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Oscillation criteria for second order neutral functional differential equations

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Abstract. In this paper we establish some new criteria for the oscillation of the second order neutral functional differential equation

$$\Big(x(t)+p_1x(t-\tau_1)-p_2x(t+\tau_2)\Big)''=q_1(t)x\big(t-\sigma_1\big)+q_2(t)x\big(t+\sigma_2\big)$$

where both delayed and advanced arguments are included.

1. Introduction

Consider the second order neutral functional differential equation

$$(1)\quad \Big(x(t)+p_1x(t-\tau_1)-p_2x(t+\tau_2)\Big)''=q_1(t)x\big(t-\sigma_1\big)+q_2(t)x\big(t+\sigma_2\big)$$

under the following assumptions which are assumed to hold throughout this paper:

(a) p_i, τ_i, σ_i, $i=1,2$ are positive constants;

(b) $q_i\in C(\mathbb{R}_+,\mathbb{R}_+)$, $i=1,2$, $\quad\mathbb{R}_+=(0,\infty)$.

The aim of this paper is to obtain some sufficient oscillation conditions, involving the coefficients and the arguments only. The results are based on suitable comparison theorems. We shall introduce a new technique which enables us generalize some known results. We also show that a certain condition imposed on the coefficients of (1) in [6] can be relaxed.

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It is known that the problem of oscillatory and asymptotic behaviour of solutions of neutral differential equations is of both theoretical and practical interest. For recent contributions regarding the theoretical part we refer to the latest books by D. D. Bainov and D. P. Mishev [1], by I. Győri and G. Ladas [7], and by L. H. Erbe, Q. Kong and B. G. Zhang [5], and references cited therein.

By a solution of (1) we mean a continuous function x on $[T_x, \infty)$ such that $x(t)+p_1x(t-\tau_1)-p_2x(t+\tau_2)$ is twice continuously differentiable and $x(t)$ satisfies (1) for $t \geqslant T_x$. As is customary, a nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Main results

We begin with the following oscillation criterion, which is the key result of this paper and several oscillation criteria for (1) will be established on the base of this theorem.

To achieve our goals we shall use comparison theory of differential equations without neutral terms.

Theorem 1. *Let $\sigma_1 > \tau_1$ and $\sigma_2 > \tau_2$. Let $q_i^* \in C([t_0,\infty),\mathbb{R}_+)$ be such that $q_i(t+\tau_i) \leqslant q_i^*(t) \leqslant \min\{q_i(t), q_i(t-\tau_i)\}$, $i=1,2$. Assume that*

(i) *The differential inequality*

$$y''(t) - \frac{q_2^*(t)}{1+p_1} y(t+\sigma_2) \geqslant 0 \tag{2}$$

has no eventually positive increasing solution.

(ii) *The differential inequality*

$$y''(t) - \frac{q_1^*(t)}{1+p_1} y(t-\sigma_1+\tau_1) \geqslant 0 \tag{3}$$

has no eventually positive decreasing solution.

(iii) *The differential inequality*

$$u''(t) + \frac{q_1^*(t)}{p_2} u(t-\sigma_1-\tau_2) + \frac{q_2^*(t)}{p_2} u(t+\sigma_2-\tau_2) \leqslant 0 \tag{4}$$

has no eventually positive solution.

Then Equation (1) *is oscillatory.*

PROOF. Assume that (1) has an eventually positive solution $x(t)$, say $x(t) > 0$ for $t \geqslant t_0$. Set

$$z(t) = x(t) + p_1 x(t-\tau_1) - p_2 x(t+\tau_2).$$

Then

$$z''(t) = q_1(t)x(t-\sigma_1) + q_2(t)x(t+\sigma_2) > 0, \quad t \geqslant t_1 \geqslant t_0,$$

which implies that the functions $z(t)$, $z'(t)$ are of one sign on $[t_2, \infty)$, $t_2 \geq t_1$. We claim that $z(t) > 0$, eventually. To prove it assume that $z(t) < 0$. Then we let

$$0 < u(t) = -z(t) = p_2 x(t+\tau_2) - p_1 x(t-\tau_1) - x(t) \leqslant p_2 x(t+\tau_2).$$

