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Some criteria for dynamical systems to be nonchaotic

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Abstract. In this paper we address the important question that under what conditions a dynamical system can be free of chaos. First we discuss some features of chaotic dynamical systems described by autonomous ODE, then we establish some criteria to determine the nonchaoticity of dynamical systems and apply them to some specific systems.

1. Introduction

It is important to know under what conditions a dynamical system can be chaotic and much has been done along this line [1]–[3]. On the other hand, it also of significance to know when a dynamical system is nonchaotic. The aim of this paper is to establish some criteria for dynamical systems not being chaotic.

2. Some properties of chaotic dynamical systems

In this section we examine some elementary properties of chaotic dynamical systems, which are useful in establishing conditions for nonchaotic behavior in a dynamical system.

Consider a dynamical system described by ODE:

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \quad f \in c^1[\mathbb{R}^n, \mathbb{R}^n]. \tag{2.1}$$

Recall that a fundamental fact about a chaotic orbit is that it is a compact set and cannot be contained in a two-dimensional manifold.

The first property is the following

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Theorem 2.1. *If $\phi(t, t_0, x_0) = (x_1(t), \ldots, x_n(t))$ is a chaotic orbit to* (2.1) *with the initial point x_0, then there exist at least three components of ϕ with the property that all of them possess an infinite number of zero points. Here a zero point for $x_i(t)$ means a time value $s \in \mathbb{R}$ such that $\dot{x}_i(s) = 0$.*

PROOF. Suppose that this is not the case. Then it can be assumed, for simplicity, that the components $x_1, (t), \ldots, x_{n-2}(t)$ have just a finite number of zero points. In this case it is easy to see that there exists a $T > 0$ such that for $t > T$,

$$\dot{x}_i(t) \neq 0, \qquad i = 1, 2, \ldots, n-2.$$

On the other hand, due to the boundedness of the orbit $\phi(t, t_0, x_0) = (x_1(t), \ldots, x_n(t))$, the ω-limit set $\omega(x_0)$ is a nonempty compact set in $\mathbb{R}^n$. For a point $p \in \omega(x_0)$, there exists a sequence $t_i \to \infty$ such that

$$\lim_{i \to \infty} \phi(t_i, t_0, x_0) = p = (p_1, p_2, \ldots, p_n).$$

Now $\dot{x}_i(t) \neq 0$ for $t > T$, $1 \leq i \leq n-2$, therefore $x_i(t)$ is increasing or decreasing. In either of these cases, one has $x_i(t) \to p_i$ as $t \to \infty$, $1 \leq i \leq n-2$. It follows that $\phi(t_i, t_0, x_0)$ approaches the two-dimensional pane $x_1 = p_1, \ldots, x_{n-2} = p_{n-2}$. Since chaotic behavior cannot occur on a two-dimensional space in case when the system is described by ODE, the above arguments show that $\phi(t, t_0, x_0)$ is not chaotic, leading to a contradiction. □

More generally, one has the following

Theorem 2.2. *Let $\phi(t) = \phi(t, t_0, x_0)$ be a chaotic orbit to* (2.1) *contained in a compact region B, then there are at most $n - 3$ functions $F_1, \ldots, F_m \in c^2[B, \mathbb{R}]$ which are functionally independent (i.e.,* $\operatorname{Rank} \partial(F_1, \ldots, F_m) = m$*) such that every equation*

$$\frac{dF_j(\phi(t))}{dt} = 0, \qquad 1 \leq j \leq m$$

possesses only a finite number of zero points.

PROOF. Suppose that there exist more than $n - 3$ functionally independent functions $F_1, \ldots, F_m \in c^2[B, \mathbb{R}]$ defined on B such that every equation

$$\frac{dF_j(\phi(t))}{dt} = 0$$

has a finite number of zero points, then there exists a number $T > 0$ such that

$$\frac{dF_j(\phi(t))}{dt} \neq 0, \qquad j = 1, \dots, m$$

for $t > T$.

