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On the total curvature of hypersurfaces in negatively curved Riemannian manifolds

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Abstract. The total curvature of hypersurfaces is estimated in certain Hadamard manifolds.

0. Introduction

Let M^n be an n-dimensional Hadamard manifold, that is, a simply connected manifold with nonpositive sectional curvature and F be an $n-1$-dimensional smooth immersed hypersurface. Denote by $A_q : T_qF \to T_qF$ the shape operator of F at $q \in F$ and set $K(q) = \det A_q$. It is well defined up to sign and when M is the Euclidean space it is called the Gauss-Kronecker curvature. We adapt the same name for K in a Hadamard manifold although it is no longer an intrinsic quantity of the hypersurface.

It is well known that in case M is the Euclidean space and F is compact we have

$$\mathrm{Vol}(S^{n-1}) \deg S = \int_F K,$$

where S denotes the Gauss map of F. As a consequence we also have

$$\int_F |K| \geq \mathrm{Vol}(S^{n-1}). \tag{1}$$

This remains true for a general Hadamard manifold in dimensions $n = 2, 3$ as a result of the Gauss–Bonnet theorem. It seems natural to expect that the above statement will hold in higher dimensions as well.

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There is another important motivation for trying to show that (1) is satisfied for a general nonpositively curved manifold. This is the so called isoperimetric conjecture (see [3], [4]).

Isoperimetric Conjecture. *Let M^n be a Hadamard manifold and $D \subset M^n$ be a compact domain with smooth boundary. Then it satisfies the Euclidean isoperimetric inequality:*

$$\mathrm{area}(\partial D) \geq d_n (\mathrm{vol}(D))^{\frac{n-1}{n}},$$

where $d_n = \mathrm{area}(S^{n-1})/(\mathrm{vol}(B^n))^{\frac{n-1}{n}}$.

This is now settled in dimension 4 by [3] and in dimension 3 by [4]. In fact, the main part of the proof in [4] is to show how (1) implies the isoperimetric inequality. Although, it was carried out in dimension 3 only, it is very likely (and is explicitly mentioned in [4]), that it generalizes to higher dimensions. This means that a possible way of proving the isoperimetric conjecture is to establish (1) for a general Hadamard manifold.

The goal of this paper is to prove inequality (1) in certain situations, thereby making a case for the validity of (1) for a general Hadamard manifold. We have the following theorem.

Theorem 1. *Let M^n be a Hadamard manifold which is rotationally symmetric at $p \in M^n$ and $p \in E \subset M^n$ be an open subset with compact closure and a smooth boundary. Then we have*

$$\int_{\partial E} |K| \geq \mathrm{Vol}(S^{n-1}), \tag{1'}$$

where K denotes the determinant of the shape operator of the boundary.

If equality occurs, then E is flat, that is, E is isometric to a subset of the Euclidean space.

Since the n-dimensional hyperbolic space H^n is rotationally symmetric at any of its point we have the following corollary.

Corollary 1. *Let F be a closed $(n-1)$-dimensional manifold and $F \hookrightarrow H^n$ be an isometric immersion. Then*

$$\int_F |K| \geq \mathrm{Vol}(S^{n-1}).$$

There are several natural questions related to inequality (1') which could be studied in the context of general Hadamard manifolds. For example, it seems natural to expect that Theorem 1 holds in general. One might also try to generalize results of Chern and Lashof [2] to Hadamard manifolds.

Although, we have precise results about certain integrals on hypersurfaces due to Chern (the curvature integra [1]) the generalized Gauss–Bonnet–Chern theorem does not seem to help in higher dimension (at least not in an obvious way).

1. Rotationally symmetric manifolds

Let M^n be a rotationally symmetric Hadamard manifold at $p \in M^n$. We use the conformal model for M, that is, we think of M^n as an open Euclidean ball around p (possibly the whole $\mathbb{R}^n$) equipped with the metric:

$$ds^2 = f(r)^2 ds_E^2, \qquad f(0) = 1, \tag{2}$$

where ds_E denotes the natural metric of the underlying Euclidean space and r denotes the Euclidean distance from p.

We call a two-dimensional submanifold a radial plane if it is a radial plane in the underlying Euclidean space. It is obviously true that:

Claim 1. *Every radial plane is totally geodesic.*

We also need the following. Let $Z(p) \in T_pM^n$ be a unit tangent vector and denote by Z the vector field which we obtained from $Z(p)$ by parallel translation along geodesics from p.