Thus

$$x(t) \geqslant \frac{1}{p_2} u(t-\tau_2), \qquad t \geqslant t_2.$$

From Equation (1) one gets

$$\begin{aligned} 0 &= u''(t) + q_1(t)x(t-\sigma_1) + q_2(t)x(t+\sigma_2) \\ &\geqslant u''(t) + \frac{q_1(t)}{p_2} u(t-\sigma_1-\tau_2) + \frac{q_2(t)}{p_2} u(t+\sigma_2-\tau_2) \\ &\geqslant u''(t) + \frac{q_1^*(t)}{p_2} u(t-\sigma_1-\tau_2) + \frac{q_2^*(t)}{p_2} u(t+\sigma_2-\tau_2). \end{aligned}$$

Hence $u(t)$ is a positive solution of (4), a contradiction. Therefore $z(t) > 0$.

We define

$$y(t) = z(t) + p_1 z(t-\tau_1) - p_2 z(t+\tau_2). \tag{5}$$

Then

$$\begin{aligned} y''(t) = {} & q_1(t)x(t-\sigma_1) + q_2(t)x(t+\sigma_2) \\ & + p_1 q_1(t-\tau_1)x(t-\sigma_1-\tau_1) + p_1 q_2(t-\tau_1)x(t+\sigma_2-\tau_1) \\ & - p_2 q_1(t+\tau_2)x(t-\sigma_1+\tau_2) - p_2 q_2(t+\tau_2)x(t+\sigma_2+\tau_2). \end{aligned}$$

Thus

$$y''(t) \geqslant q_1^*(t)z(t-\sigma_1) + q_2^*(t)z(t+\sigma_2) > 0. \tag{6}$$

Consequently $y(t)$, $y'(t)$ are of one sign, eventually. Now we shall prove that $y(t) > 0$. If not, then we repeat the procedure used above. We let

$$0 < u_1(t) = -y(t) = p_2 z(t+\tau_2) - p_1 z(t-\tau_1) - z(t) \leqslant p_2 z(t+\tau_2).$$

Hence

$$z(t) \geqslant \frac{1}{p_2} u_1(t-\tau_2)$$

and (6) implies

$$\begin{aligned} 0 &\geqslant u_1''(t) + q_1^*(t)z(t-\sigma_1) + q_2^*(t)z(t+\sigma_2) \\ &\geqslant u_1''(t) + \frac{q_1^*(t)}{p_2} u_1(t-\sigma_1-\tau_2) + \frac{q_2^*(t)}{p_2} u_1(t+\sigma_2-\tau_2). \end{aligned}$$

We get that $u_1(t)$ is a positive solution of (4), a contradiction. Next we consider the following two cases:

Case 1. Let $z'(t) < 0$ for $t \geqslant t_3$. We claim that $y'(t) < 0$ for $t \geqslant t_3$. If not then we have $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ which implies $\lim_{t\to\infty} y(t) = \infty$. On the other hand $z(t) > 0$, $z'(t) < 0$ implies that $\lim_{t\to\infty} z(t) = k < \infty$. Then applying limits on both sides of (5) we have a contradiction. Thus $y'(t) < 0$ for $t \geqslant t_3$. Using the monotonicity of $z(t)$ we now get

$$\begin{aligned} y(t-\sigma_1) &= z(t-\sigma_1) + p_1 z(t-\sigma_1-\tau_1) - p_2 z(t+\sigma_1+\tau_2) \\ &\leqslant z(t-\sigma_1) + p_1 z(t-\sigma_1-\tau_1) \leqslant z(t-\sigma_1-\tau_1)(1+p_1), \end{aligned}$$

that is $z(t-\sigma_1) \geqslant \frac{y(t-\sigma_1+\tau_1)}{1+p_1}$ which together with (6) provides

$$y''(t) \geqslant q_1^*(t)z(t-\sigma_1) \geqslant \frac{q_1^*(t)}{1+p_1} y(t-\sigma_1+\tau_1)$$

Thus $y(t)$ is a positive decreasing solution of (3), a contradiction.

