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On osculation and system of connections

By L. KOZMA (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

The aim of this paper is to give a relationship of between current notions of connection theory, namely, those of osculation and system of connections.

The osculation method comes from Finsler connection theory. O. Varga [Va] used this for introducing Cartan's connection in a Finsler space. Then A. Moór [Moór] generalized this method for the connections of the line element bundle, and recently [Koz] for arbitrary pull back bundles, thus a series of special cases can be considered at the same time. In Finslerian studies the recent paper of M. Matsumoto [Mat2] has given a new perspective for the notion of osculation. The osculation means that a connection in a pull back bundle is approximated along a fixed section by a connection for the base bundle in such a way that their geodesic curves have a contact of second order (for the case of tangent bundles).

A system of connections is, roughly speaking, a set of connections characterizable with parameters of finite number. This notion and that of a universal connection is a generalization [Mod] of the investigations for principal connections made by P.L. Garcia [Gar1]. This was utilized for gauge theory [Gar2] as well, and by means of it some information can be gained on the properties of the underlying bundle, too [Can, Del]. A detailed exposition of this theory can be found in [Mod, Cab].

Now, we can regard in a natural way the set of all osculating connection $(H^\sigma, \sigma \in \text{Sec}\ \xi_1)$ of a connection H^* for the pull back bundle $\pi_1^*(\xi_2)$, and ask whether this set is a system of connections, or not. The answer is only partially affirmative. $(H^\sigma, \sigma \in \text{Sec}\ \xi_1)$ is a system of connections iff the v-horizontal part of H^* is trivial. (See Theorem 3.)

1. Basic notions and notations

Manifolds, functions and maps are of class C^∞ as usual. $\xi = (E, B, \pi, F)$ denotes a vector bundle with a finite dimensional real fibre type F, a base manifold B, a total space E ($n+r$ dimensional), and a projection $\pi\colon E \to B$. $\operatorname{Sec}\xi$ denotes the set of all differentiable sections of ξ. When $f\colon M \to B$ is a map, $f^*(\xi)$ denotes the pull back bundle whose total space is $M \times_B E = \{(m,z) \mid m \in M, z \in E, f(m) = \pi(z)\}$ and its projection is $pr_1\colon M \times_B E \to M$. Then the second projection $pr_2\colon M \times_B E \to E$ is bundle map with the property that its restriction on a fibre of $f^*(\xi)$ is a linear isomorphism. V_zE is the kernel of $(d\pi)_z\colon T_zE \to T_{\pi(z)}B$. Then the vertical bundle $V\xi = (VE, E, \pi_V, F)$ is a subbundle of τ_E, and isomorphic to $\pi^*(\xi)$. The latter isomorphism is described as follows: $\varepsilon\colon \pi^*(\xi) \to V\xi$ $\varepsilon(z_1, z_2) =$ the tangent of the curve $z_1 + tz_2$ at 0, where $z_1, z_2 \in E$ with the property $\pi(z_1) = \pi(z_2)$.

By *a connection* of ξ we mean a splitting $H\colon \pi^*(\tau_B) \to \tau_E$ of the short exact sequence

$$0 \to V\xi \xrightarrow{\iota} \tau_E \xrightarrow{\widetilde{d\pi}} \pi^*(\tau_B) \to 0 \tag{1}$$

where $\widetilde{d\pi}\colon \tau_E \to \pi^*(\tau_B)$ is given by $\widetilde{d\pi}(A) = (\pi_E(A), d\pi(A))$ for $A \in TE$. H is also called a horizontal map, and its images $H_zE = \operatorname{Im} H|_{\{z\}\times T_{\pi(z)}B}$ are the horizontal subspaces which are complementary to the vertical subspaces: $\tau_E = V\xi \oplus H\xi$. Let $\varphi\colon I \to B$ a curve in the base space. A section $\sigma \in \operatorname{Sec}\xi$ is called parallel along φ if $d\sigma(\dot\varphi)$ are horizontal vectors, where $\dot\varphi$ denotes the tangent of φ. This means that $H(\sigma\circ\varphi, \dot\varphi) = d\sigma(\dot\varphi)$. From the elementary calculus it follows that for a given curve φ and a vector z in $E_{\varphi(0)}$ there exists — at least locally — a section σ which is parallel along φ and $z = \sigma(\varphi(0))$. $\sigma(\varphi(t))$ is called the parallel transport of z along φ, and denoted by $P_\varphi(t,z)$. The covariant derivation $\nabla\colon \mathcal{X}(B) \times \operatorname{Sec}\xi \to \operatorname{Sec}\xi$ is given as $\nabla_X\sigma = \alpha(v(d\sigma(X)))$ for all $X \in \mathcal{X}(B)$, $\sigma \in \operatorname{Sec}\xi$, where $\alpha = pr_2 \circ \varepsilon^{-1}\colon V\xi \to \xi$, and v is the vertical projection.