Claim 2. *The orthogonal distribution $Z^\perp$ is integrable and the integral manifolds are hyperplanes in the Euclidean sense.*

Proof. Denote by $\tilde{Z}$ the vector field on M^n which is parallel in the Euclidean sense and $\tilde{Z}(p) = Z(p)$. From the special form of the metric one can easily deduce that

$$Z = \frac{1}{f(r)}\tilde{Z}.$$

Since $\tilde{Z}^\perp$ is obviously integrable and $Z^\perp = \tilde{Z}^\perp$ the claim follows. □

Denote by H_Z the family of integral manifolds of $Z^\perp$. We are going to show that every hypersurface of H_Z has a definite 2nd fundamental form. More precisely, let $q \in M^n$ be an arbitrary point different from p and denote by H_q the integral manifold of $Z^\perp$ passing through q. Denote by R the radial unit vector field ($R = \frac{1}{f(r)}\frac{\partial}{\partial r}$) defined on $M^n - \{p\}$. Then we have:

Proposition 1. *If $\langle Z, R\rangle \geq 0$ at q, then the 2nd fundamental form of H_q at q with respect to the normal field Z is positive semi-definite.*

Proof. Let $T(q)$ be a unit tangent vector tangent to H_q at q. We need to show that $\langle \nabla_T Z, T\rangle \geq 0$. Denote also by T the vector field defined on the geodesic segment $[pq]$ which is obtained from $T(q)$ by parallel translation. Now, we have two globally defined unit vector fields Z and R (actually R is defined only on $M^n - \{p\}$) and a unit vector field T defined only on the geodesic segment $[pq]$.

The following computation takes place along the geodesic segment $[pq] - \{p\}$. Since $\nabla_R Z$ is zero on $M^n - \{p\}$ and T is parallel along $[pq]$ we have

$$R\langle \nabla_T Z, T\rangle = \langle \nabla_R \nabla_T Z, T\rangle = \langle \mathbf{R}(T,R)Z, T\rangle - \langle \nabla_{[T,R]} Z, T\rangle, \tag{3}$$

where $\mathbf{R}$ is the curvature tensor.

Let us decompose $Z = aR + bT + Z_1$ along $[pq] - \{p\}$, where a, b are constants (since Z, T, R are all parallel along $[pq] - \{p\}$) and Z_1 is a vector field along $[pq] - \{p\}$ orthogonal to the two-plane determined by R and T. Since we assumed that $\langle Z, R\rangle \geq 0$ a simple computation shows that $a \geq 0$. Indeed, write

$$0 \leq \langle Z, R\rangle = a + b\langle T, R\rangle$$

and

$$0 = \langle Z, T\rangle = a\langle R, T\rangle + b.$$

Observing that T, R are unit vectors, the claim follows by substituting the expression for b into the first inequality.

We can also write $[T, R]$ along $[pq] - \{p\}$ in the form $[T, R] = c(r)T + d(r)R$. We will need the fact that $c(r) > 0$. Indeed, we have $[T, R] = \nabla_T R$ since $\nabla_R T = 0$ on $[pq] - \{p\}$. Write $T = T' + dR$ along $[pq] - \{p\}$, where T' is orthogonal to R and d is some constant since Z, R are parallel along $[pq] - \{p\}$. The shape operator of every ball centerd around p is a positive multiple of the identity operator, therefore we have $\nabla_T R = \nabla_{T'} R = \alpha(r)T'$, where $\alpha(r) > 0$. Taking into account that $T' = T - dR$

the fact follows. Since radial two-planes are totally geodesic (Claim 1) the curvature term $\langle \mathbf{R}(T,R)Z_1,T\rangle = 0$ and (3) becomes

$$R\langle \nabla_T Z, T\rangle = a\langle \mathbf{R}(T,R)R,T\rangle - c(r)\langle \nabla_T Z, T\rangle. \tag{4}$$

This is an ordinary differential equation for $\langle \nabla_T Z, T\rangle$ along the geodesic segment $[pq] - \{p\}$. Since Z is a globally defined smooth vector field $\langle \nabla_T Z, T\rangle$ is defined and is differentiable on the whole of $[pq]$ with initial value 0 at p. Since $a\langle \mathbf{R}(T,R)R,T\rangle \geq 0$ and $c(r) > 0$ the solution has to be also non-negative. This concludes the proof of the proposition. $\square$

2. Proof of Theorem 1

We will need an elementary fact from linear algebra. Let C, D be two positive definite matrices such that $C \leq D$, that is, $\langle CX, X\rangle \leq \langle DX, X\rangle$ for every X. Then $\det(C) \leq \det(D)$. This follows from Hadamard's inequality which states: for a positive definite matrix $C = [c_{ij}]$ we have $\det(C) \leq c_{11}c_{22} \cdot \ldots \cdot c_{nn}$. Equality occurs if and only if C is a diagonal matrix.

The method of the proof of Theorem 1 is the same as in the Euclidean case. We are going to estimate the determinant of the Gauss map.

Let $p \in \operatorname{int} E$ and let $Z(p) \in T_pM^n$ be an arbitrary unit tangent vector at p. As before we construct the vector field Z and the family of integral manifolds H_Z. If we think of M^n as a Euclidean ball equipped with the metric (2) it is clear from previous remarks (Claim 2) that H_Z is a family of parallel hyperplanes in the Euclidean sense. Let H be the supporting hyperplane of the set E such that E lies completely on one side of H and the outward unit normals at the intersection $H \cap \partial E$ are the same as the corresponding values of the vector field Z. Set $F_Z = H \cap \partial E \neq \emptyset$ and let F be the union of F_Z for all unit vectors $Z \in T_pM^n$. Then $F \subset \partial E$ and we are going to show that

$$\int_F |K| \geq \operatorname{Vol}(S^{n-1}).$$

This clearly implies Theorem 1.

