Publ. Math. Debrecen
59 / 1-2 (2001), 45–50

Subgroups of infinite triangular matrices containing diagonal matrices

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To the memory of Zenon I. Borevich (1922–1995)

Abstract. In this note we describe the subgroups of the group of infinite upper triangular matrices $T_\infty(R)$ containing (or normalized by) the diagonal matrices, under some restrictions on the associative ring R.

0. Introduction

The problem of finding the subgroups of classical groups over rings has received considerable attention. Satisfactory answer was obtained only for some classes of subgroups with additional restrictions on rings. In number of papers Z. I. Borevich and N. A. Vavilov described subgroups containing diagonal matrices over semilocal rings, using the notion of net subgroups (see [1]–[3] and [6] for comprehensive survey). In this paper we extend these results describing the subgroups of the group of infinite upper triangular matrices, containing (or normalized by) the diagonal matrices, under some restrictions on the ring. We generalize for triangular case, results for infinite general linear group over division ring [5].

Let R be an associative ring with 1 ($1 \neq 0$), $R^\star$ be the group of invertible elements of R. Let $T_\infty(R)$ denote a group of countably infinite (indexed by positive integers $\mathbb{N}$) upper triangular matrices over R (with

Mathematics Subject Classification: 20H25, 20E07, 20E15.
Key words and phrases: infinite triangular matrix, diagonal matrix, lattice of subgroups.

elements from $R^\star$ on diagonal) and $D_\infty(R)$ be the subgroup of all diagonal matrices. By e we denote the unit matrix in $T_\infty(R)$, by e_{ij} – a matrix with the only nontrivial element 1 in i-th row and j-th column. We denote $t_{ij}(\zeta) = e + \zeta e_{ij}$, $\zeta \in R$, $i, j \in \mathbb{N}$, $d_i(\theta) = e + (\theta - 1)e_{ii}$, $\theta \in R^\star$ and $[x, y] = xyx^{-1}y^{-1}$. $D_{\mathrm{fin}}(R)$ denote the subgroup of $D_\infty(R)$ generated by all $d_i(\theta)$, where $\theta \in R^\star$, $i \in \mathbb{N}$. The group $D_{\mathrm{fin}}(R)$ can be regarded as a direct limit of the groups $D_n(R)$ of $n \times n$ diagonal matrices. Similarly, by $T_\omega(R)$ we denote a direct limit of triangular groups $T_n(R)$. $T_\omega(R)$ is a normal subgroup of $T_\infty(R)$, consisting of all matrices which have a finite number of nonzero entries above the main diagonal.

A system $\sigma = (\sigma_{ij})$ $(i, j \in \mathbb{N})$ of two sided ideals σ_{ij} of R is called a *net* if

$$(\star) \qquad \sigma_{ir} \cdot \sigma_{rj} \subseteq \sigma_{ij} \quad \text{for all } i, j, r \in \mathbb{N}.$$

It is clear that if σ, τ are nets, then a system $\sigma \cap \tau = (\sigma_{ij} \cap \tau_{ij})$ is a net too. The relation $\sigma \leq \tau$ if $\sigma_{ij} \subseteq \tau_{ij}$ define a partial order on the set of all nets. Here, we consider only *upper nets* σ for which σ_{ij} is trivial for $i > j$. If moreover $\sigma_{ii} = R$ for all $i \in \mathbb{N}$, we call σ *upper D-net.*

Let the set $M(\sigma)$ consist of all triangular matrices a, such that $a_{ij} \in \sigma_{ij}$. If σ satisfies $(\star)$, then $e + M(\sigma) = \{e + a : a \in M(\sigma)\}$ is closed under multiplication of matrices, and by $G(\sigma)$ we denote its maximal subgroup. $E(\sigma)$ denotes a subgroup of $G(\sigma)$ generated by all elementary transvections $t_{ij}(\zeta)$, where $\zeta \in \sigma_{ij}$, $i, j \in \mathbb{N}$, $i < j$.

1. Main results

Theorem 1. *Let R be an associative ring with 1, such that there exists an element $\theta \in R^\star$ for which $\theta - 1 \in R^*$ and R be additively generated by elements of $R^\star$. Let H be a subgroup of $T_\infty(R)$ containing $D_{\mathrm{fin}}(R)$. Then there exists a unique upper D-net $\sigma = (\sigma_{ij})$ of two-sided ideals of R, such that*

$$D_{\mathrm{fin}}(R) \cdot E(\sigma) \leq H \leq G(\sigma).$$

If moreover H is contained in $T_\omega(R)$, then $H = G(\sigma)$.

