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Framed $(2M+3)$-dimensional manifolds endowed with a vertical cyclic connection structure

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Abstract. Geometrical and structural properties are proved for a class of framed manifolds which are equiped with a vertical cyclic connection structure.

1. Introduction

Framed manifolds and f-structures have been initiated by K. Yano and M. Kon and have subsequently been studied intensively, see for example [1], [19], [22], [18]. We recall that if $M(\phi,\Omega,\xi_r,\eta^r,g)$ is a $(2m+q)$-dimensional manifold of this kind, then the ξ_r, for $(r=2m+1,\dots,2m+q)$, are the Reeb vector fields (in the large sense) of the f-structure, and $\eta^r={\xi_r}^\flat$ their corresponding covectors. One has the following structure equations:

$$\phi^2=-\operatorname{Id}+\sum\eta^r\otimes\xi_r,\quad \phi\xi_r=0,\quad \eta^r\circ\phi=0,\quad \eta^s(\xi_r)=\delta^s_r, \tag{1}$$

where ϕ is a (1.1) tensor field. With respect to g, one has the following relation

$$g(\phi Z,Z')+g(Z,\phi Z')=0,\qquad Z,Z'\in\Xi(M),$$

(i.e. ϕ is skew-symmetric with respect to g). The 2-form Ω of rank $2m$ has

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the following properties

$$\Omega(Z,Z') = g(\phi Z, Z'), \quad \Omega^m \wedge \eta^{2m+1} \wedge \cdots \wedge \eta^{2m+q} \neq 0, \tag{2}$$

and is called the fundamental form of the framed manifold.

In the present paper we assume that $r \in \{2m+1, 2m+2, 2m+3\}$ and for the indices a and b we have the following range $a, b \in \{1, \ldots, 2m\}$. Under these conditions and with reference to [18], we call θ^a_b, θ^r_a, and θ^r_s the horizontal, the transversal, and the vertical connection forms respectively. We will assume here that the θ^r_a vanish and that the θ^r_s are defined by a cyclic permutation of the Reeb covectors η^r, which means that

$$\theta^r_s = f_s \eta^r - f_r \eta^s, \qquad \forall\ \widehat{r, s, t} \tag{3}$$

where the f_r are scalar fields, called the principal scalars on M. In the sequel we will call

$$\eta = f_r \eta^r, \quad \text{and} \quad \eta^\sharp = \sum f_r \xi_r, \tag{4}$$

the principal pfaffian and the principal vector field of M respectively. Further, let ${D_p}^\top = \{e_a\}$ and ${D_p}^\perp = \{\xi_r\}$ be the horizontal, respectively vertical, distribution on M.

In a first step, the following properties are proved.

(i) The manifold M under consideration may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is a $2m$-dimensional submanifold tangent to ${D_p}^\top$ and $M^\perp$ is a 3-dimensional submanifold tangent to ${D_p}^\perp$, and the immersion $x : M^\top \to M$ is totally geodesic;

(ii) the Ricci tensor field $\mathcal{R}$ of $M^\perp$ is expressed by

$$\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \qquad Z \in \Xi(M);$$

(iii) ξ is harmonic and if V is any vertical vector which has the property to be a skew-symmetric Killing vector field having ξ as generative, then V is an exterior concurrent vector field and by Bochner's theorem $g(V, \xi)$ is closed;

(iv) the principal scalars f_r define an isoparametric system [20];

(v) the gradients ${df_r}^\sharp = \operatorname{grad} f_r$, define a commutative group.

In a second step, and making use of E. Cartan's structure equations involving the curvature 2-forms, one finds that the vertical curvature 2-forms Θ_r^s satisfy

$$\Theta_r^s = \left(\left(\|\xi\|^2 - \frac{{f_t}^2}{2}\right)\eta^r + f_r f_t \eta^t\right) \wedge \eta^s - \left(\left(\|\xi\|^2 - \frac{{f_t}^2}{2}\right)\eta^s + f_s f_t \eta^t\right) \wedge \eta^r, \quad \forall\ \widehat{r,s,t},$$

and consequently, following [19] the above equations prove that $M^\perp$is a conformally flat submanifold of M.

