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The discrete Tchebycheff transformation in certain spaces of sequences and its applications

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Abstract. It is proved that the discrete Tchebycheff transformation defines an isomorphism from the space $T(\mathbb{Z})$ of testing-sequences of rapid descent into the space $T(-1,1)$ of testing-functions with slow growth at the end points 1 and -1. This result is extended to spaces of generalized sequences and distributions. The discrete translation operator and the discrete convolution are also studied. Finally, the operational calculus generated is applied in solving certain finite-difference equations.

1. Introduction

The discrete Tchebycheff transformation

$$(1.1)\qquad (\mathcal{T}\Phi)(x) = \varphi(x) = \frac{1}{\pi}\Phi(0) + \frac{2}{\pi}\sum_{n=1}^{\infty}\Phi(n)T_n(x),$$

where $T_n(x) = \cos(n\arccos x)$, $n = 0,1,2,\dots$, denote the well-known Tchebycheff polynomials of the first kind ([4], [10] and [14]), can be considered as a special case of more general orthogonal series expansions. However, we think that it is worth doing a particular study of this transform because its properties are more simple than those corresponding to the Jacobi series expansions investigated by G. Gasper [5] and I. I. Hirschman [9], amongst other authors. The inverse of (1.1) is given by [16, p. 344]

$$(1.2)\qquad \left(\mathcal{T}^{-1}\varphi\right)(n) = \Phi(n) = \int_{-1}^{1}(1-x^2)^{-1/2}T_n(x)\varphi(x)dx.$$

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This direct form has been investigated in detail by P. L. Butzer and P. L. Stens [3] in classical spaces of functions and by H.-J. Glaeske and T. Runst [7] in the most general context of the finite Jacobi transform in certain spaces of generalized functions. Since $T_n(x)$ is an even function of the discrete variable n, then $\Phi(n)$ is always an even sequence. By this reason, instead of (1.1), we consider in [2] the most compact form

$$(1.3) \qquad (\mathcal{T}\Phi)(x) = \varphi(x) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Phi(n) T_n(x),$$

and discuss its classical properties, operational rules and applications.

In this paper we investigate the discrete transformation $\mathcal{T}$ in an even complex-valued sequence space $T(\mathbb{Z})$ and its dual $T'(\mathbb{Z})$. The operator $\mathcal{T}$ transforms the space $T(\mathbb{Z})$ into a space, called $T(-1,1)$, of infinitely differentiable functions $\varphi(x)$ defined on $-1 < x < 1$. Inasmuch as $T_n(x)$ does not belong to $T(\mathbb{Z})$, considered as a function of the integer variable n, the generalized discrete Tchebycheff transformation will be defined by the adjoint of the classical transform $\mathcal{T}$, that is to say, through the extension of the Parseval equation

$$(1.4) \qquad \sum_{n=-\infty}^{\infty} \Phi(n)\Psi(n) = \pi \int_{-1}^{1} (1-x^2)^{-1/2} \varphi(x)\psi(x)dx,$$

with $\varphi(x)$ and $\psi(x)$ being the respective transforms of $\Phi(n)$ and $\Psi(n)$. This procedure has been employed by L. Schwartz [15] in relation to the Fourier transform and by A. H. Zemanian [18], J. J. Betancor and M. T. Flores [1] and J. M. R. Méndez-Pérez [11] to generalize different versions of Hankel transform. It is proved that the distributional transformation $\mathcal{T}'$ is a topological isomorphism between the spaces $T'(\mathbb{Z})$ and $T'(-1,1)$. We are specially interested in stressing the results obtained when the discrete transform (1.3) is used, that is, when we deal with sequences, in contrast to other researches where the direct form (1.2) plays the main role and acts on functions. Thus, the discrete translation operator and the discrete convolution associated to the Tchebycheff series expansions are analysed, and their main properties are established. Finally, the operational calculus generated is applied in solving a class of finite-difference equations.

Now we remember that $T_n(x)$, as function of x, satisfies the differential equation [10, p. 79]

$$(1.5) \qquad (1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0,$$

whereas $T_n(x)$, as a function of n, fulfils the difference equation [10, p. 185(16)]

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \tag{1.6}$$

and the most general relation [14]

$$T_{n+l}(x) + T_{n-l}(x) = 2T_n(x)T_l(x), \tag{1.7}$$

with $l \in \mathbb{Z}$.

Along this paper, $D(-1,1)$ stands for the space of infinitely differentiable functions whose supports are contained in the open interval $(-1,1)$ and $D'(-1,1)$ is the space of Schwartz distributions on $(-1,1)$ [15]. Finally, $E(-1,1)$ represents the space of all infinitely differentiable functions on $(-1,1)$ and its dual $E'(-1,1)$ is the space of distributions on $(-1,1)$ with compact supports.

As for the notation, we choose the symbol $\langle\!\langle\ ,\ \rangle\!\rangle$ to represent a functional acting on $T(\mathbb{Z})$ and $\langle\ ,\ \rangle$ to denote a functional operating on $T(-1,1)$.

2. The testing function spaces $T(\mathbb{Z})$ and $T(-1,1)$ and their duals

$T(-1,1)$ is the space of all infinitely differentiable functions $\varphi(x)$ defined on $(-1,1)$ such that

$$\gamma_k(\varphi) = \sup_{x\in(-1,1)} |\Omega_x^k \varphi(x)| \tag{2.1}$$

exists for each nonnegative integer k, where Ω_x denotes the differential operator [10, p. 98]

$$\begin{aligned} \Omega_x = \Omega &= (1-x^2)^{1/2} D_x (1-x^2)^{1/2} D_x \\ &= (1-x^2)D^2 - xD, \quad D = \frac{d}{dx}. \end{aligned} \tag{2.2}$$

We equip $T(-1,1)$ with the topology generated by the countable multinorm (2.1). Thus, $T(-1,1)$ is a Fréchet space. By using a technique similar to the one employed by A. H. Zemanian [18] and H. J. Glaeske and A. Hess [6] in relation to Hankel and Mehler–Fock transformations, respectively, the members of the space $T(-1,1)$ can be characterized as follows.

Proposition 2.1. *$\varphi(x)$ belongs to the space $T(-1,1)$ if and only if $\varphi(x)$ is an infinitely differentiable function on the interval $(-1,1)$ and $D^k\varphi(x) = O(1)$ as $x \to 1-0$ and $x \to -1+0$, for all $k = 0,1,2,\dots$.*

$T'(-1,1)$ stands for the dual space of $T(-1,1)$. We consider only in $T'(-1,1)$ the weak $*$ topology [18, p. 21]. Then, $T'(-1,1)$ is sequentially complete.