Let $G_p : TM^n \to T_pM^n$ be the map defined by parallel translating vectors in T_qM^n to T_pM^n along the geodesic segment $[qp]$. This is clearly a differentiable map on TM^n which is linear on the fibers T_qM^n. Denote the restriction of G_p to T_qM^n by $G_{qp} : T_qM^n \to T_pM^n$.

For $q \in \partial E$ denote by $N(q)$ the outer unit normal and define the map $S : \partial E \to T_pM^n$ by $S(q) = G_p(N(q))$. This may be regarded as the generalization of the Gauss map. Then $dS : T\partial E \to TS^{n-1} \subset T_pM^n$, where S^{n-1} denotes the unit sphere in T_pM^n.

Let $q \in F$ be an arbitrary point. From the construction of F we know that there exists a unit vector $Z(p) \in T_pM^n$ such that the integral manifold H_q of the distribution $Z^\perp$ at q is tangent to ∂E, E lies completely on one side of H_q and the outward normal of ∂E coincides with Z at q. Here Z, as before, denotes the vector field obtained from $Z(p)$ by parallel translation along geodesics.

We are going to express dS in terms of the covariant derivatives. Let $T \in T_q\partial E$ be a unit vector and $\gamma : [0, \epsilon) \to \partial E$ be a curve emanating from q with $\gamma'(0) = T$. Then

$$\begin{aligned} dS(T) &= \lim_{t\to 0} \frac{G_p(N(\gamma(t))) - G_p(N(q))}{t} = \lim_{t\to 0} G_p\left(\frac{N(\gamma(t)) - Z(\gamma(t))}{t}\right) \\ &= G_p\left(\lim_{t\to 0} \frac{N(\gamma(t)) - Z(\gamma(t))}{t}\right) = G_p(\nabla_T(N - Z)). \end{aligned}$$

If we identify the tangent spaces T_pM^n and T_qM^n via the isometry G_{pq}, then $G_{qp}^{-1} \circ dS : T_q\partial E \to T_q\partial E$ is a symmetric map and

$$G_{qp}^{-1} \circ dS(T) = \nabla_T(N - Z) = \nabla_T N - \nabla_T Z.$$

The terms on the right hand side are the shape operators of ∂E and H_q at q which we denote by A_q and B_q, respectively. Since H_q "envelops" ∂E at q, that is, ∂E lies completely on one side of H_q and they have a common normal at q we conclude that $A_q \geq B_q$ in the sense that for every $T \in T_qH = T_q\partial E$ we have $\langle A_qT, T\rangle \geq \langle B_qT, T\rangle$. This implies that $G_{qp}^{-1} \circ dS$ is positive semi-definite. Since $p \in \operatorname{int} E$ from the construction of H_q it is clear that $\langle R, Z\rangle > 0$ at q. Taking into account Proposition 1 we conclude that $B_q \geq 0$, therefore

$$0 \leq G_{qp}^{-1} \circ dS \leq A_q.$$

From Hadamard's inequality one can easily get

$$0 \leq \det(dS) \leq \det(A_q) = |K|.$$

Since the Gauss map $S : F \to S^{n-1}$ is onto it implies

$$\mathrm{Vol}(S^{n-1}) \le \int_F \det(dS) \le \int_F |K| \le \int_{\partial E} |K|.$$

This concludes the proof of the inequality.

If equality occurs, then $\det(dS) = \det(A)$ at every point of F. Let $q \in \partial E$ be a point where $\mathrm{dist}(p, q)$ is maximal. We will show that q must belong to the set F. Let H_q denote the supporting hyperplane (in the Euclidean sense) of E at the point q. Since the metric is rotationally symmetric and the $\mathrm{dist}(p, q)$ is maximal, we conclude that the set E must lie on completely one side of H_q and from the definition of F the claim follows.

Clearly, the shape operator A_q at q is positive definite and from the equality case of Hadamard's inequality we conclude that $A_q = dS$ at q. This implies that H is flat at q ($B_q = 0$), that is, all the principal curvatures of H are zero at q. The vector fields Z and R introduced in the proof of Proposition 1 are equal at q, which means that $a = 1$ in (4). Since $\langle \nabla_T Z, T\rangle = 0$ at p and at q from the differential equation (4) we obtain that $\langle \nabla_T Z, T\rangle \equiv 0$ on $[pq]$, which implies that $\langle \mathbf{R}(T, R)R, T\rangle \equiv 0$ along the geodesic segment $[pq]$. Since $T \in T_q\partial E$ was arbitrary, we conclude that the radial sectional curvatures along $[pq]$ are zero. Taking into account the rotational symmetry we see that all the radial sectional curvatures are zero and since they completely determine the metric the theorem follows. $\square$

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