Case 2. Let $z'(t) > 0$ for $t \geqslant t_3$. Now we consider the following two cases:

Case *(i)* Assume that $y'(t) < 0$. Proceeding similarly as above and using the monotonicity of $z(t)$ we obtain

$$y(t-\sigma_1) \leqslant z(t-\sigma_1)(1+p_1).$$

Then using (6) and monotonicity of $y(t)$ we get

$$y''(t) \geqslant q_1^*(t)z(t-\sigma_1) \geqslant \frac{q_1^*(t)}{1+p_1}y(t-\sigma_1) \geqslant \frac{q_1^*(t)}{1+p_1}y(t-\sigma_1+\tau_1)$$

and again $y(t)$ is a positive, decreasing solution of (3), a contradiction.

Case *(ii)* Assume that $y'(t) > 0$. Then

$$y(t+\sigma_2) \leqslant z(t+\sigma_2)(1+p_1),$$

which in view of (6) implies

$$y''(t) \geqslant q_2^*(t)z(t+\sigma_2) \geqslant \frac{q_2^*(t)}{1+p_1}y(t+\sigma_2),$$

that is (2) possesses a positive increasing solution, a contradiction. The proof is complete. □

Remark. If the functions $q_i(t)$, $i = 1, 2$ are decreasing then we can put $q_i^*(t) = q_i(t)$.

Remark. Grace imposed in [6] for the differential equation (1) with constant functions $q_i(t) = q_i$, $i = 1, 2$ the condition $1 + p_1 - p_2 > 0$ on the coefficients. As we can see from our proof (Case (ii)) this condition is not needed.

Remark. Applying existing conditions sufficient for the inequalities (2), (3), (4) to have no abovementioned solutions we immediately obtain various oscillation criteria for (1).

We shall provide two new oscillation criteria for Equation (1).

Theorem 2. *Let $\sigma_1 > \tau_1$. Assume that*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2} (t+\sigma_2-s)q_2^*(s)\,ds > 1+p_1 \tag{7}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (s-t+\sigma_1-\tau_1)q_1^*(s)\,ds > 1+p_1 \tag{8}$$

and that the differential inequality (4) *has no eventually positive solution. Then Equation* (1) *is oscillatory.*

Proof. Conditions (7), (8) are sufficient for (2) to have no increasing solution and for (3) to have no decreasing solution, respectively (see e.g. [2] or [10]). □

Remark. Taking into account the result of Kusano and Naito [10], we see that the absence of positive solutions of (4) can be replaced by the assumption that the corresponding equation

$$u''(t) + \frac{q_1^*(t)}{p_2} u(t - \sigma_1 - \tau_2) + \frac{q_2^*(t)}{p_2} u(t + \sigma_2 - \tau_2) = 0 \tag{4e}$$

is oscillatory.

Imposing additional conditions on the coefficients and arguments of Equation (1) we obtain the following sufficient conditions for (2) and (3) to have the desired properties.

Theorem A. *Assume that there exists a function* $a_2 \in C^1((t_0, \infty))$ *such that*

$$a_2(t) > 0, \quad a_2'(t) \leqslant 0, \tag{9}$$

and

$$q_2^*(t) \geqslant (1 + p_1) a_2(t) a_2(t + \sigma_2/2). \tag{10}$$

If the first order differential inequality

$$v'(t) - a_2(t + \sigma_2/2) v(t + \sigma_2/2) \geqslant 0 \tag{11}$$

has no eventually positive solutions, then (2) *has no eventually positive increasing solution.*

For the proof see [3].