Finally, the structure 2-form Ω of M is presymplectic. Then, if X is any horizontal vector field and ${}^\flat X$ $(= -i_X\Omega)$ means the symplectic isomorphism, and in addition the 1-form ${}^\flat X$ is ϕ-closed, it follows that Ω is invariant by X. In consequence of this, X is a 2-covariant recurrent vector field, which in the case under consideration is expressed by

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X, \qquad \lambda \in \Lambda^0 M.$$

2. Preliminaries

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator with respect to the metric tensor g. We assume that M is oriented and ∇ is the Levi–Civita connection of g. Let ΓTM be the set of sections of the tangent bundle, and

$$\flat : TM \xrightarrow{\flat} T^*M \quad \text{and} \quad \sharp : TM \xleftarrow{\sharp} T^*M$$

the isomorphisms defined by g (i.e. $^\flat$ is the index lowering operator, and $^\sharp$ is the index raising operator).

Following [14], we denote by

$$A^q(M, TM) = \Gamma\,\mathrm{Hom}(\Lambda^q TM, TM),$$

the set of vector valued q-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM). \tag{5}$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M, and is also called the soldering form of M [4]. Since ∇ is symmetric one has that $d^\nabla(dp) = 0$.

A vector field Z which satisfies

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M, \tag{6}$$

is defined to be an exterior concurrent vector field [16] (see also [13]). The 1-form π in (6) is called the concurrence form and is defined by

$$\pi = \lambda Z^\flat, \qquad \lambda \in \Lambda^0 M. \tag{7}$$

In this case, if $\mathcal{R}$ is the Ricci tensor of ∇, one has

$$\mathcal{R}(Z, V) = \varepsilon(n-1)\lambda g(Z, V) \tag{8}$$

$(\varepsilon = \pm 1,\ V \in \Xi(M),\ n = \dim M)$.

A function $\mathbb{R}^n \to \mathbb{R}$ is isoparametric [20] if $\|\nabla f\|^2$ and $\operatorname{div}(\nabla f)$ are functions of f $(\nabla f = \operatorname{grad} f)$.

Let $\mathcal{O} = \{e_A \mid A = 1, \dots n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \operatorname{covect}\{\omega^A\}$ be its associated coframe. Then E. Cartan's structure equations can be written in indexless manner as

$$\nabla e = \theta \otimes e, \tag{9}$$

$$d\omega = -\theta \wedge \omega, \tag{10}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{11}$$

In the above equations θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-forms on M).

3. The main theorem

Let $M(\phi, \Omega, \xi_r, \eta^r, g)$ be a $(2m+3)$-dimensional C^∞-manifold with soldering form dp and carrying an f-structure ϕ [22], that is a tensor field of type (1.1) of rank $2m$ which satisfies

$$\phi^3 + \phi = 0, \tag{12}$$

$$\phi^2 = -\operatorname{Id} + \sum \eta^r \otimes \xi_r, \quad \phi\xi_r = 0, \quad \eta^r \circ \phi = 0, \tag{13}$$

$$g(Z, Z') = g(\phi Z, \phi Z') + \sum \eta^r(Z)\eta^r(Z'), \tag{14}$$

where Id is the identity morphism of M.

If in addition the fundamental 2-form Ω of M satisfies

$$\Omega(Z, Z') = g(\phi Z, Z'), \quad \Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \neq 0, \tag{15}$$

then M is known [22] to be a framed f-manifold.

