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A generalization of Lucas' congruence for q-binomial coefficients

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Abstract. In this paper, we generalize the Lucas' congruence for q-binomial coefficients.

1. Introduction

The famous Lucas' property for binomial coefficients is

$$(1) \qquad \binom{n}{r} \equiv \prod_{i\geq 0} \binom{n_i}{r_i} \pmod{p},$$

where and throughout this paper p is a prime, n, r are integers, their expansions in base p are given by $n = \sum_{i\geq 0} n_i p^i$ and $r = \sum_{i\geq 0} r_i p^i$, with $0 \leq n_i,\ r_i \leq p-1$ (only a finite number of the n_i's are non-zero). This relation has been generalized to many unidimensional or bidimensional sequences, such as Apéry numbers. Also, many authors have studied the values of these sequences modulo a prime power, see [1]–[8]. In this paper, we investigate whether identity (1) holds when replacing the binomial coefficient $\binom{n}{r}$ by the Gaussian binomial coefficient, i.e., the q-binomial

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coefficient $\binom{n}{r}_q$ defined by the following formula,

$$\binom{n}{r}_q = \begin{cases} \dfrac{q^n-1}{q^r-1}\dfrac{q^{n-1}-1}{q^{r-1}-1}\cdots\dfrac{q^{n-r+1}-1}{q-1}, & \text{if } 0 < r \le n, \\ 1, & \text{if } r = 0, \\ 0, & \text{if } r < 0 \text{ or } r > n. \end{cases}$$

However, for $\binom{n}{r}_q$, (1) is not always true. For example, if $q \not\equiv 1 \pmod p$, $(q,p)=1$, taking $n = p\,\mathrm{ord}_p(q) - 1$, $r = 1$, it is easy to verify that

$$\binom{n}{1}_q \not\equiv \binom{\mathrm{ord}_p(q)-1}{0}_q \binom{p-1}{1}_q \pmod p,$$

where $\mathrm{ord}_p(q)$ is the order of q modulo p, i.e., the smallest positive integer f such that

$$q^f \equiv 1 \pmod p.$$

Although (1) fails in general for q-binomial coefficients, Fray [2] proved an interesting result: let $d = \mathrm{ord}_p(q)$, $n = n_0 + d\sum_{i\ge1} a_i p^i$, $r = r_0 + d\sum_{i\ge1} b_i p^i$, with $0 \le n_0,\ r_0 < d$, $0 \le a_i,\ b_i \le p$, $i \ge 1$, then

$$\binom{n}{r}_q \equiv \binom{n_0}{r_0}_q \prod_{i\ge1}\binom{a_i}{b_i} \pmod p.$$

In this paper, we first obtain the following

Theorem 1. *If* $q \ne 1$, $n = n_0 + n_1 p$, $r = r_0 + r_1 p$, $0 \le n_0,\ r_0 \le p-1$, $n_1, r_1 \ge 0$, *then*

$$\binom{n}{r}_q \Big/ \binom{n_1}{r_1}_{q^p} \equiv \binom{n_0}{r_0}_q \left(\mathrm{mod}\, \frac{q^p-1}{q-1}\right). \tag{2}$$

In particular, let $q \to 1$, (2) *become* (1), *i.e., Lucas' property* (1) *is a direct consequence of Theorem 1.*

Proof. It is obvious that (2) is true if $n_1 < r_1$. Let $n_1 \ge r_1$, if $r_0 > n_0$, then

$$\binom{n_0}{r_0}_q = 0. \tag{3}$$

On the other hand, one has

$$(4)\quad \binom{n}{r}_q = \prod_{1\le i\le r} \frac{q^{n-i+1}-1}{q^i-1}$$
$$= \frac{\prod_{0\le j\le r_1}(q^{(n_1-j)p}-1)}{\prod_{1\le j\le r_1}(q^{jp}-1)} \frac{\prod_{n-i+1\not\equiv 0 \pmod p}(q^{n-i+1}-1)}{\prod_{i\not\equiv 0 \pmod p}(q^i-1)}.$$

The first fraction in (4) is equal to

$$(5)\qquad \binom{n_1}{r_1}_{q^p}(q^{(n_1-r_1)p}-1) \equiv 0 \left(\operatorname{mod} \frac{q^p-1}{q-1}\right).$$

