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# A new application of quasi power increasing sequences

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*Dedicated to Professor Mátyás Arató on his 70th birthday*

**Abstract.** The aim of this note is to extend a theorem of H. Bor [2], which utilizes the concept of almost increasing sequences, to the class of quasi $\beta$-power increasing sequences $0 < \beta < 1$, which is a wider class of sequences, and no additional assumption will be taken on that used by Bor.

## 1. Introduction

Let $\sum_{n=1}^{\infty} a_n$ be a given series with partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n := \sum_{\nu=0}^{n} p_\nu \to \infty.$$

The sequence-to-sequence transformation

$$t_n := \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$$

defines the sequence $\{t_n\}$ of the $(\overline{N}, p_n)$ mean of the sequence $\{s_n\}$, generated by the sequence $\{p_n\}$ (see [4]). The series $\sum_{n=1}^{\infty} a_n$ is said to be

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summable $|\overline{N}, p_n|_k$, $k \ge 1$, if (see [1])

$$\sum_{n=1}^{\infty} \Big(\frac{P_n}{p_n}\Big)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special case $p_n = 1$ the $|\overline{N}, p_n|_k$ summability reduces to the $|C, 1|_k$ summability (see [3]).

Concerning the $|C, 1|_k$ summability factors Mishra and Srivastava [5] had proved a theorem for nondecreasing sequences, and Bor [2] extended it to $|\overline{N}, p_n|_k$ summability under a weaker condition, namely he considers the so-called "almost increasing sequences". A positive sequence $\{b_n\}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}$ and two positive constants $A$ and $B$ such that $A\, c_n \le b_n \le Bc_n$.

The theorem of Bor reads as follows:

**Theorem A.** *Let $\{X_n\}$ be an almost increasing sequence and let the condition*

$$\sum_{n=1}^{m} \frac{1}{n} |s_n|^k = O(X_m) \tag{1.1}$$

*satisfied. If the sequences $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions*

$$|\Delta\lambda_n| \le \beta_n, \tag{1.2}$$

$$\beta_n \to 0, \tag{1.3}$$

$$\sum_{n=1}^{\infty} n |\Delta\beta_n| X_n < \infty, \tag{1.4}$$

$$|\lambda_n| X_n = O(1), \tag{1.5}$$

*furthermore if $\{p_n\}$ is a positive sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = O(X_m), \tag{1.6}$$

*then the series $\sum_{n=1}^{\infty} a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \ge 1$.*

We observe that if $\{X_n\}$ is a positive nondecreasing sequence and $p_n$=1 for all $n$, then Theorem A reduces to that of Mishra and Srivastava [5].

In order to release our first result we need the definition of the quasi $\beta$-power increasing sequence. A positive sequence $\{\gamma_n\}$ is said to be quasi $\beta$-power increasing sequence if there exists a constant $K = K(\gamma) \geq 1$ such that

$$Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m \tag{1.7}$$

holds for all $n \geq m \geq 1$.

## 2. Results

**Theorem.** *Let $\{X_n\}$ be a quasi $\beta$-power increasing sequence for some $0 < \beta < 1$. If all of the conditions from* (1.1) *to* (1.6) *are satisfied, then the series $\sum_{n=1}^{\infty} a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

Our next proposition affirms that the class of almost increasing sequences is a strict subclass of the quasi $\beta$-power increasing sequences if $\beta > 0$.

**Proposition.** (i) *Every almost increasing sequence is quasi $\beta$-power increasing for any nonnegative $\beta$, but the converse is not true if $\beta > 0$.*

(ii) *Moreover for any positive $\beta$ there exists a quasi $\beta$-power increasing sequence tending to infinity, but it is not almost increasing.*

## 3. Proofs

Proof of Theorem. The proof given by Bor has the following arrangement. First it is proved that under the conditions on $\{X_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ given in Theorem A, the following inequalities

$$n\beta_n X_n = O(1), \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \tag{3.2}$$

hold (see Lemma 3 in [2]).

The assumption that $\{X_n\}$ is an almost increasing sequence is utilized only in the proof of the inequalities (3.1) and (3.2).