We now list some properties of these spaces:

(i) The operation $\varphi \longrightarrow \Omega\varphi$, where Ω_x is given by (2.2), gives rise to a continuous linear mapping of $T(-1,1)$ into itself. Therefore, the mapping $f \longrightarrow \Omega^* f$, defined by

$$\langle \Omega^* f, \varphi\rangle = \langle f, \Omega\varphi\rangle, \quad f \in T'(-1,1),\ \varphi \in T(-1,1), \tag{2.3}$$

is also a continuous linear mapping of $T'(-1,1)$ into itself. Note that Ω^* is the adjoint operator of Ω [12]

$$\Omega_x^* = \Omega^* = D_x(1-x^2)^{1/2} D_x (1-x^2)^{1/2} = (1-x^2)D^2 - 3xD - 1.$$

(ii) The operation $\varphi \longrightarrow x^l\varphi$, $l = 0,1,2,\dots$, is a continuous linear mapping of $T(-1,1)$ into itself. Consequently, the operation $f \longrightarrow x^l f$ defined by means of

$$\langle x^l f(x), \varphi(x)\rangle = \langle f(x), x^l\varphi(x)\rangle, \quad f \in T'(-1,1),\ \varphi \in T(-1,1), \tag{2.4}$$

is too a continuous linear mapping of $T'(-1,1)$ into itself. To see the first part of the assertion, observe that we can write

$$\Omega_x\left[x^l\varphi(x)\right] = x^l\Omega_x\varphi(x) + 2lx^{l-1}(1-x^2)D\varphi(x) + \left[l(l-1)x^{l-2} - l^2x^l\right]\varphi(x)$$

for any $l = 0,1,2,\dots$. By an inductive procedure on k we can establish, in general, that

$$\begin{aligned}\Omega_x^k\left[x^l\varphi(x)\right] &= \sum_{i=0}^{k} \Pi_{l,i}^{(k)}(x)\Omega_x^{k-i}\varphi(x) \\ &\quad + (1-x^2)\sum_{i=0}^{k-1}\Pi_{l-1,i}^{(k)}(x) D_x \Omega_x^{k-i-1}\varphi(x)\end{aligned} \tag{2.5}$$

where $\Pi_{j,i}^{(k)}(x)$ are polynomials of j-th degree. On the other hand, by integrating in (2.2) we find that

$$\int_{-1}^{x}(1-t^2)^{-1/2}\Omega_t\varphi(t)dt=(1-x^2)^{1/2}D_x\varphi(x), \tag{2.6}$$

since $\lim_{x\to-1}(1-x^2)^{1/2}D_x\varphi(x)=0$ in view of Proposition 2.1. If we substitute (2.6) in (2.5), we obtain

$$\gamma_k\left[x^l\varphi(x)\right]\le\sum_{i=0}^{k}C_i(k,l)\gamma_i(\varphi), \tag{2.7}$$

where the positive constants C_i only depend on k and l. This completes our proof. The second part of the assertion is inferred from the above result.

(iii) Let f be an integrable function such that $(1-x^2)^{-1/2}f(x)$ is absolutely integrable on $-1<x<1$. Then, f generates a regular generalized function f in $T'(-1,1)$ through

$$\langle f,\varphi\rangle=\int_{-1}^{1}(1-x^2)^{-1/2}f(x)\varphi(x)dx,\qquad \varphi\in T(-1,1). \tag{2.8}$$

Indeed,

$$|\langle f,\varphi\rangle|\le\gamma_0(\varphi)\int_{-1}^{1}(1-x^2)^{-1/2}|f(x)|dx.$$

Since $\varphi\in T(-1,1)$ fulfils the condition

$$\int_{-1}^{1}(1-x^2)^{-1/2}|\varphi(x)|dx\le\pi\gamma_0(\varphi),$$

every member of $T(-1,1)$ gives rise to a regular distribution in $T'(-1,1)$ by means of (2.8). Moreover, two members of $T(-1,1)$ that generate the same member of $T'(-1,1)$ must be identical. Hence, $T(-1,1)$ can be considered a subspace of $T'(-1,1)$.

(iv) $D(-1,1)$ is a subspace of $T(-1,1)$ and the topology of $D(-1,1)$ is stronger than that induced on it by $T(-1,1)$. Consequently, the restriction of any $f\in T'(-1,1)$ to $D(-1,1)$ is a member of $D'(-1,1)$.

Note that $\varphi(x) = x^2$ belongs to $T(-1,1)$ and $\varphi(x)$ tends towards 1 as $x \to 1-0$ and $x \to -1+0$. Then, it is clear that the ball $\{\psi : \psi \in T(-1,1),\ \gamma_0(\psi - \varphi) < 1/2\}$ does not contain any member of $D(-1,1)$. Therefore, $D(-1,1)$ is not dense in $T(-1,1)$. On the other hand, $T(-1,1) \subset E(-1,1)$ and $E'(-1,1)$ is a subspace of $T'(-1,1)$.

(v) The Tchebycheff polynomial $T_n(x) \in T(-1,1)$ considered as function of the continuous variable x. Certainly, by invoking (1.5) we have

$$\gamma_k\left(T_n(x)\right) = \sup_{x\in(-1,1)} |n^{2k} T_n(x)| \le n^{2k}, \quad -1 < x < 1.$$

Next, we introduce the space $T(\mathbb{Z})$, where $\mathbb{Z}$ denotes the set of integer numbers. $T(\mathbb{Z})$ consists of all the even complex-valued sequences $\Phi(n)$ defined on $\mathbb{Z}$ such that

$$\tau_{k,l}(\Phi) = \sup_{n\in\mathbb{Z}} |n^{2k}\omega_n^l \Phi(n)| < \infty, \tag{2.9}$$

for any pair of nonnegative integers k and l, where ω_n stands for the difference operator

$$\omega_n \Phi(n) = \omega\Phi(n) = \frac{\Phi(n+1) + \Phi(n-1)}{2}. \tag{2.10}$$

Note that this operator is formally selfadjoint. When endowed with the topology generated by the family of seminorms $\{\tau_{k,l}\}_{k,l\in\mathbb{N}}, T(\mathbb{Z})$ becomes a Fréchet space.

