartheta^2]$ and denote by $D_{\mathcal{O}}$ the discriminant of this order. We are going to consider the existence of power integral basis, and more generally the existence of elements of a given index, in the order $\mathcal{O}$ which often coincides with $\mathbb{Z}_K$. We have

$$\frac{g^6\sqrt{|D_K|}}{\sqrt{|D_{\mathcal{O}}|}} \in \mathbb{Z}.$$

Let I_0 be a given, non-zero positive integer and consider the solutions $\alpha \in \mathbb{Z}_K$ of

$$I(\alpha) = I_0. \tag{4}$$

Set

$$I_1 = \frac{g^{15} I_0 \sqrt{|D_K|}}{\sqrt{|D_{\mathcal{O}}|}} \in \mathbb{Z}.$$

Denote by ϑ_i $(1 \le i \le 3)$ the conjugates of ϑ over M and set $\rho = -\vartheta_2 - \vartheta_3$.

Lemma 1 (cf. [2], Theorem 2.1)**.** *If $\alpha \in \mathbb{Z}_K$ is a solution of equation* (4), *and $x_0, x_1, x_2, y_0, y_1, y_2 \in \mathbb{Z}$ are the coefficients of α in the representation* (3), *then*

$$N_{K/M}((x_1 + \omega y_1) - \rho(x_2 + \omega y_2)) = \mu \tag{5}$$

$$N_{L/Q}(y_0 + y_1\vartheta + y_2\vartheta^2) = d \tag{6}$$

where $\mu \in \mathbb{Z}_M$, $d \in \mathbb{Z}$, such that $d \cdot N_{M/Q}(\mu)$ divides I_1.

Under our assumptions on the field K, denote by $\rho = \rho_1,\ \rho_2,\ \rho_3$ the conjugates of ρ over L and let $X = x_1 + \omega y_1$, $Y = x_2 + \omega y_2$ be an arbitrary, but fixed solution of (5). Choose the indices $\{r, s, t\} = \{1, 2, 3\}$ according to

$$|X - \rho_r Y| \le |X - \rho_s Y| \le |X - \rho_t Y|. \tag{7}$$

Set

$$c_m = \begin{cases} 2, & \text{if } -m \equiv 1 \pmod 4 \\ 1, & \text{if } -m \equiv 2, 3 \pmod 4 \end{cases}$$

$$c_1 = 9c_m^3|\mu|, \quad c_2 = \min(|\rho_r - \rho_s|, |\rho_r - \rho_t|), \quad c_3 = |\rho_r - \rho_s| \cdot |\rho_r - \rho_t|$$

$$c_4 = \max\left\{\frac{2|\mu|^{1/3}}{c_2}, \frac{4c_m|\mu|}{c_3\sqrt{m}}\right\}, \quad c_5 = \left(\frac{8|\mu|}{c_2c_3}\right)^{1/3}.$$

Finally put

$$F(x,y)=\prod_{j=1}^{3}(x-\rho_j y)\in\mathbb{Z}[x,y].$$

Under these assumptions we have the following theorem:

Lemma 2 (cf. [2], Theorem 2.2). *Let* $X=x_1+\omega y_1$, $Y=x_2+\omega y_2\in\mathbb{Z}_M$ *be a solution of* (5) *according to* (7)*. Suppose* $|Y|>c_4$*. We have*

$$x_1y_2=x_2y_1.$$

Further, in case $-m\equiv 1 \pmod 4$*:*

$$\begin{cases} \text{if } |2x_2+y_2|\geq 2c_5, & \text{then } |F(2x_1+y_1,2x_2+y_2)|\leq c_1\\ \text{if } |y_2|\geq 2c_5/\sqrt{m}, & \text{then } |F(y_1,y_2)|\leq c_1/(\sqrt{m}\,)^3, \end{cases}$$

and in case $-m\equiv 2,3 \pmod 4$*:*

$$\begin{cases} \text{if } |x_2|\geq 2c_5, & \text{then } |F(x_1,x_2)|\leq c_1,\\ \text{if } |y_2|\geq c_5/\sqrt{m}, & \text{then } |F(y_1,y_2)|\leq c_1/(\sqrt{m}\,)^3. \end{cases}$$

Remark. In [2] we considered the fields $K=\mathbb{Q}(\vartheta,i\sqrt{m}\,)$, where ϑ is a root of $f(x)=x^3-ax^2-(a+3)x-1$ and m is a square-free positive integer. By Theorem 3.1 of [2] if $a\geq 3$ and $m\geq m_0$, then there is no power integral basis in the order $\mathcal{O}$ of K.