Denote $\mu_t\colon E \to E$ the multiplication by $t \in R$ in the fibres of ξ. The homogeneity condition for a horizontal map H says as

$$H(\mu_t(z), v) = d\mu_t(H(z,v)) \tag{2}$$

holds for all $z \in E$, $v \in TB$ and $t \in R$. When H satisfies (2) and is differentiable anywhere, then we get a *linear connection.* It is important that for a linear connection the parallel transport exists along the entire curve φ, and is linear.

2. Osculation of connections for $\pi_1^*(\xi_2)$

Let $\xi_1 = (E_1, B, \pi_1, F_1)$ and $\xi_2 = (E_2, \pi_2, B, F_2)$ be two real vector bundles with finite rank over the same base manifold B. Consider a connection H^* of the pull back bundle $\pi_1^*(\xi_2)$, i.e. a splitting $H^* : pr_1^*(\tau_{E_1}) \to \tau_{E_1 \times_B E_2}$ of the next short exact sequence

$$0 \to V\pi_1^*(\xi_2) \to \tau_{E_1 \times_B E_2} \to pr_1^*(\tau_{E_1}) \to 0$$

Let $\sigma \in Sec\xi_1$ be a fixed section. Then the map

$$H^\sigma : \pi_2^*(\tau_B) \to \tau_{E_2}$$

defined as

$$H^\sigma(z, v) := dpr_2(H^*(\beta_\sigma(z), d\sigma(v)))$$

is a horizontal map for ξ_2, i.e. a connection in ξ_2, where the map $\beta_\sigma : E_2 \to E_1 \times_B E_2$ is given as $\beta_\sigma(z) = (\sigma(\pi_2(z)), z)$.

This connection H^σ is called as an *osculating connection of H^* along $\sigma \in Sec\xi_1$* [Koz]. It is easy to show that if H^* is a linear connection in $\pi_1^*(\xi_2)$, then for any osculating connection H^σ along $\sigma \in Sec\xi_1$ is linear as well.

In order to investigate relationships between the structures at different levels we define a lift and a sunk of some sections.

Let $\eta \in \mathrm{Sec}\,\xi_2$ be an arbitrary section of ξ_2. The section $\eta^\uparrow$ of $\pi_1^*(\xi_2)$ defined as $\eta^\uparrow(z) = (z, \eta(\pi_1(z)))$ for all $z \in E^1$ is called as a lift of η. Let now $\sigma \in Sec\xi_1$ be a fixed, and $\Sigma \in \mathrm{Sec}\,\pi_1^*(\xi_2)$ an arbitrary section. Then the map $\Sigma_\sigma^\downarrow = pr_2 \circ \Sigma \circ \sigma$ is called as a sunk of Σ along $\sigma \in Sec\xi_1$.

The following theorem states [Koz] that the parallelity of sections is hereditary through lifting and sinking of sections with respect to osculation.

Theorem 1. *Let $\sigma \in Sec\xi_1$ be a fixed section, and $\varphi : I \to B$ an arbitrary curve in B. If $\Sigma \in \mathrm{Sec}\,\pi_1^*(\xi_2)$ is parallel along $\sigma \circ \varphi$ with respect to H^*, then the sunk $\Sigma_\sigma^\downarrow \in \mathrm{Sec}\,\xi_2$ is parallel along φ with respect to the osculating connection H^σ, too.*

On the other hand, if $\eta \in \mathrm{Sec}\,\xi_2$ is parallel along the curve φ with respect to H^σ , then the lift $\eta^\uparrow \in \mathrm{Sec}\,\pi_1^*(\xi_2)$ is also parallel along $\sigma \circ \varphi$ with respect to H^*.

This theorem shows us that the notion of osculation could be defined in terms of parallelism. Namely, starting from a parallelism structure P^* of the pull back bundle $\pi_1^*(\xi_2)$, fixing a section $\sigma \in Sec\xi_1$, an osculation parallelism structure P^σ can be defined as follows:

$$P_\varphi^\sigma := pr_2 \circ P_{\sigma\circ\varphi}^* \circ \beta_\sigma(z)$$

It can be verified that this type of osculation means the same as the previous one. A similar argument is valid for covariant derivations under the next theorem:

Theorem 2. *Denote ∇^* and ∇^σ the covariant derivations belonging to H^* and H^σ, resp. Then for any $X \in \mathcal{X}(B)$, $\eta \in \operatorname{Sec}\xi_2$ and $\Sigma \in \operatorname{Sec}\pi_1^*(\xi_2)$*

$$(\nabla^*_{d\sigma(X)}\eta^\uparrow)^\downarrow_\sigma = \nabla^\sigma_X\eta$$

$$(\nabla^*_{d\sigma(X)}\Sigma)^\downarrow_\sigma = \nabla^\sigma_X\Sigma^\downarrow_\sigma$$

hold.