The study of the space $T(\mathbb{Z})$ runs parallel to that of the space $T(-1,1)$. Thus, it can be easily proved

Proposition 2.2. *An even complex-valued sequence $\Phi(n)$, with $n \in \mathbb{Z}$, belongs to $T(\mathbb{Z})$ when and only when $\Phi(n)$ is of rapid descent as $|n| \to \infty$.*

$T'(\mathbb{Z})$ symbolizes the dual space $T(\mathbb{Z})$. We shall assign to $T'(\mathbb{Z})$ the weak $*$ topology. As a matter of fact, it is well-known that the dual of $T(\mathbb{Z})$ is the space of slow increasing sequences, in other words, the members of $T'(\mathbb{Z})$ are of slow growth at infinity.

(vi) The operation $\Phi \longrightarrow \omega\Phi$ is a continuous linear mapping of $T(\mathbb{Z})$ into itself. Hence, the mapping $F \longrightarrow \omega F$, defined by

$$\langle\!\langle \omega F, \Phi \rangle\!\rangle = \langle\!\langle F, \omega\Phi \rangle\!\rangle, \quad F \in T'(\mathbb{Z}),\ \Phi \in T(\mathbb{Z}), \tag{2.11}$$

is also a continuous linear mapping of $T'(\mathbb{Z})$ into itself.

(vii) Every sequence $\{F(n)\}_{n\in\mathbb{Z}}$ of complex numbers such that $\sum_{n=-\infty}^{\infty} |F(n)| \leq \infty$ gives rise to a regular member in $T'(\mathbb{Z})$ by means of

$$\langle\!\langle F, \Phi \rangle\!\rangle = \sum_{n=-\infty}^{\infty} F(n)\Phi(n), \qquad \Phi \in T(\mathbb{Z}). \tag{2.12}$$

The linearity being obvious, its continuity is inferred from

$$|\langle\!\langle F, \Phi \rangle\!\rangle| \leq \tau_{0,0}(\Phi) \sum_{n=-\infty}^{\infty} |F(n)| = C\tau_{0,0}(\Phi).$$

In particular, each member Φ of the space $T(\mathbb{Z})$ generates a regular member in $T'(\mathbb{Z})$. Indeed, the series

$$\sum_{n=-\infty}^{\infty} |\Phi(n)| \leq [\tau_{0,0}(\Phi) + \tau_{1,0}(\Phi)] \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2},$$

is absolutely convergent. Notice that two sequences of $T(\mathbb{Z})$ that give rise to the same regular member of $T'(\mathbb{Z})$ have to be identical. Therefore, $T(\mathbb{Z})$ can be identified with certain subspace of $T'(\mathbb{Z})$ and the inclusion $T(\mathbb{Z}) \subset T'(\mathbb{Z})$ is justified.

(viii) Now, on the contrary to that happens in (iv), for certain $x \in (-1, 1)$, the Tchebycheff polynomials $T_n(x) \notin T(\mathbb{Z})$ considered as a function of the discrete variable n. In fact, the sequence $T_n(1-)$ is not of rapid descent at infinity since

$$\tau_{k,0}\{T_n(1-)\} = \sup_{n\in\mathbb{Z}} |n^{2k}| = \infty, \quad k = 1, 2, 3, \ldots$$

(ix) The operation $\Phi \longrightarrow n^{2r}\Phi$, $r = 0, 1, 2, \ldots$, is a continuous linear mapping of $T(\mathbb{Z})$ into itself. Therefore, the operation $F \longrightarrow n^{2r}F$ defined by

$$\langle\!\langle n^{2r}F, \Phi \rangle\!\rangle = \langle\!\langle F, n^{2r}\Phi \rangle\!\rangle, \quad F \in T'(\mathbb{Z}),\ \Phi \in T(\mathbb{Z}), \tag{2.13}$$

is also a continuous linear mapping of $T'(\mathbb{Z})$ into itself.

To see this note that, for every $k \in \mathbb{N}$, $\tau_{k,0}(n^{2r}\Phi) = \tau_{k+r,0}(\Phi)$. On the other hand, when $l = 1$ the expression $n^{2k}\omega_n\left[n^{2r}\Phi(n)\right]$ can be written

$$\frac{1}{2}\left[\left(\frac{n}{n+1}\right)^{2k}(n+1)^{2(k+r)}\Phi(n+1)+\left(\frac{n}{n-1}\right)^{2k}(n-1)^{2(k+r)}\Phi(n-1)\right],$$

if $n \neq \pm 1$, and takes the value $2^{2r}\Phi(2) = 2^{-k}\,2^{2(k+r)}\Phi(2)$ if $n = \pm 1$. Hence,

$$\tau_{k,1}(n^{2r}\Phi) \leq C_k \tau_{k+r,0}(\Phi),$$

C_k being a certain positive constant. The general case

$$\tau_{k,l}(n^{2r}\Phi) \leq C_{k,l}\tau_{k+r,0}(\Phi),$$

with $C_{k,l} > 0$, follows by induction on l. Moreover, $n^{2r}\Phi(n)$ is clearly an even sequence.

Remark 2.1. We can argue as in the proof of the property (ii) to establish the following assertion:

If $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ has no zeros on $-1 \leq x \leq 1$, then the operator $\varphi(x) \to (P(x)/Q(x))\,\varphi(x)$ is a continuous linear mapping of $T(-1,1)$ into itself. Consequently, the operator $f(x) \to (P(x)/Q(x))\,f(x)$, which is defined on $T'(-1,1)$ by

$$\big\langle (P(x)/Q(x))\,f(x), \varphi(x)\big\rangle = \big\langle f(x), (P(x)/Q(x))\,\varphi(x)\big\rangle, \quad \varphi \in T(-1,1),$$

is also a continuous linear mapping of $T'(-1,1)$ into itself. In short, $P(x)/Q(x)$ is a multiplier for both the spaces $T(-1,1)$ and $T'(-1,1)$.

Remark 2.2. It is easily seen, by invoking Proposition 2.1, that the space $T(-1,1)$ is closed under the product of functions. Even more, we can proceed as in the proof of property (ii) to establish that the members of the space $T(-1,1)$ are multipliers of the same space, that is to say, the operator $\varphi \longrightarrow \psi\varphi$, with $\varphi, \psi \in T(-1,1)$, is a continuous linear mapping of $T(-1,1)$ into itself. Hence, the operator $f \longrightarrow \psi f$, where $f \in T'(-1,1)$ and $\psi \in T(-1,1)$, defined by

$$\langle \psi f, \varphi\rangle = \langle f, \psi\varphi\rangle, \quad \varphi \in T(-1,1)$$

is also a continuous linear mapping of $T'(-1,1)$ into itself.

3. The generalized discrete Tchebycheff transformation

Firstly we investigate the behaviour of the classical Tchebycheff transformations (1.3) and (1.2) in the spaces $T(-1,1)$ and $T(\mathbb{Z})$.