3. Results

Let

$$f_n(x)=x^3-nx^2-(n+1)x-1 \tag{8}$$

where $n\in\mathbb{N}$. If $n\geq 3$, then $f_n(x)$ is totally real. Let $\vartheta=\vartheta_n$ be a root of $f_n(x)$ and let m be a square-free positive integer. Consider the two-parametric family $K=\mathbb{Q}(\vartheta,i\sqrt{m}\,)$ of totally complex sextic fields. Define ω as in (2) and set $\mathcal{O}=\mathbb{Z}[1,\vartheta,\vartheta^2,\omega,\omega\vartheta,\omega\vartheta^2]$ with discriminant $D_{\mathcal{O}}$ as before. We also use $L=\mathbb{Q}(\vartheta)$ and $M=\mathbb{Q}(i\sqrt{m}\,)$. Put

$$m_0=\begin{cases} 36, & \text{if } -m\equiv 1 \pmod 4,\\ 9, & \text{if } -m\equiv 2,3 \pmod 4. \end{cases}$$

Theorem 1. *Assume that $n \geq 7$ and $m \geq m_0$. Then the order $\mathcal{O}$ has no power integral basis.*

The proof of this theorem uses arguments similar to [2], but we will consider the small parameters, as well. It means that we will deal with the cases, which do not satisfy $n \geq 7$ or $m \geq m_0(n)$. Note that if $n \leq 2$, then (8) is not totally real, so we have to deal additionally only with the cases, when $n = 3, 4, 5, 6$. Using similar tools as in the proof of Theorem 1, we can show that for $m \geq m_0(n)$ the order $\mathcal{O}$ admits no power integral basis.

Theorem 2. *Set*

$$m_0(3) = \begin{cases} 143, & \text{if } -m \equiv 1 \pmod 4, \\ 36, & \text{if } -m \equiv 2, 3 \pmod 4. \end{cases}$$

$$m_0(4) = \begin{cases} 59, & \text{if } -m \equiv 1 \pmod 4, \\ 15, & \text{if } -m \equiv 2, 3 \pmod 4. \end{cases}$$

$$m_0(5) = \begin{cases} 42, & \text{if } -m \equiv 1 \pmod 4, \\ 11, & \text{if } -m \equiv 2, 3 \pmod 4. \end{cases}$$

$$m_0(6) = \begin{cases} 36, & \text{if } -m \equiv 1 \pmod 4, \\ 9, & \text{if } -m \equiv 2, 3 \pmod 4. \end{cases}$$

For $n = 3, 4, 5, 6$, $m \geq m_0(n)$, the order $\mathcal{O}$ has no power integral basis.

Further, for $n = 3, 4, 5, 6$ by performing direct computation for the small values of m we get:

Theorem 3. *If $n = 3, 4, 5, 6$ and $2 \leq m_0 < m_0(n)$, then $\mathcal{O}$ has no power integral basis.*

In the proof of our statements we shall use the result of Mignotte and Tzanakis [12] on the solutions of the Thue equations corresponding to the family (8). Note that in [12] the equation is solved only for sufficiently large parameters, but later on the result was extended to the range $n \geq 3$ (private communication).

Lemma 3 (cf. [12], Theorem). *If $n \geq 3$, then all solutions of the equation*

$$N_{L/Q}(x - \vartheta y) = \pm 1 \qquad (x, y \in \mathbb{Z})$$

are $(x, y) = (\pm 1, 0), (0, \pm 1), (\pm 1, \mp 1), (\pm 1, \mp n), (\pm(n+1), \pm 1)$.

Proof of Theorem 1. In our situation we apply Lemma 1 and Lemma 2 with $I_0 = I_1 = 1$. Denote by $\vartheta = \vartheta_1 < \vartheta_2 < \vartheta_3$ the roots of (8) and put $\rho = -\vartheta_2 - \vartheta_3$ that is $\rho_j = \vartheta_j - n$ for $1 \le j \le 3$. We have

$$\frac{-n}{n+1} \le \vartheta_1 \le \frac{-(n-1)}{n}$$
$$\frac{-1}{n-1} \le \vartheta_2 \le \frac{-1}{n}$$
$$n+1 \le \vartheta_3 \le n+2.$$

The above inequalities imply:

$$\rho_2 - \rho_1 = \vartheta_2 - \vartheta_1 \ge 1 - \left(\frac{1}{n-1} + \frac{1}{n}\right)$$
$$\rho_3 - \rho_2 = \vartheta_3 - \vartheta_2 \ge n + 1 + \frac{1}{n}$$
$$c_2 \ge 1 - \left(\frac{1}{n-1} + \frac{1}{n}\right)$$
$$c_3 \ge (n+1)\left(1 - \left(\frac{1}{n-1} + \frac{1}{n}\right)\right)$$
$$c_4 \le \max\left\{\frac{2}{c_2}, \frac{4c_m}{c_3\sqrt{m}}\right\} \le \max\left\{\frac{2}{c_2}; \frac{8}{c_3}\right\}$$
$$c_5 \le \left(\frac{8}{c_2 c_3}\right)^{1/3}$$

In case $n \ge 7$, we have $c_4 < 3$ and $c_5 < 1.3$.