An upper net σ is called *finite*, if there exists $n_0 \in \mathbb{N}$, such that $\sigma_{ij} = (0)$ for all $j > i > n_0$. From Theorem 1 it follows

Corollary 1. *If σ in Theorem 1 is finite and $D_\infty(R) \leq H \leq T_\infty(R)$, then $H = G(\sigma) = D_\infty(R) \cdot E(\sigma)$.*

In general, the inequality $D_\infty(R) \cdot E(\sigma) \neq G(\sigma)$ holds. For example for D-net $\tau = \operatorname{diag}(\bar{\tau}, \bar{\tau}, \dots)$, where $\bar{\tau} = \begin{pmatrix} R & R \\ (0) & R \end{pmatrix}$.

Our proof works without changes for finite matrices and gives another proof of the following result of Z. I. Borevich [2, Theorem 7].

Theorem 2. *Let R be an associative ring with 1, such that there exists an element $\theta \in R^\star$ for which $\theta - 1 \in R^*$ and let R be additively generated by elements of $R^\star$. Let H be a subgroup of the group of $n \times n$-triangular matrices $T_n(R)$ over R containing the diagonal matrices $D_n(R)$. Then there exists a unique upper $D-$net $\sigma = (\sigma_{ij})$ $(1 \leq i, j \leq n)$ of two-sided ideals of R, such that $H = G(\sigma)$.*

Theorem 2 shows that all subgroups of Borel subgroup containing maximal split torus are net subgroups.

In fact, our proof of Theorem 1 gives the following generalization.

Theorem 3. *If under assumptions of Theorem 1 on ring R the subgroup H of $T_\infty(R)$ is normalized by $D_{\text{fin}}(R)$, then there exist a unique upper D-net $\sigma = (\sigma_{ij})$ of ideals of R, such that $E(\sigma) \leq H \leq G(\sigma)$.*

The following proposition characterizes the group $G(\sigma)$.

Proposition 1. $G(\sigma) = T_\infty(R) \cap (e + M(\sigma))$.

Proof. We show that if $a = (a_{ij}) \in G(\sigma)$, then $a^{-1} = (a'_{ij}) \in G(\sigma)$. Since $a^{-1} \cdot a = e$, we have for $i < j$ the equality $a'_{ii} a_{i,i+1} + a'_{i,i+1} a_{i+1,i+1} = 0$. So $a'_{i,i+1} \in \sigma_{i,i+1}$, because $a_{i+1,i+1}$ is invertible and by induction from equation $a'_{ii} a_{ij} + \dots + a'_{ij} a_{jj} = 0$, it follows $a'_{ij} \in \sigma_{ij}$. □

Lemma 1. *If $\zeta \in R^\star$, then for all $i, j \in \mathbb{N}$ we have $d_i(\zeta^{-1}) t_{ij}(\alpha) d_i(\zeta) = t_{ij}(\alpha\zeta)$ and $d_j(\zeta) t_{ij}(\alpha) d_j(\zeta^{-1}) = t_{ij}(\zeta\alpha)$.*

Proposition 2. *Let R be an associative ring with 1, such that there exists an element $\theta \in R^\star$, for which $\theta - 1 \in R^\star$. If the subgroup H of $T_\infty(R)$ is normalized by $D_{\text{fin}}(R)$ and $a \in H$, then $t_{ij}(a_{ij}) \in H$ for all $i, j \in \mathbb{N}$.*

Proof. We fix $i < j$. If $a \in H$ then $b = [a_{-1}, d_i(\theta)] \in H$. We have $b_{ij} = a'_{ii}(\theta - 1) a_{ij}$. If we put $c = [b_{-1}, d_j(\theta^{-1})]$ we will have $[d_i(\theta), c] =$

$t_{ij}((\theta-1)c_{ij}) \in H$. Since $c_{ij} = b'_{ii}b_{ij}(\theta-1)$ by Lemma 1 it follows $t_{ij}(a_{ij}) \in H$. $\square$

Now we can prove Theorem 1. Let $D_{\text{fin}}(R) \leq H \leq T_\infty(R)$. We put (for $i<j$) $\sigma_{ij} = \{\alpha \in R : t_{ij}(\alpha) \in H\}$ and $\sigma_{ii} = R$ for $i \in \mathbb{N}$. From Proposition 2 and Lemma 1, σ_{ij} are two-sided ideals of R. A system $\sigma = (\sigma_{ij})$ forms a D-net because $[t_{ir}(\alpha), t_{rj}(\beta)] = t_{ij}(\alpha\beta)$. It is clear that $D_{\text{fin}}(R) \cdot E(\sigma) \leq H$ and from Proposition 2 we have $H \leq G(\sigma)$.

The Corollary 1 and Theorem 2 follows easily from the fact that in finite case the equality holds $D_\infty(R) \cdot E(\sigma) = G(\sigma)$. The proof of Theorem 3 is obvious.