With respect to the cobasis $\mathcal{O}^* = \operatorname{covect}\{\omega^a, \eta^r\}$ of $\mathcal{O} = \operatorname{vect}\{e_a, \xi_r\}$ ($1 \leq a \leq 2m$; $2m+1 \leq r \leq 2m+3$), the 2-form Ω is expressed by the standard form

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*}, \quad i^* = i + m. \tag{16}$$

Making use of (9) and (13), one finds the known Kaehlerian relations

$$\theta^i_j = \theta^{i^*}_{j^*}, \quad \theta^{i^*}_j = \theta^{j^*}_i. \tag{17}$$

We recall [18] that one may split the tangent space $T_p(M)$ of M at every point $p \in M$ as

$$T_p(M) = D_p{}^{\top} \oplus D_p{}^{\perp}, \tag{18}$$

where $D_p{}^{\top} = \{e_a \mid a \in \{1, \ldots, 2m\}\}$ and $D_p{}^{\perp} = \{\xi_r\}$ are two complementary orthogonal distributions, called the horizontal and the vertical distribution respectively. As a consequence of this decomposition, one may write the soldering form as

$$dp = dp^{\top} \oplus dp^{\perp}, \tag{19}$$

where $dp^{\top} = dp \mid_{D^{\top}}$ and $dp^{\perp} = dp \mid_{D^{\perp}}$. By reference to [18] (see also [12]), the connection forms θ^a_b, θ^r_s, and θ^a_r are called the horizontal, the vertical, and the transversal connection forms respectively. In the present paper we assume that the θ_r^a vanish and that the vertical connection forms are defined by a cyclic permutation of the Reeb covectors η^r, that is:

$$\theta_s^r = f_s\eta^r - f_r\eta^s, \qquad \forall\ \widehat{r,s,t} \quad \text{(cyclic)}. \tag{20}$$

In the above relations, the f_r are scalar fields, called the principal scalars on M, and setting

$$\eta = f_r\eta^r, \quad \eta^\sharp = \xi = \sum f_r\xi_r, \tag{21}$$

η and ξ are called the principal pfaffian and the principal vector field respectively. Taking into account that

$$\theta_r^a = 0, \tag{22}$$

one derives by (10) and (20) that

$$d\eta^r = \eta \wedge \eta^r. \tag{23}$$

This shows that the Reeb covectors are η^r exterior recurrent forms [3]. In addition, exterior differentiation of (23) and taking into account (21), yields

$$df_r = f_r\eta, \tag{24}$$

which expresses that η is an exact form. Since one has that

$$(d\eta^r) \neq 0, \quad \eta^r \wedge d\eta^r = 0,$$

it follows according to a known definition [6] that in the case under discussion the Reeb covectors are of class 2. Let now

$$\varphi^\perp = \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \tag{25}$$

and

$$\varphi^\top = \omega^1 \wedge \cdots \wedge \omega^{2m} \tag{26}$$

be the simple unit forms which correspond to the distributions ${D_p}^\perp$ and ${D_p}^\top$ respectively. Taking the exterior derivative of (25) and (26), and in view of (20) and (22), one derives that

$$d\varphi^\perp = 0 \tag{27}$$

and

$$d\varphi^\top = 0. \tag{28}$$

Hence, in terms of well known terminology [9], the above equations show that $\varphi^{\perp}$ and $\varphi^{\top}$ are integral invariants of ${D_p}^{\perp}$ and ${D_p}^{\top}$ respectively. Therefore, by the theorem of Frobenius, we conclude that the manifold M under consideration may be viewed as the local Riemannian product

$$M = M^{\top} \times M^{\perp}, \tag{29}$$

where $M^{\top}$ is a $2m$-dimensional manifold tangent to $D^{\top}$ and $M^{\perp}$ is a 3-dimensional manifold tangent to $D^{\perp}$ $(= \{\xi_r\})$.