Theorem 3.1. *The discrete Tchebycheff transformation* $\mathcal{T}$*, defined by* (1.3)*, is an isomorphism from* $T(\mathbb{Z})$ *into* $T(-1,1)$*. Its inverse is* $\mathcal{T}^{-1}$*, as given by* (1.2).

Proof. Let $\Phi \in T(\mathbb{Z})$. Observe that the function $(\mathcal{T}\Phi)(x) = \varphi(x)$ exists and is infinitely differentiable on $-1 \le x \le 1$. On the other hand, formally applying the operator $(1-x^2)^{1/2}D_x$ under the summation sign in (1.3), we obtain

$$\sum_{n=-\infty}^{\infty} \Phi(n) n \sin(n \arccos x). \tag{3.1}$$

This series is bounded by

$$\big[\tau_{1,0}(\Phi) + \tau_{2,0}(\Phi)\big] \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2},$$

that is to say, (3.1) converges uniformly on $-1 \le x \le 1$. By applying again the operator $(1-x^2)^{1/2}D_x$ in (3.1), the resulting series

$$\sum_{n=-\infty}^{\infty} \Phi(n) n^2 T_n(x)$$

converges uniformly on $-1 \le x \le 1$, as well. From this and (2.2) we see that the differential operator Ω_x may be applied in (1.3) term by term, to get for an arbitrary nonnegative integer k

$$\Omega_x^k \varphi(x) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Phi(n)(-1)^k n^{2k} T_n(x), \tag{3.2}$$

in view of (1.5). Finally, from (3.2) we are immediately led to

$$\gamma_k(\varphi) = \gamma_k\Big(\mathcal{T}\Phi\Big) \le C\left[\tau_{k,0}(\Phi) + \tau_{k+1,0}(\Phi)\right],$$

for a certain positive constant C. This implies that $\mathcal{T}$ is a continuous linear mapping of $T(\mathbb{Z})$ into $T(-1,1)$.

To prove the converse, assume that $\varphi \in T(-1,1)$. For any pair of nonnegative integers k and l we can write

$$\begin{aligned} n^{2k}\omega_n^l \Phi(n) &= n^{2k}\int_{-1}^{1}(1-x^2)^{-1/2}\varphi(x)\left[\omega_n^l T_n(x)\right]dx \\ &= n^{2k}\int_{-1}^{1}(1-x^2)^{-1/2}\varphi(x)x^l T_n(x)dx, \end{aligned} \tag{3.3}$$

in accordance with (1.6). Furthermore, bearing in mind (1.5) the right-hand side of (3.3) adopts the form

$$(-1)^k\int_{-1}^{1}(1-x^2)^{-1/2}\varphi(x)x^l\left[\Omega_x^k T_n(x)\right]dx. \tag{3.4}$$

By integrating by parts $2k$ times, we find

$$n^{2k}\omega_n^l \Phi(n) = (-1)^k\int_{-1}^{1}(1-x^2)^{-1/2}T_n(x)\Omega_x^k\left[x^l\varphi(x)\right]dx. \tag{3.5}$$

This expression has a sense due to property (ii) in the above paragraph. Finally, by invoking (2.7) we derive from (3.5)

$$\tau_{k,l}(\Phi) = \tau_{k,l}\left(\mathcal{T}^{-1}\varphi\right) \le \pi\gamma_k(x^l\varphi) \le \pi\sum_{i=0}^{k} C_i(k,l)\gamma_i(\varphi),$$

because of $|T_n(x)| \le 1$ on $-1 \le x \le 1$. Consequently, $\mathcal{T}^{-1}$ is also a continuous linear mapping from $T(-1,1)$ into $T(\mathbb{Z})$. Inasmuch as $\mathcal{T}$ and $\mathcal{T}^{-1}$ are inverses of each other, the proof of the assertion is complete. □

The generalized discrete Tchebycheff transformation

$$\begin{aligned} \mathcal{T}' : T'(\mathbb{Z}) &\longrightarrow T'(-1,1) \\ F(n) &\longrightarrow (\mathcal{T}'F)(x) \end{aligned}$$

is defined through

$$\pi\big\langle(\mathcal{T}'F)(x),(\mathcal{T}\Phi)(x)\big\rangle = \big\langle\!\big\langle F(n),\Phi(n)\big\rangle\!\big\rangle, \tag{3.6}$$

for any $F \in T'(\mathbb{Z})$ and $\Phi \in T(\mathbb{Z})$. So (3.6) appears as an extension of the mixed Parseval equation (1.4) to distributions.

Theorem 3.2. *The generalized discrete Tchebycheff transformation* $\mathcal{T}'$*, defined by* (3.6)*, is an isomorphism from* $T'(\mathbb{Z})$ *into* $T'(-1,1)$.

Proof. This result can be seen as a consequence of Theorem 3.1 and [18, Theorem 1.10-1]. Notice that the inverse $\mathcal{T}'^{-1}$ is easily supplied by setting $(\mathcal{T}'F)(x) = f(x)$ and $(\mathcal{T}\Phi)(x) = \varphi(x)$.Then, (3.6) may be rewritten

$$\left\langle\!\!\left\langle (\mathcal{T}'^{-1}f)(n), (\mathcal{T}^{-1}\varphi)(n) \right\rangle\!\!\right\rangle = \pi \left\langle f(x), \varphi(x) \right\rangle, \tag{3.7}$$

for any $f \in T'(-1,1)$ and $\varphi \in T(-1,1)$. □

Corollary 3.1. *The classical discrete Tchebycheff transformation* $\mathcal{T}$*, when acting on* $T(\mathbb{Z})$*, is a special case of the generalized transformation* $\mathcal{T}'$ *as defined by* (3.6)–(3.7)*, in other words,*

$$\mathcal{T}(F) = \mathcal{T}'(F), \tag{3.8}$$

for an arbitrary $F \in T(\mathbb{Z})$.