From the well-known property of the greatest common divisor:
$(q^s-1, q^t-1) = q^{(s,t)}-1$, it follows that

$$\left(\prod_{\substack{1\le i\le r\\ i\not\equiv 0 \pmod p}} (q^i-1),\quad \frac{q^p-1}{q-1}\right) = 1.$$

Combining (4) and (5),

$$(6)\qquad \binom{n}{r}_q \Big/ \binom{n_1}{r_1}_{q^p} \equiv 0 \left(\operatorname{mod} \frac{q^p-1}{q-1}\right).$$

Here we used the following property of divisibility for integers: if $c \mid a$, $(c, P) = 1$, $a \equiv 0 \pmod P$, then $\frac{a}{c} \equiv 0 \pmod P$. Comparing (3) with (6), we deduce (2). If $r_0 \le n_0$, then

$$(7)\quad \binom{n}{r}_q \Big/ \binom{n_1}{r_1}_{q^p} = \prod_{\substack{1\le i\le r\\ n-i+1\not\equiv 0 \pmod p}} (q^{n-i+1}-1) \Big/ \prod_{\substack{1\le i\le r\\ i\not\equiv 0 \pmod p}} (q^i-1)$$
$$\equiv \prod_{i=1}^{r_0} \frac{q^{n_0-i+1}-1}{q^i-1} \prod_{i=1}^{p-1} \frac{(q^i-1)^{r_1}}{(q^i-1)^{r_1}} = \binom{n_0}{r_0}_q \left(\operatorname{mod} \frac{q^p-1}{q-1}\right).$$

Here in (7) we used the following property of divisiblity for integers: if $b \mid a$, $d \mid c$, $a \equiv c \pmod P$, $b \equiv d \pmod P$, $(b, P) = 1$, then $\frac{a}{b} \equiv \frac{c}{d} \pmod P$. Therefore (2) is true. □

Next, we want to generalize (2) to modulo $(q^{p^b}-1)/(q-1)$ for any integer $b \geq 1$, in order to do this, we recall the following remarkable observation of Kummer in 1885: for any prime p and positive integers $n \geq r \geq 0$, the exact power of p that divides the binomial coefficient $\binom{n}{r}$ is given by the number of "carries" when adding r and $n-r$ in base p. Therefore, if n and r are expanded in base p as the beginning of this paper, then $p \nmid \binom{n}{r}$ means $n_i \geq r_i$ for each $i \geq 0$.

Theorem 2. *Define n'_j to be the least non-negative residue of $n \pmod{p^j}$, $j \geq 1$, if p does not divide $\binom{n}{r}$, then for any positive integer b,*

$$\binom{n}{r}_q \equiv \binom{n'_b}{r'_b}_q \binom{[n/p]}{[r/p]}_{q^p} \Big/ \binom{[n'_b/p]}{[r'_b/p]}_{q^p} \left(\bmod \frac{q^{p^b}-1}{q-1}\right) \tag{8}$$

in particular, if $b=1$, (8) becomes (2).

Proof. Let $n = n'_b + n''_b p^b$, $r = r'_b + r''_b p^b$, and $n''_b, r''_b \geq 0$, then

$$\binom{n}{r}_q = \prod_{1\leq i\leq r} \frac{q^{n-i+1}-1}{q^i-1} = \prod_{\substack{1\leq i\leq p^b \\ 0\leq j\leq r''_b-1}} \frac{q^{(n''_b-j-1)p^b+i}-1}{q^{jp^b+i}-1} \tag{9}$$

$$\times \frac{\prod_{1\leq i\leq n'_b}(q^{n''_b p^b+i}-1)}{\prod_{1\leq i\leq r'_b}(q^{r''_b p^b+i}-1)\prod_{1\leq i\leq n'_b-r'_b}(q^{(n''_b-r''_b)p^b+i}-1)}, \tag{10}$$

if $r''_b = 0$, the right product in (9) is 1; if $r''_b > 0$, the product could be split into two parts, i.e., over $i \equiv 0 \pmod p$ and $i \not\equiv 0 \pmod p$, respectively, the second part is congruent to 1 modulo $(q^{p^b}-1)/(q-1)$, here again we use the property of divisiblity for integers, hence the product is congruent to

$$\binom{n''_b p^{b-1}}{r''_b p^{b-1}}_{q^p} \left(\bmod \frac{q^{p^b}-1}{q-1}\right). \tag{11}$$