The rest of the proof uses only the conditions (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6), furthermore the inequalities (3.1) and (3.2). Consequently we can declare that in [2] it is proved implicitly that the assumptions from (1.1) to (1.6) and the inequalities (3.1) and (3.2) imply that the series $\sum a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \ge 1$. So we have only to verify that the inequalities (3.1) and (3.2) also hold under the conditions (1.3), (1.4) and our assumption, that is, if it is assumed only that the sequence $\{X_n\}$ is quasi $\beta$-power increasing sequence with some $0<\beta<1$, instead of being almost increasing one.

Since $0<\beta<1$ thus, by (1.7), for any $\nu \ge n$

$$nX_n \le K\nu X_\nu$$

also holds, whence, by (1.3) and (1.4), it follows that

$$nX_n\beta_n \le nX_n \sum_{\nu=n}^{\infty} |\Delta\beta_\nu| \le K \sum_{\nu=n}^{\infty} \nu X_\nu |\Delta\beta_\nu| < \infty.$$

This clearly verifies (3.1).

To prove (3.2) we make the following consideration:

$$\begin{aligned}\sum_{n=1}^{\infty} X_n\beta_n &\le \sum_{n=1}^{\infty} X_n \sum_{\nu=n}^{\infty} |\Delta\beta_\nu| = \sum_{\nu=1}^{\infty} |\Delta\beta_\nu| \sum_{n=1}^{\nu} X_n \\ &\le \sum_{\nu=1}^{\infty} |\Delta\beta_\nu| \sum_{n=1}^{\nu} n^\beta X_n n^{-\beta} \\ &\le \sum_{\nu=1}^{\infty} |\Delta\beta_\nu| K\nu^\beta X_\nu \sum_{n=1}^{\nu} n^{-\beta} \\ &\le K \sum_{\nu=1}^{\infty} |\Delta\beta_\nu| K(\beta)\nu X_\nu,\end{aligned}$$

where $K(\beta)$ is a constant depending only on $\beta$. Hence, by (1.4), we get (3.2).

Herewith, as above we have discussed, our Theorem is proved. □

Proof of Proposition. First we prove the statement (i). If the sequence $\{\gamma_n\}$ is almost increasing, that is, if

$$A\ c_n \le \gamma_n \le B\ c_n,$$

holds for all $n$ with an increasing sequence $\{c_n\}$, then for any $n \geq m \geq 1$

$$\gamma_m \leq B\ c_m \leq B\ c_n \leq \frac{B}{A}\gamma_n \tag{3.3}$$

also upholds, whence (1.7) follows obviously for any $\beta \geq 0$ with $K := \frac{B}{A}$. This shows that any almost increasing sequence is quasi $\beta$-power increasing for any $\beta \geq 0$.

The converse is clearly not true, see e.g. the sequence $\gamma_n := n^{-\beta}$, $\beta > 0$. Then $\gamma_n \to 0$, thus it is obviously not an almost increasing sequence.

Herewith the statement (i) is verified.

In order to prove assertion (ii) we define the following sequence $\{\gamma_n\}$. Let us assume that $\beta > 0$ and $\mu_m := 2^{2^m}$. Then let

$$\gamma_n := \begin{cases} \mu_m, & \text{if } n = \mu_m, \\ \mu_m^{1+\beta} n^{-\beta}, & \text{if } \mu_m < n \leq m\mu_m, \\ \mu_m m^{-\beta}, & \text{if } m\mu_m < n < \mu_{m+1}. \end{cases} \tag{3.4}$$

It is evident that $\gamma_n$ tends to infinity, and

$$\frac{\gamma_{\mu_m}}{\gamma_{m\mu_m}} = m^\beta. \tag{3.5}$$

As we have seen above, every almost increasing sequence satisfies the inequality (3.3), thus (3.5) demonstrates that the sequence $\{\gamma_n\}$ given in (3.4) is not an almost increasing one, namely $m^\beta \to \infty$ if $m \to \infty$.

Next we show that our sequence is quasi $\beta$-power increasing. This follows from that the sequence $\{n^\beta \gamma_n\}$ between $\mu_m$ and $m\mu_m$ is constant, and from $m\mu_m$ to $\mu_{m+1}$ it is increasing. Consequently the whole sequence $\{\gamma_n\}$ is quasi $\beta$-power increasing.

The proof is complete. □

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