Arrange the conjugates of any $\gamma \in K$ so that $\gamma^{(j+3)}$ is the conjugate of $\gamma^{(j)}$ over M $(j = 1, 2, 3)$, $\vartheta^{(j)} = \vartheta^{(j+3)} = \vartheta_j$ and $\omega^{(j)} = \omega$, $\omega^{(j+3)} = \bar{\omega}$ (conjugate of ω over M) $(j = 1, 2, 3)$.

Set

$$\beta = g \cdot \alpha,$$
$$\beta^{(j)} = x_0 + x_1 \vartheta^{(j)} + x_2 {\vartheta^{(j)}}^2 + y_0 \omega^{(j)} + y_1 \omega^{(j)} \vartheta^{(j)} + y_2 \omega^{(j)} {\vartheta^{(j)}}^2,$$

$$\beta_{jk} = \beta^{(j)} - \beta^{(k)}, \quad 1 \le j,\ k \le 6.$$

The proof of Theorem 2.1 of [2] implies that the expressions

$$\begin{aligned} d_1 &= |\beta_{14}\beta_{25}\beta_{36}|, \\ d_2 &= |\beta_{12}\beta_{23}\beta_{31}\beta_{45}\beta_{56}\beta_{64}|, \\ d_3 &= \beta_{15}\beta_{16}\beta_{24}\beta_{26}\beta_{34}\beta_{35} \end{aligned} \tag{9}$$

attain integer values with $d_1 \cdot d_2 \cdot d_3 = \pm I_1$. In our case $I_1 = 1$, hence $d_1 = d_2 = 1$ (cf. equations (5), (6)) and $d_3 = \pm 1$.

A. Consider now the solutions with $|Y| \le c_4$. In view of $m \ge m_0$ it implies $y_2 = 0$ and $|x_2| \le 2$ both for $-m \equiv 1 \pmod 4$ and for $-m \equiv 2, 3 \pmod 4$. For $y_2 = 0$ equation (6) reduces to

$$N_{L/Q}(y_0 + y_1\vartheta) = \pm 1,$$

whence by Lemma 3 we have $(y_0, y_1) = (\pm 1, 0), (0, \pm 1), (\pm 1, \mp 1), (\pm 1, \mp n)$, $(\pm(n+1), \pm 1)$.

A1. For $y_0 = \pm 1$, $y_1 = 0$ equation (5) becomes

$$N_{K/M}(x_1 - \rho x_2) = \pm 1.$$

Using again, Lemma 3 we obtain

$$(x_1, x_2) = (\pm 1, 0), (\mp n, \pm 1), (\pm(n+1), \mp 1), (\pm(n^2+1), \mp n), (\pm 1, \pm 1).$$

For these values we tested the third factor (9). Using symmetric polynomials we calculated coefficients of the third factor, and substituted the above five pairs of (x_1, x_2).

a11. In case $-m \equiv 1 \pmod 4$:

1. For $y_0 = \pm 1$, $x_1 = \pm 1$, $x_2 = 0$ we have

$$\begin{aligned} d_3 = &-m^3 - 6m^2 - 9m + 23 + (-6m^2 - 18m + 6)n \\ &+ (-2m^2 - 15m + 5)n^2 + (-2 - 6m)n^3 + (-m - 1)n^4. \end{aligned}$$

2. For $y_0 = \pm 1$, $x_1 = \mp n$, $x_2 = \pm 1$ we have

$$\begin{aligned} d_3 = &-m^3 - 2m^2 - m + 23 + (2m^2 + 2m + 6)n + (-2m^2 - 3m + 5)n^2 \\ &+ (2m - 2)n^3 + (-m - 1)n^4. \end{aligned}$$

3. For $y_0 = \pm 1$, $x_1 = \pm(n+1)$, $x_2 = \mp 1$ we have

$$\begin{aligned} d_3 &= -m^3 + 10m^2 - 25m + 23 + (-2m^2 + 10m + 6)n \\ &\quad + (-2m^2 + 9m + 5)n^2 + (-2m - 2)n^3 + (-m - 1)n^4. \end{aligned}$$