Noting that (11) is also true for $r''_b = 0$; the fraction in (10) could be denoted by $\prod = \prod_1 / \prod_2 \prod_3$, and

$$\prod\nolimits_1 = \prod_{1\leq i\leq n'_b} \frac{q^{n''_b p^b+i}-1}{q^i-1} \prod_{1\leq i\leq n'_b}(q^i-1), \tag{12}$$

the first product on the right of (12) could be split into two parts, over $i \equiv 0 \pmod p$ and $i \not\equiv 0 \pmod p$, respectively. The first part is equal to $\binom{[n/p]}{[n'_b/p]}_{q^p}$, as for the second part, since the numerator is congruent to the denominator modulo $(q^{p^b}-1)/(q-1)$, hence

$$\prod\nolimits_1 \Big/ \binom{[n/p]}{[n'_b/p]}_{q^p} \equiv \prod_{1\le i\le n'_b} (q^i-1) \left(\operatorname{mod} \frac{q^{p^b}-1}{q-1}\right). \tag{13}$$

Similarly we could deal with $\prod_2$ and $\prod_3$, i.e.,

$$\prod\nolimits_2 \Big/ \binom{[r/p]}{[r'_b/p]}_{q^p} \equiv \prod_{1\le i\le r'_b} (q^i-1) \left(\operatorname{mod} \frac{q^{p^b}-1}{q-1}\right), \tag{14}$$

$$\prod\nolimits_3 \Big/ \binom{[(n-r)/p]}{[(n'_b-r'_b)/p]}_{q^p} \equiv \prod_{1\le i\le n'_b-r'_b} (q^i-1) \left(\operatorname{mod} \frac{q^{p^b}-1}{q-1}\right). \tag{15}$$

Combining (13), (14) and (15), we have

$$\prod \equiv \frac{\binom{[n/p]}{[n'_b]}_{q^p}}{\binom{[r/p]}{[r'_b]}_{q^p}\binom{[(n-r)/p]}{[(n'_b-r'_b)/p]}_{q^p}} \; \frac{\prod_{1\le i\le n'_b}(q^i-1)}{\prod_{1\le i\le r'_b}(q^i-1)\prod_{1\le i\le n'_b-r'_b}(q^i-1)} \tag{16}$$

$$= \binom{[n/p]}{[r/p]}_{q^p} \binom{n'_b}{r'_b}_q \Big/ \binom{[n'_b/p]}{[r'_b/p]}_{q^p} \binom{[(n-n'_b)/p]}{[(r-r'_b)/p]}_{q^p} \quad \left(\operatorname{mod} \frac{q^{p^b}-1}{q-1}\right),$$

from (11) and (16), we deduce (8). □

We have three immediate consequences:

Corollary 1. *If p does not divide $\binom{n}{r}$, then for any integer $b \ge 1$.*

$$\binom{n}{r} = \binom{n'_b}{r'_b}\binom{[n/p]}{[r/p]} \Big/ \binom{[n'_b/p]}{[r'_b/p]} \pmod{p^b}.$$

This is a result of Andrew Granville [4, Proposition 2].

Corollary 2. *If p does not divide $\binom{n}{r}$ and $n \equiv r \pmod{p^b}$, then*

$$\binom{n}{r}_q \equiv \binom{[n/p]}{[r/p]}_{q^p} \left(\operatorname{mod} \frac{q^{p^b}-1}{q-1}\right).$$

Corollary 3. *If p does not divide $\binom{n}{r}$ and $n \equiv r \pmod{p^k}$ where $k \geq b-1$, then*

$$\binom{n}{r}_q \equiv \binom{[n/p^{k+1-b}]}{[r/p^{k+1-b}]}_{q^p} \left(\operatorname{mod} \frac{q^{p^b}-1}{q-1}\right).$$

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