Proof. Let $F, \Phi \in T(\mathbb{Z})$. By resorting to property (vii) in Section 2, expression (3.8) has a meaning. To see the equality, recall that $F(n)$ generates a regular member in $T'(\mathbb{Z})$ by means of (2.12)

$$\pi\langle \mathcal{T}'F, \mathcal{T}\Phi\rangle = \langle\!\langle F, \Phi \rangle\!\rangle = \sum_{n=-\infty}^{\infty} F(n)\Phi(n). \tag{3.9}$$

By applying the Parseval equality (1.4) to the right-hand side in (3.9), we get that

$$\begin{aligned} \langle \mathcal{T}'F, \mathcal{T}\Phi\rangle &= \int_{-1}^{1} (1-x^2)^{-1/2} f(x)\varphi(x)dx \\ &= \int_{-1}^{1} (1-x^2)^{-1/2} (\mathcal{T}F)(x)(\mathcal{T}\Phi)(x)dx \end{aligned} \tag{3.10}$$

where $(\mathcal{T}F)(x) = f(x)$ and $(\mathcal{T}\Phi)(x) = \varphi(x)$ belong to the space $T(-1,1)$ by virtue of Theorem 3.1. Finally, taking into account (2.8), the right-hand side in (3.10) can be written $\langle \mathcal{T}F, \mathcal{T}\Phi\rangle$. Therefore,

$$\langle \mathcal{T}'F, \mathcal{T}\Phi\rangle = \langle \mathcal{T}F, \mathcal{T}\Phi\rangle,$$

which is what we wished to prove. □

Remark 3.1. Since $T_n(x) \notin T(\mathbb{Z})$ considered as function of n, for some $-1 \leq x \leq 1$, it is impossible to define the generalized transformation $\mathcal{T}'$ of $F \in T'(\mathbb{Z})$ through

$$(\mathcal{T}'\, F)(x) = \frac{1}{\pi} \langle\!\langle F(n), T_n(x) \rangle\!\rangle,$$

because this expression has no sense. By this reason, we are obliged to adopt the most general definition (3.6).

Remark 3.2. Note that the Parseval relation (1.4) holds for any Φ and Ψ in $T(\mathbb{Z})$, where $\varphi(x) = (\mathcal{T}\Phi)(x)$ and $\psi(x) = (\mathcal{T}\Psi)(x)$. This formula is readily obtained by substituting $\varphi(x) = (\mathcal{T}\Phi)(x)$ into the right-hand side of (1.4) and then bearing in mind that the signs of the summation and the integral can be interchanged.

In the next assertion we compile the main operational rules of the discrete Tchebycheff transformations $\mathcal{T}$ and $\mathcal{T}'$.

Proposition 3.1. *If $\Phi \in T(\mathbb{Z})$, then*

$$\mathcal{T}\left\{\omega_n^k \Phi(n)\right\}(x) = x^k (\mathcal{T}\Phi)(x), \quad k = 0, 1, 2, \ldots \tag{3.11}$$

$$\mathcal{T}\Big\{\big(-n^2\big)^k\, \Phi(n)\Big\}(x) = \Omega_x^k (\mathcal{T}\Phi)(x), \quad k = 0, 1, 2, \ldots \tag{3.12}$$

and when $F \in T'(\mathbb{Z})$, we find

$$\mathcal{T}'\left\{\omega_n^k F(n)\right\}(x) = x^k (\mathcal{T}'F)(x), \quad k = 0, 1, 2, \ldots \tag{3.13}$$

$$\mathcal{T}'\Big\{\big(-n^2\big)^k\, F(n)\Big\}(x) = \Omega_x^{*k} (\mathcal{T}'F)(x), \quad k = 0, 1, 2, \ldots \tag{3.14}$$

Proof. Let $\Phi \in T(\mathbb{Z})$. To obtain (3.11), assume firstly that $k = 1$. Then, from definitions (1.3) and (2.10), we get

$$\begin{aligned}
\mathcal{T}\left\{\omega_n \Phi(n)\right\} &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\Phi(n+1) + \Phi(n-1)\right] T_n(x) \\
&= \frac{1}{2\pi} \left\{ \sum_{n=-\infty}^{\infty} \Phi(n) T_{n-1}(x) + \sum_{n=-\infty}^{\infty} \Phi(n) T_{n+1}(x) \right\} \\
&= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Phi(n)\, x T_n(x) = x\, (\mathcal{T}\Phi),
\end{aligned}$$

by using (1.6). Since $T(\mathbb{Z})$ is closed under the difference operator ω_n (Remember (vi) of Section 2), we may repeatedly apply this result to deduce (3.11). To see (3.12), by invoking property (ix) of the paragraph 2, the series

$$\sum_{n=-\infty}^{\infty} (-n^2)^k \Phi(n) T_n(x)$$

is bounded by

$$\left[\tau_{k+1,0}(\Phi) + \tau_{k,0}(\Phi)\right] \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2},$$

in other words, it converges uniformly on $-1 \leq x \leq 1$. By this reason, making use previously of (1.5), we are authorized to interchange the differential operator Ω_x and the summation sign, and so on until we reach

$$\begin{aligned} \mathcal{T}\left\{(-n^2)^k\, \Phi(n)\right\}(x) &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Phi(n) \left[\Omega_x^k T_n(x)\right] \\ &= \Omega_x^k \left(\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Phi(n) T_n(x) \right) = \Omega_x^k\,(\mathcal{T}\Phi). \end{aligned}$$

Let $F \in T'(\mathbb{Z})$. To verify (3.13), assume again that $k = 1$ and invoke our definitions (3.7) and (2.11) to write for every $\Phi \in T(\mathbb{Z})$

$$\begin{aligned} \left\langle \mathcal{T}' \left\{\omega_n F(n)\right\}, \mathcal{T}\Phi(n) \right\rangle &= \pi^{-1} \left\langle\!\left\langle \omega_n F(n), \Phi(n) \right\rangle\!\right\rangle = \pi^{-1} \left\langle\!\left\langle F(n), \omega_n \Phi(n) \right\rangle\!\right\rangle \\ &= \left\langle \mathcal{T}' F(n), \mathcal{T}\left\{\omega_n \Phi(n)\right\} \right\rangle. \end{aligned} \tag{3.15}$$

Finally, according to (3.11) and (2.4), the right-hand side in (3.15) becomes

$$\left\langle \mathcal{T}' F(n), x\left\{\mathcal{T}\Phi(n)\right\} \right\rangle = \left\langle x\left\{\mathcal{T}' F(n)\right\}, \mathcal{T}\Phi(n) \right\rangle. \tag{3.16}$$

By combining (3.15) and (3.16) we deduce (3.13) when $k = 1$, the general formula being confirmed by repeating the process. In order to establish (3.14) it suffices to bear in mind (3.7), (2.13), (3.12) and (2.3). □

Remark 3.3. It is worth to notice that the operational rules (3.11) and (3.13) involve the finite difference operator ω_n, unlike the results achieved in [7, Propositions 5.1 and 5.1'] concerning the Jacobi differential operator [7, (2.2)].

4. The discrete convolution in the spaces $T(\mathbb{Z})$ and $T'(\mathbb{Z})$ Applications

In [2] we introduced the discrete Tchebycheff translation Θ_l by means of

$$\Theta_l\Phi(m) = \frac{\Phi(l+m)+\Phi(l-m)}{2}, \qquad l,m\in\mathbb{Z}, \tag{4.1}$$

for every $\Phi\in T(\mathbb{Z})$.