4. For $y_0 = \pm 1$, $x_1 = \pm(n^2+1)$, $x_2 = \mp n$ we have

$$\begin{aligned} d_3 &= -m^3 - 6m^2 - 9m + 23 + (12m^2 + 36m + 6)n \\ &\quad + (-2m^2 - 42m + 5)n^2 + (4m^2 + 24m - 2)n^3 \\ &\quad + (-2m^2 - 31m - 1)n^4 + 16mn^5 - 6mn^6 + 4mn^7 - mn^8. \end{aligned}$$

5. For $y_0 = \pm 1$, $x_1 = \pm 1$, $x_2 = \pm 1$ we have

$$\begin{aligned} d_3 &= -m^3 - 26m^2 - 169m + 23 + (-36m^2 - 468m + 6)n \\ &\quad + (-26m^2 - 662m + 5)n^2 + (-12m^2 - 624m - 2)n^3 \\ &\quad + (-2m^2 - 411m - 1)n^4 - 192mn^5 - 62mn^6 - 12mn^7 - mn^8. \end{aligned}$$

a12. In case $-m \equiv 2, 3 \pmod 4$:

1. For $y_0 = \pm 1$, $x_1 = \pm 1$, $x_2 = 0$ we have

$$\begin{aligned} d_3 &= -64m^3 - 96m^2 - 36m + 23 + (-96m^2 - 72m + 6)n \\ &\quad + (-32m^2 - 60m + 5)n^2 + (-2 - 24m)n^3 + (-4m - 1)n^4. \end{aligned}$$

2. For $y_0 = \pm 1$, $x_1 = \mp n$, $x_2 = \pm 1$ we have

$$\begin{aligned} d_3 &= -64m^3 - 32m^2 - 4m + 23 + (32m^2 + 8m + 6)n \\ &\quad + (-32m^2 - 12m + 5)n^2 + (8m - 2)n^3 + (-4m - 1)n^4. \end{aligned}$$

3. For $y_0 = \pm 1$, $x_1 = \pm(n+1)$, $x_2 = \mp 1$ we have

$$\begin{aligned} d_3 &= -64m^3 + 160m^2 - 100m + 23 + (-32m^2 + 40m + 6)n \\ &\quad + (-32m^2 + 60m + 5)n^2 + (-8m - 2)n^3 + (-4m - 1)n^4. \end{aligned}$$

4. For $y_0 = \pm 1$, $x_1 = \pm(n^2+1)$, $x_2 = \mp n$ we have

$$\begin{aligned} d_3 &= -64m^3 - 96m^2 - 36m + 23 + (192m^2 + 144m + 6)n \\ &\quad + (-32m^2 - 168m + 5)n^2 + (64m^2 + 96m - 2)n^3 \\ &\quad + (-32m^2 - 124m - 1)n^4 + 64mn^5 - 24mn^6 + 16mn^7 - 4mn^8. \end{aligned}$$

5. For $y_0 = \pm 1$, $x_1 = \pm 1$, $x_2 = \pm 1$ we have

$$\begin{aligned} d_3 = &-64m^3 - 416m^2 - 676m + 23 + (-576m^2 - 1872m + 6)n \\ &+ (-416m^2 - 2648m + 5)n^2 + (-192m^2 - 2496m - 2)n^3 \\ &+ (-32m^2 - 1644m - 1)n^4 - 768mn^5 - 248mn^6 - 48mn^7 - 4mn^8. \end{aligned}$$

None of them can attain the values ± 1.

A2. For $y_1 = \varepsilon = \pm 1, \pm n$, $y_2 = 0$ equation (5) gets the form

$$N_{K/M}((x_1 + \varepsilon\omega) - \rho x_2) = \mu,$$

where $x_1 \in \mathbb{Z}$; $x_2 = 0, \pm 1, \pm 2$; $|\mu| = 1$. This equation can be written in the form

$$\prod_{i=1}^{3} \big((x_1 + \varepsilon\omega) - \rho_i x_2\big) = \mu.$$

Set $\gamma_i = x_1 + \varepsilon\omega - \rho_i x_2$ $(i = 1, 2, 3)$. Then

$$\operatorname{Im}(\gamma_i) = \operatorname{Im}(\varepsilon\omega) = \begin{cases} \dfrac{\varepsilon\sqrt{m}}{2}, & \text{if } -m \equiv 1 \pmod 4 \\ \varepsilon\sqrt{m}, & \text{if } -m \equiv 2, 3 \pmod 4). \end{cases}$$

In view of $m \geq 9$ it implies $\frac{\sqrt{m}}{2} > 1$, so $|\operatorname{Im}(\gamma_i)| > 1$. Because of above condition $|\gamma_i| \geq |\operatorname{Im}(\gamma_i)| > 1$ $(i = 1, 2, 3)$, thus $|\gamma_1\gamma_2\gamma_3| > 1$ contradicting $\gamma_1\gamma_2\gamma_3 \neq \mu$, if $|\mu| = 1$. We have no solutions in these cases, either.