Remark 3.1. As the tangent space $T_p(M)$, the soldering form dp may be split as

$$dp = dp^{\top} + dp^{\perp},$$

where $dp^{\top}$ and $dp^{\perp}$ are the horizontal and the vertical components of dp respectively. In the case under discussion, operating on $dp^{\top}$ and $dp^{\perp}$ by the exterior covariant derivative operator d^{∇}, one finds

$$d^{\nabla}(dp^{\perp}) = 0, \quad d^{\nabla}(dp^{\top}) = 0, \tag{30}$$

which, since ∇ is the Levi–Civita connection, leads to

$$d^{\nabla}(dp) = 0.$$

Using (20), (21), and (22), one gets

$$\nabla \xi_r = f_r dp^{\perp} - \eta^r \otimes \xi, \tag{31}$$

and one derives

$$[\xi_r, \xi_s] = f_s \xi_r - f_r \xi_s. \tag{32}$$

In view of (24), the covariant differential of $[\xi_r, \xi_s]$ can be expressed as

$$\nabla[\xi_r, \xi_s] = \eta \otimes [\xi_r, \xi_s] - [\xi_r, \xi_s]^{\flat} \otimes \xi, \tag{33}$$

with which one can check Jacobi's identity

$$\sum_{r,s,t} \big[\xi_r, [\xi_s, \xi_t]\big] = 0, \qquad \forall\ \widehat{r,s,t}.$$

Next, operating on (21) with ∇, and using (20) and (21), one derives that

$$\nabla\xi = \|\xi\|^2 dp^{\perp}; \tag{34}$$

consequently, following a well known definition [2] one may consider ξ as a concurrent vector field on $M^{\perp}$. This implies [15] (see also [13]) that ξ is an exterior concurrent vector field on $M^{\perp}$. Since $\|\xi\|^2 = \sum {f_r}^2$, one gets at once by (24) that

$$d\|\xi\|^2 = 2\|\xi\|^2\eta. \tag{35}$$

Therefore, since $d^{\nabla}(dp^{\perp}) = 0$, operating on (34) by d^{∇} yields

$$d^{\nabla}(\nabla\xi) = \nabla^2\xi = 2\|\xi\|^2\eta\wedge dp^{\perp}. \tag{36}$$

Hence, by reference to [13], the Ricci tensor field $\mathcal{R}$ of $M^{\perp}$ is expressed by

$$\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \qquad Z \in \Xi(M). \tag{37}$$

Next, by (24) one may write

$$(df_r)^{\sharp} = f_r\xi_r, \qquad (df_r)^{\sharp} = \operatorname{grad} f_r, \tag{38}$$

and after further elaboration, one derives that

$$\left[(df_r)^{\sharp}, (df_s)^{\sharp}\right] = 0, \qquad \forall\, \widehat{r, s, t}. \tag{39}$$

Accordingly we may say that the vector fields $(df_r)^{\sharp}$, $(df_s)^{\sharp}$, and $(df_t)^{\sharp}$ define a commutative group.

Next, by (24) one has that

$$\|\operatorname{grad} f_r\|^2 = \|\xi\|^2 {f_r}^2,$$

and since

$$\operatorname{div} Z = \operatorname{tr}\nabla Z, \qquad Z \in \Xi(M),$$

one derives that

$$\operatorname{div}\operatorname{grad} f_r = {f_r}^3 + \|\xi\|^2 {f_r}^2, \quad \|\xi\|^2 = \sum {f_r}^2.$$

Hence, noticing that $[\operatorname{grad} f_r, \operatorname{grad} f_r] = 0$ and on behalf of [20], we conclude from the above relations that the scalars f_r define an isoparametric system.

In another perspective, we recall that the star operator $*$ on an oriented n-dimensional Riemannian manifold (M,g) is an isometric bundle isomorphism between ΛT^*M and itself, and maps $\Lambda^q T^*M$ isomorphically to $\Lambda^{n-q}T^*M$ (see also [14]).

Coming back to the case under consideration, one has

$$\Lambda^q T^*M \to \Lambda^{2m+3-q}TM. \tag{40}$$

With the usual notation, we denote the codifferential of a p-form by $\delta = (-1)^p *^{-1} d*$, where $*^{-1} = (-1)^{n(n-p)}$ (p is the degree of the form, n is the dimension of the manifold, thus $\delta\omega$ is of degree $p-1$; see also [14]). Then, in the case under consideration, one deduces that

$$d\delta\eta = 0. \tag{40}$$

Since η is a closed pfaffian, there follows at once that

$$\Delta\eta = 0. \tag{42}$$

This shows that η is a harmonic pfaffian (and consequently $\eta^\sharp$ is a harmonic vector field). Finally, consider the immersion $x : M^\top \to M$. As it is well known, the second quadratic forms l_r associated with x are defined by

$$l_r = -\langle dp^\top, \nabla\xi_r\rangle. \tag{43}$$

Then, by reference to (31), it can be seen that the l_r vanish, and consequently the immersion $x : M^\top \to M$ is totally geodesic.