In next assertion the main properties of this operator are listed

Proposition 4.1.

(i) *The operation $\Phi \longrightarrow \Theta_l\Phi$ is a continuous linear mapping from the space $T(\mathbb{Z})$ into itself, for every $l\in\mathbb{Z}$.*

(ii) $\mathcal{T}(\Theta_l\Phi)(x) = T_l(x)(\mathcal{T}\Phi)(x)$, *for every $\Phi\in T(\mathbb{Z})$ and $l\in\mathbb{Z}$.*

(iii) *The mapping $F\to\Theta_l F$, given by*

$$\langle\!\langle \Theta_l F(m),\Phi(m)\rangle\!\rangle = \langle\!\langle F(m),\Theta_l\Phi(m)\rangle\!\rangle, \qquad \Phi\in T(\mathbb{Z}),$$

is too a continuous linear mapping from $T'(\mathbb{Z})$ into itself, for each $l\in\mathbb{Z}$.

(iv) $\mathcal{T}'(\Theta_l F)(x) = T_n(x)(\mathcal{T}'F)(x)$, $\quad F\in T'(\mathbb{Z})$.

Proof. Let $\Phi\in T(\mathbb{Z})$. To verify (i), notice firstly that an inductive argument on $s\in\mathbb{N}$ allows us to obtain

$$\omega^s\Theta_l\Phi(n) = \frac{1}{2^s}\left\{\sum_{j=0}^{s}\binom{s}{j}\Phi(n+l+s-2j) + \sum_{j=0}^{s}\binom{s}{j}\Phi(n-l+s-2j)\right\},$$

for every $l,n\in\mathbb{Z}$. Then, making use of this formula, we can conclude that

$$\tau_{r,s}\Big(\Theta_l\Phi\Big) = \sup_{n\in\mathbb{Z}}\Big|n^{2r}\omega_n^s\Big(\Theta_l\Phi(n)\Big)\Big| \le C_{r,s,l}\tau_{r,0}(\Phi),$$

for every $r,s\in\mathbb{N}$ and $l\in\mathbb{Z}$, where $C_{r,s,l}$ is certain positive constant.

To prove (ii) it is sufficient to recall the corresponding definitions, make a change of indices and invoke (1.7) to get

$$\begin{aligned}\mathcal{T}(\Theta_l\Phi)(x) &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\Phi(n+l)+\Phi(n-l)}{2}T_n(x)\\ &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\Phi(n)\frac{T_{n+l}(x)+T_{n-l}(x)}{2} = T_l(x)(\mathcal{T}\Phi)(x).\end{aligned}$$

Properties (iii) and (iv) are simple consequences of the above results. □

The convolution $\Phi \circ \Psi$ of $\Phi \in T(\mathbb{Z})$ and $\Psi \in T(\mathbb{Z})$, is defined through

$$(\Phi \circ \Psi)(l) = \sum_{n=-\infty}^{\infty} \Phi(n)(\Theta_l \Psi)(n), \qquad l \in \mathbb{Z}.$$

By taking into account the interchange formula ([2, (2.5)]), since the pointwise multiplication is a closed operation in $T(-1,1)$, we can see that the $\circ$-convolution defines a bilinear and continuous mapping from $T(\mathbb{Z}) \times T(\mathbb{Z})$ into $T(\mathbb{Z})$.

We now characterize the convolution operators on $T(\mathbb{Z})$ as those continuous linear mappings from $T(\mathbb{Z})$ into itself that commut with the translations Θ_l, $l \in \mathbb{Z}$, or with the difference operator ω, or with the $\circ$-convolution.

Proposition 4.2. *Let L be a continuous linear mapping from $T(\mathbb{Z})$ into itself. The following properties are equivalent.*

- (i) *There exists a unique $\Phi \in T(\mathbb{Z})$ such that $L(\Psi) = \Phi \circ \Psi$, for every $\Psi \in T(\mathbb{Z})$.*
- (ii) *L commuts with the operator ω, that is, $L(\omega\Phi) = \omega L(\Phi)$, for every $\Phi \in T(\mathbb{Z})$.*
- (iii) *L commuts with Θ_l, $l \in \mathbb{Z}$, that is to say, $L(\Theta_l \Phi) = \Theta_l L(\Phi)$, for every $\Phi \in T(\mathbb{Z})$.*
- (iv) *L commuts with the $\circ$-convolution, in other words, $L(\Phi \circ \Psi) = \Phi \circ L(\Psi)$, for every $\Phi, \Psi \in T(\mathbb{Z})$.*

Proof. (i) $\Longrightarrow$ (ii) This property can be immediately inferred from (3.11) and the interchange formula [2, (2.5)].

(ii) $\Longrightarrow$ (iii) Let $\Phi \in T(\mathbb{Z})$ and $l \in \mathbb{Z}$. According to Proposition 4.1, (ii), we have

$$\mathcal{T}(\Theta_l \Phi)(x) = T_l(x)\mathcal{T}(\Phi)(x), \qquad x \in (-1,1).$$

Hence, from (3.11) and Theorem 3.1 we deduce

$$\Theta_l \Phi = T_l(\omega)\Phi.$$

Then, according to (ii),

$$L(\Theta_l\Phi) = T_l(\omega)L(\Phi) = \Theta_l L(\Phi).$$

Thus (iii) is established.

(iii) $\Longrightarrow$ (iv) Let $\Phi, \Psi \in T(\mathbb{Z})$. We can write

$$(\Phi \circ \Psi)(l) = \sum_{n=-\infty}^{\infty} \Phi(n)(\Theta_l\Psi)(n), \qquad l \in \mathbb{Z}, \tag{4.2}$$

where the convergence of the series is understood in $T(\mathbb{Z})$. Indeed, let $\alpha, \beta \in \mathbb{N}$. By resorting to Theorem 3.1 and the inequality $|T_n^{(j)}(x)| \leq |n|^{2j}$, $j \in \mathbb{N}$ and $x \in (-1,1)$, involving the derivatives of the Tchebycheff polynomials [14, p. 51], for certain $C > 0$ and $s \in \mathbb{N}$, we obtain

$$\begin{aligned}\sup_{l\in\mathbb{Z}} \left| l^{2\beta}\omega_l^\alpha(\Theta_l\Psi)(n)\right| &\leq C \max_{0\leq m\leq s} |D^m(T_n(x)\mathcal{T}(\Psi)(x))| \\ &\leq C\Big(|n|^{2s}+1\Big), \qquad n \in \mathbb{Z}.\end{aligned}$$

Then, for every $m \in \mathbb{Z}$,

$$\begin{aligned}&\sup_{l\in\mathbb{Z}} \left| l^{2\beta}\omega_l^\alpha\Big(\sum_{|n|\geq m} \Phi(n)(\Theta_l\Psi)(n)\Big)\right| \\ &\leq C\big[\tau_{s+1,0}(\Phi) + \tau_{s,0}(\Phi) + \tau_{1,0}(\Phi) + \tau_{0,0}(\Phi)\big] \sum_{|n|\geq m} \frac{1}{1+n^2}.\end{aligned}$$

Hence the series in (4.2) converges in $T(\mathbb{Z})$.