On the other hand, the chaotic behavior of $\phi(t)$ means that $\phi(t)$ is bounded, so every $F_j(\phi(t))$ has a finite limit value c_i as $t \to \infty$, due to the monotonicity for $t > T$. Therefore $\phi(t)$ approaches the level set $F^{-1}(c)$, where $F = (F_1, \dots, F_m)$, $c = (c_1, \dots, c_m)$. Because of functional independence of $F = (F_1, \dots, F_m)$ on B, $c = (c_1, \dots, c_m)$ is a regular value of the map F, and this means that the level set $F^{-1}(c)$ is a manifold of less than three dimensions. Because chaotic behavior cannot occur on a manifold of less than three dimensions, one gets a contradiction, thus completing the proof. □

3. Nonchaotic conditions for three-dimensional dynamical systems

For three-dimensional dynamical systems described by ODE, we can present more detailed results on nonchaoticity.

Consider the system

$$\text{(3.1)} \qquad \begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3). \end{cases}$$

We first have the following

Theorem 3.1. *Suppose that one of the f_i's is of the form*

$$f_i = x_i g(x_1, x_2, x_3),$$

and

$$f_j \neq 0 \quad \text{for } x_i \neq 0,\ j \neq i,$$

then the system (3.1) *does not exhibit chaotic behavior.*

Proof. It is easy to see that the plane $x_i = 0$ is an invariant manifold of (3.1), therefore it is enough to consider the case when the orbit $\phi(t)$ is not contained in this plane; this implies that the orbit $\phi(t)$ has a component $x_i(t) \neq 0$ which consequently means that $\dot{x}_j(t) \neq 0$. In view of Theorem 2.1, $\phi(t)$ is not chaotic. This completes the proof. □

Theorem 3.2. *If there exists a c^1-function $V(x)$ defined on $\mathbb{R}^3$ satisfying*

$$\operatorname{grad} V(x) \bullet (f_1, f_2, f_3) \leq 0 \ (\geq 0), \tag{3.2}$$

and the set of critical points of V consists of isolated points, curves or surfaces, then (3.1) *is not chaotic.*

Proof. Let $\phi(t)$ be a nontrivial orbit to (3.1). The inequality (3.2) guarantees that the function $V(\phi(t))$ is monotonously decreasing or increasing, therefore $V(\phi(t))$ has a limit $-\infty < c < \infty$ or $c = \pm\infty$. In the first case, $\phi(t)$ is contained in the level set $V^{-1}(c)$ and consequently cannot be chaotic. In the later case, $\phi(t)$ is an unbounded orbit. Therefore no chaotic behavior occurs in (3.1). $\square$

Similarly, it is easy to prove the following

Theorem 3.3. *Suppose that there exists an invariant manifold M to* (3.1) *and a c^1-function $V(x)$ defined on $\mathbb{R}^3$ such that*

$$\operatorname{grad} V(x) \bullet (f_1, f_2, f_3) \neq 0, \quad x \notin M,$$

then (3.1) *is not chaotic.*

Consider the third order dynamical system

$$\dddot{x} = f(x, \dot{x}, \ddot{x}). \tag{3.3}$$

We have the following

Corollary 3.4. *If f is of the form*

$$f = \ddot{x} g(x, \dot{x}, \ddot{x}) \tag{3.4}$$

then (3.3) *is not chaotic.*

Proof. Rewrite (3.3) as

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = f(x, y, z). \end{cases} \tag{3.5}$$

By (3.4) one has $f = zg(x, y, z)$. In view of Theorem 3.1, one gets the conclusion. $\square$

4. Examples

In this section we discuss by means of the above results a few quadratic dynamical systems which [4] failed to cope with.

Example 4.1. Consider the following system:

$$\dot{x} = xy + y^2, \quad \dot{y} = z, \quad \dot{z} = x. \tag{4.1}$$

Statement 4.1. *The system* (4.1) *does not exhibit chaotic behavior.*

This assertion is not as easy to prove as may be expected, and it needs delicate analysis as seen in the following

Proof. It is easy to see that the trivial orbit to (4.1) is just the equilibrium point $O = (0, 0, 0)$. The proof consists of three steps.

Step 1. Let $\phi(t) = (x(t), y(t), z(t))$ be a nontrivial orbit of (4.1). Suppose that $x(t^*) = 0$ and $y(t^*) = 0$, then t^* is the simple zero point of $y(t)$. If this is not the case, then there exists a sequence $t_i \to t^*$ as $i \to \infty$ such that $y(t_i) = y(t^*) = 0$. It follows that

$$\dot{y}(t^*) = \lim_{i\to\infty} \frac{y(t_i) - y(t^*)}{t_i - t^*} = 0.$$

This means by virtue of the third equation of (4.1) that $z(t^*) = 0$. Keeping in mind that $x(t^*) = 0$ and $y(t^*) = 0$, we see that $\phi(t) = (x(t), y(t), z(t))$ should be the trivial orbit, leading to a contradiction.