Corollary 1. *Assume that* (9)–(10) *hold. If*

$$\liminf_{t \to \infty} \int_t^{t+\sigma_2/2} a_2(s + \sigma_2/2)\, ds > \frac{1}{\mathrm{e}} \tag{12}$$

then (2) *has no eventually positive increasing solution.*

Proof. It is known (see Theorem 2.4.1 in [10]) that (12) is sufficient for (11) to have no eventually positive solutions. So, the assertion of this corollary follows from Theorem A. □

Theorem B. *Assume that there exists a function* $a_1 \in C^1((t_0,\infty))$ *such that*

$$a_1(t) > 0, \quad a_1'(t) \geqslant 0, \tag{13}$$

and

$$q_1^*(t) \geqslant (1+p_1)a_1(t)a_1[t-(\sigma_1-\tau_1)/2]. \tag{14}$$

If the first order differential inequality

$$v'(t) + a_1[t-(\sigma_1-\tau_1)/2]v[t-(\sigma_1-\tau_1)/2] \geqslant 0 \tag{15}$$

has no eventually positive solutions, then (3) *has no eventually positive decreasing solution.*

For the proof see [3].

Corollary 2. *Assume that* (13)–(14) *hold. If*

$$\liminf_{t\to\infty} \int_{t-(\sigma_1-\tau_1)/2}^{t} a_1[s-(\sigma_1-\tau_1)/2]\, ds > \frac{1}{\mathrm{e}} \tag{16}$$

then (3) *has no eventually positive decreasing solution.*

Proof. It is known (see Theorem 2.4.1 in [10]) that (16) is sufficient for (15) to have no eventually positive solutions. So, the assertion of this corollary follows from Theorem B. □

In the sequel we deal with the case of constant coefficients of Equation (1), namely for the differential equation

$$\Big(x(t) + p_1 x(t-\tau_1) - p_2 x(t+\tau_2)\Big)'' = q_1 x\big(t-\sigma_1\big) + q_2 x\big(t+\sigma_2\big). \tag{17}$$

Note that for Equation (17) $q_i^*(t) = q_i$, $i = 1, 2$ and moreover the corresponding differential inequality (4), namely

$$u''(t) + \frac{q_1}{p_2} u\big(t-\sigma_1-\tau_2\big) + \frac{q_2}{p_2} u\big(t+\sigma_2-\tau_2\big) \leq 0$$

is oscillatory (see e.g. [4]).

Theorem 3. *Assume that* $\sigma_1 > \tau_1$ *and*

$$\left(\frac{q_2}{1+p_1}\right)^{1/2}\left(\frac{\sigma_2\,\mathrm{e}}{2}\right) > 1 \tag{18}$$

and

$$\left(\frac{q_1}{1+p_1}\right)^{1/2}\left(\frac{(\sigma_1-\tau_1)\,\mathrm{e}}{2}\right) > 1. \tag{19}$$

Then Equation (17) *is oscillatory.*

Proof. We let $a_2(t) = \sqrt{\frac{q_2}{1+p_1}}$, then conditions (9)–(10) are satisfied and (12) reduces to (18), therefore by Corollary 1 the corresponding differential inequality (2) has no eventually positive increasing solution.

If we set $a_1(t) = \sqrt{\frac{q_1}{1+p_1}}$, then conditions (13)–(14) are satisfied and (16) takes the form (19) and by Corollary 2 the corresponding differential inequality (3) has no eventually positive decreasing solution. Thus, by Theorem 1 Equation (17) is oscillatory. □

Remark. As we have mentioned above, Grace in [6] has presented an oscillation criterion for Equation (17), but in addition to the assumptions of Theorem 5 he has imposed the following condition:

$$p_1 + 1 > p_2.$$

We have used a different technique which has enabled us to omit this condition.

Remark. The main result (Theorem 1) of this paper permits us to obtain various oscillation criteria for Equation (1). We have illustrated this fact by several theorems and corollaries. Moreover we are able to study asymptotic properties of solutions of Equation (1) even if not all assumptions of Theorem 1 are satisfied. If the differential inequality (2) has an eventually positive increasing solution then the conclusion of Theorem 1 is replaced by "Every solution x of Equation (1) is either oscillatory or else $|x(t)| \to \infty$ as $t \to \infty$".

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