Since L is a continuous linear mapping from $T(\mathbb{Z})$ into itself, from (iii) we deduce that

$$\begin{aligned}L(\Phi \circ \Psi)(l) &= \sum_{n=-\infty}^{\infty} \Phi(n)L(\Theta_n\Psi)(l) = \sum_{n=-\infty}^{\infty} \Phi(n)(\Theta_n L(\Psi))(l) \\ &= (\Phi \circ (L\Psi))(l), \qquad l \in \mathbb{Z}.\end{aligned}$$

Thus (iv) is proved.

(iv) $\Longrightarrow$ (i) We define the sequence Φ_0 by

$$\Phi_0(n) = 0, \quad n \in \mathbb{Z}\setminus\{0\}, \quad \text{and} \quad \Phi_0(0) = 1.$$

It is clear that $\Phi_0 \in T(\mathbb{Z})$. Then $L(\Phi_0) \in T(\mathbb{Z})$. Since $\Phi_0 \circ \Phi = \Phi$, for every $\Phi \in T(\mathbb{Z})$, from (iv) we infer that

$$L(\Phi) = L(\Phi_0 \circ \Phi) = L(\Phi_0) \circ \Phi, \qquad \Phi \in T(\mathbb{Z}).$$

The interchange formula [2, (2.5)] implies that if $\Phi \in T(\mathbb{Z})$ and $L(\Psi) = \Phi \circ \Psi$, $\Psi \in T(\mathbb{Z})$, then $\Phi = L(\Phi_0)$. □

Now we can define the convolution $F \circ \Phi$ of $F \in T'(\mathbb{Z})$ with $\Phi \in T(\mathbb{Z})$ by means of

$$(F \circ \Phi)(n) = \langle\!\langle F(l), \Theta_n \Phi(l) \rangle\!\rangle. \tag{4.3}$$

Note that (4.3) is well-defined by virtue of property (i) in Proposition 4.1.

Proposition 4.3.

(i) *Let $F \in T'(\mathbb{Z})$ and $\Phi \in T(\mathbb{Z})$. Then, $F \circ \Phi \in T'(\mathbb{Z})$ and, for every $\Psi \in T(\mathbb{Z})$, next equality holds*

$$\langle\!\langle F \circ \Phi, \Psi \rangle\!\rangle = \langle\!\langle F, \Phi \circ \Psi \rangle\!\rangle. \tag{4.4}$$

Moreover, the mapping $F \longrightarrow F \circ \Phi$ is continuous from the space $T'(\mathbb{Z})$ into itself.

(ii)

$$\mathcal{T}'(F \circ \Phi) = (\mathcal{T}'F)(\mathcal{T}\Phi), \tag{4.5}$$

for every $\Phi \in T(\mathbb{Z})$ and $F \in T'(\mathbb{Z})$.

(iii) *Let $F \in T'(\mathbb{Z})$. Then $F \circ \Phi \in T(\mathbb{Z})$, for every $\Phi \in T(\mathbb{Z})$, if and only if $F \in T(\mathbb{Z})$.*

(iv) *$\omega(F \circ \Phi) = (\omega F) \circ \Phi = F \circ (\omega \Phi)$, for all $F \in T'(\mathbb{Z})$ and $\Phi \in T(\mathbb{Z})$.*

(v) *$\Theta_l(F \circ \Phi) = (\Theta_l F) \circ \Phi = F \circ (\Theta_l \Phi)$, for every $l \in \mathbb{Z}$, $F \in T'(\mathbb{Z})$ and $\Phi \in T(\mathbb{Z})$.*

Proof. We start proving (i). Since $F \in T'(\mathbb{Z})$ there exist $C > 0$ and $r \in \mathbb{N}$ such that

$$|(F \circ \Phi)(n)| \leq C \sup_{\substack{l \in \mathbb{Z} \\ 0 \leq s, k \leq r}} \left| l^{2s} \omega_l^k (\Theta_n \Phi)(l) \right|, \qquad n \in \mathbb{Z}.$$

Then, according to Theorems 3.1 and 4.1, (ii), we can write

$$|(F \circ \Phi)(n)| \le C \sup_{\substack{x\in(-1,1)\\ 0\le k\le r}} \Big| D^k \Big(T_n(x)\mathcal{T}(\Phi)(x) \Big) \Big|, \qquad n \in \mathbb{Z},$$

for certain $C > 0$ and $r \in \mathbb{N}$. Hence, by invoking [14, p. 51, 1.5.35], we conclude that

$$|(F \circ \Phi)(n)| \le C\Big(|n|^{2r} + 1\Big), \qquad n \in \mathbb{Z}.$$

Thus, we show that $F \circ \Phi$ is a slowly decreasing sequence. In accordance with (2.12), $F \circ \Phi$ defines an element of $T'(\mathbb{Z})$ by

$$\langle\!\langle F \circ \Phi, \Psi \rangle\!\rangle = \sum_{n=-\infty}^{\infty} (F \circ \Phi)(n)\Psi(n), \qquad \Psi \in T(\mathbb{Z}).$$

To establish (4.4) it is sufficient to argue as in the proof of the implication (iii) $\Longrightarrow$ (iv) in Proposition 4.2.

Finally, since the $\circ$-convolution defines a continuous bilinear mapping from $T(\mathbb{Z}) \times T(\mathbb{Z})$ into $T(\mathbb{Z})$, for every $\Phi \in T(\mathbb{Z})$, the mapping $F \to F \circ \Phi$ is continuous from the space $T'(\mathbb{Z})$ into itself.