B. We still have to check the solutions with $|Y| > c_4$. In case $-m \equiv 1 \pmod 4$ by Lemma 2 either $|y_2| < \frac{2c_5}{\sqrt{m}} < 1$, that is $y_2 = 0$, or in the opposite case if $|y_2| \geq \frac{2c_5}{\sqrt{m}} > 0$, then

$$|F(y_1, y_2)| \leq \frac{c_1}{\sqrt{m}^3} = \frac{9c_m^3}{\sqrt{m}^3} \leq \frac{72}{6^3} < 1$$

implying $y_1 = y_2 = 0$ (for $m \geq m_0$) and contradiction with $|y_2| > 2$. Hence $y_2 = 0$. We get the same result for $-m \equiv 2, 3 \pmod 4$ with $2c_5/\sqrt{m}$ replaced by $c_5/\sqrt{m}$. Now equation (6) reduces to

$$N_{L/Q}(y_0 + \vartheta y_1) = \pm 1$$

whence by Lemma 3 we conclude

$$(y_0, y_1) = (\pm 1, 0);\ (0, \pm 1);\ (\pm 1, \mp 1),\ (\pm 1, \mp n),\ (\pm(n+1), \pm 1).$$

B1. The case $y_0 = \pm 1,\ y_1 = 0$ was already considered in *A1.*

B2. If $y_1 = \varepsilon = \pm 1, \pm n,\ y_2 = 0$, then by Lemma 2 we get $x_2 = 0$. Then equation (5) becomes

$$N_{K/M}(x_1 + \varepsilon\omega) = \pm 1.$$

It means that $|x_1 + \varepsilon\omega| = 1$. Both for $-m \equiv 1 \pmod 4$) and for $-m \equiv 2, 3 \pmod 4$ we obtained the same result, namely the above equation has no solution over $\mathbb{Z}$. Theorem 1 is proved. □

4. Small values of parameters

The proof of Theorem 2, the cases $n = 3, 4, 5, 6,\ m \geq m_0(n)$ is similar to the proof of Theorem 1. This requires a simple but tedious calculation and considering several cases. We omit the details of the proof of Theorem 2. Consider now a coordinate system displaying the points $(m, n) \in \mathbb{Z}^2$. Note that for $n = 3, 4, 5, 6, n \geq 7$ we display the values of $m_0(n)$ which is different for $-m \equiv 1 \pmod 4$ and $-m \equiv 2, 3 \pmod 4$.

The shaded part of this diagram shows the pairs (m, n) covered by Theorems 1, 2.

It is a very interesting problem to examine the cases, when $n \geq 3$ and $m < m_0(n)$. Also in this case we apply Lemma 2.

In case $|Y| > c_4$ we have to solve the Thue inequalities of Lemma 2. This can be done by KASH [1]. This way we get a couple of possible vectors $(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4$. For fixed y_1, y_2 the corresponding value of y_0 can be found from equation (6) which is then a cubic polynomial equation in y_0.

In case $|Y| \leq c_4$ we can determine the possible values of (x_2, y_2) by this condition. For fixed $Y = x_2 + \omega y_2$ equation (5) in Lemma 1 is a cubic polynomial equation in $X = x_1 + \omega y_1$, which can be solved by MAPLE. We were looking for the solutions of X, in which x_1 and y_1 are integers. The value of y_0 we can get from equation (6) similarly as above.

Finally, the possible tuples $(x_1, x_2, y_0, y_1, y_2)$ were tested if the corresponding $\alpha \in \mathcal{O}$ satisfies

$$I(\alpha) = 1.$$

This test eliminated all possible solutions.

5. Computational experiences

The computation involved in our results were performed in Maple, except for solving Thue equations which was done by Kash. Note that the "thue" procedure of Maple is only supposed to find small (and not all) solutions of Thue inequalities, but unfortunately in some cases it failed to find even the small ones.

The total CPU time (used mainly for the proof of Theorem 3) was about 5 hours on an IBM compatible 233MHZ Pentium II PC.

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PÉTER OLAJOS
INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF DEBRECEN
H–4010 DEBRECEN P. O. BOX 12
HUNGARY

E-mail: olaj@tigris.klte.hu

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