3. Remarks

A. Our Theorem 1 says that every subgroup of $T_\infty(R)$ containing $D_{\text{fin}}(R)$ lies in a uniquely determined interval $D_{\text{fin}}(R) \cdot E(\sigma) \leq H \leq G(\sigma)$. If $R = K$- is a field, such intervals have a very simple description. It is clear that every field is additively generated by its invertible elements. The existence of invertible element θ such that $\theta - 1$ is invertible is obvious for fields of characteristic 0 and $p > 2$. If $p = 2$, such elements exist if K has a dimension > 1 over $\mathbb{F}_2$ (as a vector space). It means that assumptions of Theorem 1 are fulfilled if $K \neq \mathbb{F}_2$.

We consider 3 cases.

(i) for all $i \in \mathbb{N}$: $\sigma_{i,i+1} = K$,

(ii) the set $\{i : \sigma_{i,i+1} = (0)\}$ is infinite,

(iii) the set $\{i : \sigma_{i,i+1} = (0)\}$ is finite.

(i) Since $\sigma_{i,i+1}\sigma_{i+1,i+2} \subseteq \sigma_{i,i+2}$ we have $\sigma_{i,i+2} = K$. This means that $\sigma_{ij} = K$ for all $j \geq i$. We have $D_{\text{fin}}(K) \cdot E(\sigma)$ is equal $T_\omega(K)$, and $G(\sigma) = T_\infty(K)$. Since $T_\omega(K)$ is normal subgroup of $T_\infty(K)$, the lattice of subgroups H such that $T_\omega(K) \leq H \leq T_\infty(K)$ is isomorphic to the lattice of subgroups of a quotient group $T_\infty(K)/T_\omega(K)$.

(ii) Suppose that if $\sigma_{i,i+1} = (0)$ then $\sigma_{kj} = (0)$ for all $k < i$, $j > i$. So, for some infinite sequence of natural numbers $(n_1, n_2, \dots)$ the group $D_{\text{fin}}(K) \cdot E(\sigma)$ is isomorphic to an infinite direct product of finite dimensional triangular groups $T_{n_1}(K)$, $T_{n_2}(K)$, $\dots$, which is normal in $G(\sigma)$ – an infinite cartesian product of these groups. This gives an uncountable

number of intervals in the lattice of subgroups. Here $T_1(K) \simeq K^\star$. Otherwise, the description is more complicated. An example of D-net σ, which is not of the above form is given by $\sigma_{ij} = K$ for $i = j$ or $i = 1$ and $\sigma_{ij} = (0)$ for other indices i, j.

(iii) Suppose that if $\sigma_{i,i+1} = (0)$ then $\sigma_{kj} = (0)$ for all $k < i$, $j > i$. Then all subgroups corresponding to D-net σ have a form $T_{n_1}(K) \times \cdots \times T_{n_s}(K) \times H_1$ where $T_\omega(K) \le H_1 \le T_\infty(K)$ (for any H_1 we obtain a countable number of subgroups). Otherwise the description is more complicated.

It follows that all subgroups of $T_\infty(K)$ containing $D_{\mathrm{fin}}(K)$ lie near the groups connected with D-nets. It is clear that more detailed description in this case needs another tools and methods. The use of net subgroups reduces, in fact, the problem of finding such subgroups to the problem of describing the lattice of subgroups of $T_\infty(K)$, containing $T_\omega(K)$.

B. Let π be a permutation of the set of positive integers ($\pi \in S(\mathbb{N})$). By π we denote also the infinite matrix corresponding to the permutation π. We say that two nets τ and σ are conjugate, if $\tau = \pi\sigma\pi^{-1}$ for some $\pi \in S(\mathbb{N})$. Let $T_\infty^\pi(R) = \pi T_\infty(R)\pi^{-1}$. We have $D_{\mathrm{fin}}(R) = \pi D_{\mathrm{fin}}(R)\pi^{-1}$, $E(\tau) = \pi E(\sigma)\pi^{-1}$ and $G(\tau) = \pi G(\sigma)\pi^{-1}$. It is clear, that our theorems give also description of subgroups of $T_\infty^\pi(R)$ containing (or normalized by) $D_{\mathrm{fin}}(R)$ with the net τ instead of the net σ.

C. The proof of Theorem 1 is valid also in the case of the group $T(\mathfrak{n}, R)$, of all upper triangular matrices of dimension $\mathfrak{n}$ (where $\mathfrak{n}$ is arbitrary infinite ordinal) with finite number of nonzero elements in every column. Similar results for subgroups of infinite block upper triangular matrices, containing block elementary matrices, are obtained in [4].

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(Received March 17, 2000; revised December 19, 2000)