To see (ii), let $F \in T'(\mathbb{Z})$ and $\Phi \in T(\mathbb{Z})$. Then, for every $\Psi \in T(\mathbb{Z})$, making use of (3.6), [2, (2.5)] and Remark 2.2, we deduce that

$$\begin{aligned}
\pi\langle \mathcal{T}'(F \circ \Phi), \mathcal{T}\Psi \rangle &= \langle\!\langle F \circ \Phi, \Psi \rangle\!\rangle \\
&= \sum_{n=-\infty}^{\infty} (F \circ \Phi)(n)\Psi(n) = \sum_{n=-\infty}^{\infty} \langle\!\langle F(l), \Theta_n \Phi(l) \rangle\!\rangle \Psi(n) \\
&= \Big\langle\!\!\Big\langle F(l), \sum_{n=-\infty}^{\infty} \Psi(n)\Theta_n\Phi(l) \Big\rangle\!\!\Big\rangle = \langle\!\langle F, \Phi \circ \Psi \rangle\!\rangle \\
&= \pi\langle \mathcal{T}'F, (\mathcal{T}\Phi)(\mathcal{T}\Psi) \rangle = \pi\langle (\mathcal{T}'F)(\mathcal{T}\Phi), \mathcal{T}\Psi \rangle.
\end{aligned}$$

The proof of (4.5) is now complete.

To verify (iii) observe that, as it was pointed out previously, if F and $\Phi \in T(\mathbb{Z})$ then $F \circ \Phi$ is also in $T(\mathbb{Z})$. Conversely, suppose now that $F \circ \Phi \in T(\mathbb{Z})$, for each $\Phi \in T(\mathbb{Z})$. Then, according to (4.5) we have that

$$\mathcal{T}'(F)\mathcal{T}(\Phi) \in T(-1,1), \qquad \Phi \in T(\mathbb{Z}). \tag{4.6}$$

By taking into account Proposition 2.1, it is not hard to see that the functions $\phi_1(x) = e^x$ and $\phi_2(x) = e^{-x}$, $x \in [-1, 1]$, are in $T(-1, 1)$. Hence, from (4.6) we can infer that $\mathcal{T}'(F) \in T(-1, 1)$. By Theorem 3.1 we conclude that $F \in T(\mathbb{Z})$.

Finally, properties (iv) and (v) are rapidly inferred of the operational rules (3.11) and (4.3), and the inversion formula (Theorem 3.2). $\square$

To illustrate the use of the discrete Tchebycheff transformation , we propose to solve the following problems involving certain difference equations

(a) To find the solution $U(n) \in T'(\mathbb{Z})$ of

$$P(\omega)U(n) = G(n), \tag{4.7}$$

where $G(n)$ is a known member of $T'(\mathbb{Z})$ and $P(z)$ denotes a polynomial having no roots on $-1 \le x \le 1$. By applying $\mathcal{T}'$ to (4.7) and setting $u(x) = (\mathcal{T}'U)(x)$ and $g(x) = (\mathcal{T}'G)(x)$, we get in line with (3.13)

$$P(x)\, u(x) = g(x). \tag{4.8}$$

By virtue of Theorem 3.2, u and g belong to $T'(-1, 1)$. On the other hand, in agreement with Remark 2.1, $1/P(x)$ is a multiplier for both the spaces $T(-1, 1)$ and $T'(-1, 1)$. Then, taking the inverse Tchebycheff transformation, we are led to the desired solution

$$U(n) = \mathcal{T}'^{-1}\left\{\frac{1}{P(x)} g(x)\right\}. \tag{4.9}$$

In accordance with (3.7) and (3.8) we can express the solution (4.9) in terms of the known sequence G

$$\begin{aligned}\langle\!\langle U(n), \Phi(n)\rangle\!\rangle &= \pi\Big\langle \frac{1}{P(x)} g(x), \varphi(x)\Big\rangle = \pi\Big\langle g(x), \frac{1}{P(x)}\varphi(x)\Big\rangle \\ &= \Big\langle\!\!\Big\langle G(n), \mathcal{T}^{-1}\Big\{\frac{1}{P(x)}\varphi(x)\Big\}(n)\Big\rangle\!\!\Big\rangle,\end{aligned}$$

where Φ is an arbitrary member of $T(\mathbb{Z})$ and $\varphi(x) = (\mathcal{T}\Phi)(x)$. Consequently, in view of (1.2), the solution $U(n)$ of (4.7) is given through the functional in $T'(\mathbb{Z})$

$$\langle\!\langle U(n), \Phi(n)\rangle\!\rangle = \Big\langle\!\!\Big\langle G(n), \int_{-1}^{1} (1 - x^2)^{-1/2}\, \frac{\varphi(x)}{P(x)} T_n(x)dx\Big\rangle\!\!\Big\rangle.$$

(b) To determine the solution $u(m,n)$ in $T'(\mathbb{Z})$, for every $n \in \mathbb{N}$, of the partial difference equation ($m \in \mathbb{Z}$)

$$\frac{1}{4}u(m+2,n) + u(m+1,n) + \frac{3}{2}u(m,n) + u(m-1,n) + \frac{1}{4}u(m-2,n) - u(m,n+2) = 0, \tag{4.10}$$

satisfying the conditions

$$u(m,0) = A(m) \in T'(\mathbb{Z}), \qquad u(m,1) = 0. \tag{4.11}$$

Note that the use of the difference operator ω_n allows us to rewrite the equation (4.10) as follows

$$\omega_m^2 u(m,n) + 2\omega_m u(m,n) + u(m,n) - u(m,n+2) = 0. \tag{4.12}$$

Now by setting $U(x,n) = \mathcal{T}' u(m,n)$ and applying the operational rule (3.11), the problem (4.12) becomes

$$(x^2 + 2x + 1)U(x,n) - U(x,n+2) = 0,$$

which is an ordinary difference equation whose general solution is

$$U(x,n) = (C_1 + C_2 n)(x+1)^n.$$

Initial conditions (4.11) suggest to choose $C_1 = -C_2 = a(x)$, where $a(x) = (\mathcal{T}'A)(x)$. Thus, the transform solution is

$$U(x,n) = (1-n)(x+1)^n a(x).$$

By resorting to Theorem 3.1 we can obtain the solution in the form

$$u(m,n) = \mathcal{T}'^{-1}\{U(x,n)\} = (1-n)\int_{-1}^{1}(1-x^2)^{-1/2} a(x) T_m(x)(x+1)^n dx,$$

or, bearing in mind (4.5) and [13, p. 452], in this alternative form involving the discrete convolution

$$u(m,n) = (1-n)2^n B(1/2, n+1/2) \times \left[{}_3F_2(-m, m, 1/2; 1/2, n+1; 1) \circ A(m) \right],$$

where B and ${}_3F_2$ represent the Beta function and the hypergeometric function, respectively.

To see other applications of the discrete Tchebycheff transform, we refer to [2].

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