Summarizing, we can formulate the following

Theorem 3.1. *Let $M(\phi,\Omega,\xi_r,\eta^r,f_r,g)$ be a $(2m+3)$-dimensional manifold endowed with a vertical cyclic connection structure and with vanishing transversal connection forms. Let η, $\xi(=\eta^\sharp)$, and f_r be the principal pfaffian, the principal vector field, and the principal scalars on M; and let ${D_p}^\top$ and ${D_p}^\perp = \{\xi_r\}$ be the horizontal and the vertical distributions respectively on M.*

Then any such manifold may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is a $2m$-dimensional presymplectic submanifold tangent to ${D_p}^\top$ and $M^\perp$ is a 3-dimensional submanifold tangent to ${D_p}^\perp$.

The following properties are proved.

(i) *The immersion* $x : M^\top \to M$ *is totally geodesic;*

(ii) *the principal vector field* ξ *is an exterior concurrent vector field on* $M^\perp$*, i.e.*

$$\nabla^2 \xi = 2\|\xi\|^2 \eta \wedge dp^\perp,$$

and this implies

$$\mathcal{R}(\xi, Z) = -4\|\xi\|^2 g(\xi, Z), \qquad Z \in \Xi(M),$$

where $\mathcal{R}$ *denotes the Ricci tensor field of* $M^\perp$*;*

(iii) *the principal pfaffian* η *is harmonic;*

(iv) *the vector fields* $df_r{}^\sharp$ *define a commutative group, and the scalars* f_r *define an isoparametric system.*

4. Corollaries

Making use of E. Cartan's structure equations, involving the curvature 2-forms (11), one derives by (20), (23), and (24) that the vertical curvature forms Θ_r^s satisfy

$$\begin{aligned}\Theta_r^s = &\left(\left(\|\xi\|^2 - \frac{{f_t}^2}{2}\right)\eta^r + f_r f_t \eta^t\right) \wedge \eta^s \\ &- \left(\left(\|\xi\|^2 - \frac{{f_t}^2}{2}\right)\eta^s + f_s f_t \eta^t\right) \wedge \eta^r, \quad \forall\, \widehat{r, s, t}.\end{aligned} \tag{44}$$

Then, by reference to [19], the above expressions for Θ_r^s affirm that the vertical submanifold $M^\perp$ of M is a conformally flat submanifold of M.

In another perspective, let

$$V = V^r \xi_r, \quad r \in \{2m+1, 2m+2, 2m+3\}, \tag{45}$$

be any vertical vector field on $M^\perp$, and assume that V is a skew-symmetric Killing vector field, having ξ as generative [16] (see also [12]), thus

$$\nabla V = V \wedge \xi, \tag{46}$$

where $\wedge$ denotes the wedge product of vector fields

$$V \wedge \xi = \eta \otimes V - V^\flat \otimes \xi.$$

Since by (31) one gets

$$\nabla V = dV^r \otimes \xi_r + g(V,\xi)dp^{\perp} - V^{\flat} \otimes \xi, \tag{47}$$

then comparison of (46) and (47) gives

$$dV^{\flat} = \eta \wedge V^{\flat}, \tag{48}$$

which by (48) is in agreement by Rosca's lemma [16], [17] (see also [12]). Moreover, since V is a Killing vector field and the vector field $\xi(=\eta^{\sharp})$, is harmonic, one finds by (21) that

$$dg(V,\xi) = 0, \tag{49}$$

and (49) is in agreement with Bochner's theorem [21], and thus yields a confirmation for the correctness of our computations. In addition, by (34) and (46), one calculates that