Step 2. Suppose that $x(t^*) = 0$ and $y(t^*) = 0$, then

$$x(t) = o(y(t)), \quad \text{as } t \to t^*. \tag{4.2}$$

In fact, from

$$\lim_{t\to t^*} \frac{x(t)}{y(t)} = \lim_{t\to t^*} \frac{\dot{x}(t)}{\dot{y}(t)} = \lim_{t\to t^*} \frac{x(t)y(t) + y^2(t)}{z(t)}$$

and the nontriviality of $\phi(t) = (x(t), y(t), z(t))$ that means $z(t^*) \neq 0$, we can see that

$$\lim_{t\to t^*} \frac{x(t)}{y(t)} = 0,$$

thus obtaining the statement.

Step 3. Now we show that if $x(t^*) = 0$ for some $t^* > 0$, then $x(t) > 0$ for $t > t^*$.

There are two cases to be discussed.

Case i: $y(t^*) \neq 0$. In this case, $\dot{x}(t^*) = y^2(t^*) > 0$. Now suppose that there exists another $t^\# > t^*$ such that $x(t^\#) = 0$ and $x(t) > 0$ for $t \in (t^*, t^\#)$, then an elementary analysis shows that there exists a neighborhood $(s, t^\#]$ such that

$$\dot{x}(t) \leq 0, \qquad t \in (s, t^\#). \tag{4.3}$$

If $y(t^\#) \neq 0$, then $\dot{x}(t^\#) = y^2(t^\#) > 0$, contradicting (4.3). If $y(t^\#) = 0$, then by (4.2) and Step 1, there exists a neighborhood of $t^\#$, A, such that

$$\dot{x}(t) = x(t)y(t) + y^2(t) > 0, \qquad t \in A - \{t^\#\}.$$

In particular, we have

$$\dot{x}(t) > 0, \qquad t \in A \cap (s, t^\#),$$

which again is leading to a contradiction with (4.3). Therefore no such $t^\#$ exists.

Case ii: $y(t^*) = 0$. In this case, again by (4.2) and Step 1, there exists a neighborhood of t^*, B, such that

$$\dot{x}(t) = x(t)y(t) + y^2(t) > 0, \qquad t \in B - \{t^\#\}.$$

It follows from this inequality that there exists an $s \in B$ such that $x(s) > 0$. Now suppose that there exists $s^\# > s$, such that $x(t) \neq 0$ for $t \in [s, s^\#)$ and $x(s^\#) = 0$, then there exists a $s^* \in [s, s^\#)$ such that

$$\dot{x}(t) \leq 0, \qquad t \in (s^*, s^\#].$$

Now, by the same arguments as in Case i, we come to the conclusion that no such $s^\#$ exists.

In view of all the above arguments, it is easy to see that $x(t) > 0$ for t large enough. By Theorem 2.1, we see that $\phi(t)$ is not chaotic. Thus, because $\phi(t)$ can be arbitrary, the systems considered are not chaotic. $\square$

Example 4.2. Consider the following system:

$$\dot{x} = x^2 + xy, \quad \dot{y} = yz, \quad \dot{z} = x. \tag{4.2}$$

Statement 4.2. *The system* (4.2) *is not chaotic.*

Proof. Noting that the plane $x = 0$ is an invariant manifold of (4.2), one gets the conclusion by virtue of Theorem 3.1. □

Example 4.3. Consider the following system:

$$\dot{x} = xy + xz, \quad \dot{y} = xy, \quad \dot{z} = y. \tag{4.3}$$

Because the plane $y = 0$ is an invariant manifold of (4.3), it is easy to get the following statement in view of Theorem 3.1:

Statement 4.3. *System* (4.3) *does not exhibit chaotic behavior.*

5. Conclusion

To develop some methods for detecting whether a dynamical system is chaotic or nonchaotic is a significant and very interesting topic which deserves much investigation. In this paper we present some criteria that are not difficult but efficient in coping with a class of dynamical systems described by ODE, and this is verified by successful application to the problem of nonchaoticity of quadratic systems posed in [4]. It is expected that more efficient methods and techniques will be developed in the near future.

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