An analogous relationship is valid for curvature structures through osculation. In [Koz] there are given the relationships for horizontal and vertical projections, and are regarded the metrical aspects of this osculation method as well.

3. A relationship between osculation and system of connections

In the study of system of connections we follow the ideas of M. Modugno [Mod] and A. Cabras [Cab].

Definition 1. Let ξ and $\tilde{\xi}$ be two vector bundles over the same base manifold B. A triple $(\xi, \tilde{\xi}, k)$ is called *a system of connections* for ξ with phase bundle $\tilde{\xi}$ and evaluation map k, where $k : \pi^*(\tilde{\xi} \oplus \tau_B) \to \tau_E$ is bundle morphism, where E is the total space of ξ.

Any member of a connection system arises by performing the evaluation map of the system along an arbitrary chosen section $\tilde{\xi}$. Namely, for any $\eta \in \operatorname{Sec}\tilde{\xi}$ the map k^η defined as

$$k^\eta(z, v) = k(z, \eta(\pi(z)), v) \quad \text{for all} \quad z \in E,\ v \in TB$$

gives a horizontal map for ξ. This means a member of the connection system.

L. Mangiarotti and M. Modugno proved in 1985 [Mag] in case of principal bundles, that for a given system of connections there exists a unique connection k^* for the pull back bundle $\tilde{\pi}^*(\xi)$ for which the following diagram is commutative for any $\eta \in \operatorname{Sec}\tilde{\xi}$

$$\begin{array}{ccc} (\tilde{E}\times_B E) \times_{\tilde{E}} T\tilde{E} & \xrightarrow{k^*} & T(\tilde{E} \times_B E) \\ \eta^* \uparrow & & \downarrow dpr_2 \\ E \times_B TB & \xrightarrow{k^\eta} & TE \end{array} \tag{3}$$

where $\eta^*(z, v) = (\eta(\pi(z)), z, d\eta(v))$. This connection k^* is called the *universal connection of k*. It was also proved [Cab] that the universal connection reflects the common properties of the members of the connection system.

The relationship (3) yields that the same coupling operates between k^* and k^η as it does between H^* and H^σ. Thus we can put the following simple question: whether the family of all osculating connections H^σ — regarding every possible choice of $\sigma \in Sec\xi_1$ — for a given connection H^* of the pull back bundle $\pi_1^*(\xi_2)$ forms a system of connections, or not. The answer is only partially affirmative.

Theorem 3. *The set of all osculating connections $\{H^\sigma, \sigma \in \operatorname{Sec}\xi_1\}$ is just the set of the members of a system of connections with phase bundle ξ_1 if and only if the v–horizontal part of H^* is trivial, i.e.*

$$(4) \qquad dpr_2(H^*(Z,U)) = 0 \quad \text{for all} \quad Z \in E_1 \times_B E_2, \quad U \in VE_1.$$

In this case H^* is just the universal connection of the system.

PROOF. $\{H^\sigma, \sigma \in \operatorname{Sec}\xi_1\}$ is a system of connections with phase bundle ξ_1 iff the following definition of an evaluation map is unique:

$$k(z_2, z_1, v) = H^\sigma(z_2, v), \quad \text{where} \quad \sigma(\pi_2(z_2)) = z_1.$$

This is just independent from the special choice of σ iff for any $\sigma_1, \sigma_2 \in \operatorname{Sec}\xi_1$ satisfying $\sigma_1(\pi(z_2)) = \sigma_2(\pi(z_2)) = z_1$ the equality $H^{\sigma_1}(z_2, v) = H^{\sigma_2}(z_2, v)$ is also valid, i.e.

$$dpr_2(H^*((z_1, z_2), d\sigma_1(v))) = dpr_2(H^*((z_1, z_2), d\sigma_2(v))).$$

It means that

$$dpr_2(H^*((z_1, z_2), d\sigma_1(v) - d\sigma_2(v))) = 0.$$

$d\sigma_1(v) - d\sigma_2(v)$ is a vertical vector in τ_{E_1}, thus the relationship

$$dpr_2(H^*((z_1, z_2), U)) = 0$$

is to satisfy for any U vertical vector. Q.e.d.

Corollary. *The universal connection k^* of a system of connections has a trivial v–horizontal part. The osculating connections of a universal connection are just members of the connection system.*

We note that a connection (for Finslerian case) satisfying (4) is called by M. MATSUMOTO [Mat1] as a “flat vertical” connection. In Finslerian context this assumption means exactly that the v–horizontal subspaces of H^* coincide with the v-induced vertical subspaces.

It is an open problem whether there exists another (broader) phase bundle for $\{H^\sigma, \sigma \in \operatorname{Sec}\xi_1\}$ when the assumption of this Theorem does not hold.

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L. KOZMA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H–4010 DEBRECEN PF. 12
HUNGARY

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