$$[V,\xi] = g(V,\xi)\xi, \tag{50}$$

and the above equation means that V defines an infinitesimal conformal transformation of ξ. Operating now on (46) by the operator d^{∇} and in view of (34), one gets

$$d^{\nabla}(\nabla V) = \nabla^2 V = \|\xi\|^2 V^{\flat} \wedge dp^{\perp},$$

which shows that V is an exterior concurrent vector field on $M^{\perp}$ with $\|\xi\|^2$ as concurrent scalar, and by (6) one may write

$$\mathcal{R}(V,Z) = -2\|\xi\|^2 g(V,Z).$$

On the other hand, by (17) and (22), one finds that

$$d\Omega = 0. \tag{51}$$

Since Ω has constant rank, this means that Ω is a presymplectic form on M. We notice that in this case $\mathrm{Ker}(\Omega)$ coincides with the vertical distribution ${D_p}^{\perp} = \{\xi_r\}$ of M, which is also called the characteristic distribution of Ω.

Denote now with the usual notation

$$\Omega^{\flat}: \quad TM \to T^*M: \quad Z \to -i_Z\Omega = {}^{\flat}Z, \tag{52}$$

the symplectic isomorphism defined by Ω [8]. Since Ω is closed, any vector field X with the property that ${}^\flat X$ is closed, defines an infinitesimal automorphism of Ω, i.e.

$$\mathcal{L}_X \Omega = 0. \tag{53}$$

Assume that X is a horizontal vector field on M, i.e.

$$X = X^a e_a, \qquad a \in \{1, \ldots, 2m\}.$$

Then, by (52) one has

$${}^\flat X = \sum (X^{i^*} \omega^i - X^i \omega^{i^*}), \qquad i \in \{1, \ldots, m\},\ i^* = i + m, \tag{54}$$

and by the structure equations (10) one gets by exterior differentiation of ${}^\flat X$

$$d\,{}^\flat X = -(dX^{i^*} + X^a \theta_a^{i^*}) \wedge \omega^i - (dX^i + X^a \theta_a^i) \wedge \omega^{i^*}. \tag{55}$$

Hence, in order for ${}^\flat X$ to be a ϕ-closed form [16], one must write

$$\begin{cases} dX^i + X^a \theta_a^i = -\lambda \omega^{i^*}, \\ dX^{i^*} + X^a \theta_a^{i^*} = \lambda \omega^i, \end{cases} \tag{56}$$

where λ is a scalar. Taking now the covariant differential of the vector field X, one deduces by (56) and the structure equations (9) that

$$\nabla X = \lambda \phi dp. \tag{57}$$

This shows that X is a ϕ-concurrent vector field. Further, operating on the vector valued 1-form ϕdp by the operator d^∇, one calculates that

$$d^\nabla (\phi dp) = 0,$$

and therefore it follows from (57) that

$$\nabla^2 X = \frac{d\lambda}{\lambda} \otimes \nabla X. \tag{58}$$

Hence, the above equation proves that the vector field X is, according to well known terminology [10], a 2-covariant recurrent vector field with closed recurrence form.

Summarizing, we proved the following

Theorem 4.1. *The vertical submanifold $M^{\perp}$ of the manifold M under consideration is conformally flat, and the vertical skew-symmetric Killing vector field V is an exterior concurrent vector field which morover also defines an infinitesimal conformal transformation of the principal vector field ξ. The structure 2-form Ω of M is presymplectic, and if X is any horizontal vector field for which in addition ${}^{\flat}X(=-i_X\Omega)$ is ϕ-closed, then Ω is invariant by X, i.e. $\mathcal{L}_X\Omega=0$; moreover, X also has the following 2 properties:*

a) *X is a ϕ-concurrent vector field, i.e.*

$$\nabla X=\lambda\phi dp;$$

b) *X is a 2-covariant recurrent vector field with closed recurrence form, i.e.*

$$\nabla^2 X=\frac{d\lambda}{\lambda}\otimes\nabla X.$$

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