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On R. Baer's generalized hyperbolic planes

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To the 200th anniversary of János Bolyai's birth

Abstract. We provide a first order axiomatization for Baer's [2] generalized hyperbolic planes, and state some open problems.

1. Introduction

R. Baer [2] described a generalized model of hyperbolic planes, defined inside an arbitrary projective plane Π, whose lines will be denoted by lowercase, and points by uppercase letters, in which there is:

(i) an involutory mapping (a polarity) π mapping every point in Π to a line in Π, and every line in Π to a point in Π, for which $P \in l$ if and only if $\pi(l) \in \pi(P)$ (here $\in$ stands for the point-line incidence relation);

(ii) a partition of the elements of Π into three classes: interior, absolute, and exterior, satsifying the following conditions:

(a) P is absolute if and only if $P \in \pi(P)$; l is absolute if and only if $\pi(l) \in l$;
(b) P is interior if and only if $\pi(P)$ is exterior;
(c) if P is interior and $P \in l$, then l is interior;
(d) If l and k are interior, and $\pi(k) \in l$, then l and k meet in an interior point.

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The interior points and lines of such a projective plane Π form a generalized hyperbolic plane.

The description presented above, which coincides with the definition given by Baer, is not an axiomatics of generalized hyperbolic planes, but rather a description of the models that will be called so.

The purpose of this note is to provide a first-order axiomatization of Baer's generalized hyperbolic planes, and to ask some questions whose answer would illuminate the meaning of the main result obtained in Baer's paper. A like-minded description of a different class of generalized hyperbolic planes, which allows finite models as well, was presented in [1].

2. The axiom system

Our axiom system will be expressed in a two-sorted first-order language $\mathcal{L}$, with variables for points and lines (upper- and lower-case letters), with two binary relations, $\in$ and $\perp$, with point-line, respectively line-line arguments, with $P \in l$ to be read "point P lies on line l", and with $g \perp h$ to be read "line g is perpendicular to line h", and a ternary relation with line variables, α, with $\alpha(g,h,k)$ to be read as "g, h, k are three different lines, that meet in an absolute point". We shall use the following abbreviation

$$\overline{\alpha}(g,h,k) :\leftrightarrow \alpha(g,h,k) \vee k = g \vee k = h.$$

The axioms are (we have omitted the universal quantifiers in all axioms that are universal):

A1. $(\forall PQ)(\exists^{=1} l)\, P \neq Q \rightarrow P \in l \wedge Q \in l$,

A2. $(\exists ABC)(\forall l)\, A \neq B \wedge \neg(A \in l \wedge B \in l \wedge C \in l)$,

A3. $\alpha(g,h,k) \rightarrow \alpha(g,k,h) \wedge \alpha(k,g,h)$,

A4. $\alpha(g,h,k) \rightarrow g \neq h$,

A5. $\alpha(g,h,k) \wedge P \in g \rightarrow P \notin h$,

A6. $\alpha(g,h,k) \wedge \alpha(g,h,l) \wedge k \neq l \rightarrow \alpha(g,k,l)$,

A7. $\alpha(g,h,k) \rightarrow \neg(g \perp l \wedge h \perp l)$,

A8. $h \perp k \rightarrow k \perp h$,

A9. $(\forall g)(\forall P)(\exists^{=1} h)\, P \in h \wedge h \perp g$,

A10. $(\forall gh)(\exists P)\, g \perp h \to P \in g \wedge P \in h$,

A11. $(\forall gh)(\exists k)(\exists P)(\exists l)\, g \neq h \to [(P \in g \wedge P \in h) \vee \alpha(g,h,k)$
$\vee\, (l \perp g \wedge l \perp h)]$,

A12. $g \neq h \wedge l \perp g \wedge l \perp h \wedge l' \perp g \wedge l' \perp h \to l = l'$,

A13. $(\forall abca'b'c')(\exists l)\, \alpha(a,b,c) \wedge \alpha(a',b',c') \to \overline{\alpha}(a,b,l) \wedge \overline{\alpha}(a',b',l)$,

A14. $(\forall abc)(\forall P)(\exists l)\, \alpha(a,b,c) \to \overline{\alpha}(a,b,l) \wedge P \in l$,

A15. $(\forall abcl)(\exists^{=1} g)\, \alpha(a,b,c) \wedge \neg\overline{\alpha}(a,b,l) \to \overline{\alpha}(a,b,g) \wedge g \perp l$.

The axioms may be given the following intuitive interpretation:
A1 states that every pair of distinct points can be joined by a unique line; A2 states that there are three non-collinear points (note that, by A1 there is a line l joining A and B, which are different, thus C must be different from both A and B); A3 states that the relation α is symmetric; A4 states that if three lines are in the relation α, then they are all (by symmetry) different; A5 states that lines that are in the relation α do not intersect; A6 expresses a certain transitivity of α; A7 states that if three lines meet at an absolute point, then no two have a common peprpendicular; A8 states the symmetry of the perpendicularity relation; A9 states the existence and uniqueness of the perpendicular from a given point to a given line; A10 states that perpendicular lines intersect; A11 states that any two different lines either have an (inner) point in common, or meet at an absolute point, or have a common perpendicular; A12 states that two different lines may have at most one common perpendicular; A13 states that absolute points can be connected by an (interior) line; A14 states that an absolute point and an (interior) point can be joined by a line; A15 states that there is exactly one perpendicular passing through an absolute point and perpendicular to a line that does not go through that absolute point.

To see that all models of this axiom system are precisely the generalized hyperbolic planes of Baer, we define, in any model $\mathfrak{M}$ of our axiom system, new points and new lines, which together with the points and lines of $\mathfrak{M}$ form a projective plane, in which we can define a polarity π that satisfies the conditions (i) and (ii) above.

Let $\mathfrak{M}$ be a model of $\Sigma = \{\text{A1–A15}\}$. Lines and points that will be referred to in the sequel are in $\mathfrak{M}$. We shall denote by $l \cap l'$ the point of intersection of two lines l and l' whenever these intersect, by $\varphi(A,B)$ the line, which exists and is unique by A1, that passes through the points A and B, whenever $A \neq B$, by $\gamma(A,l)$ the perpendicular through A to l (which exists and is unique by A9), and by $\psi(A,l)$ the point $\gamma(A,l) \cap l$, which exists by A10. So far we know that $\gamma(A,l) \neq l$ only if $A \notin l$, so we know $\psi(A,l)$ is well defined in this case. However

Lemma 1. *For all lines l, we have $\neg(l \perp l)$.*

Proof. Let l be a line with $l \perp l$, and let P be a point not on l (which must exist by A2). Let $Q = \psi(P,l)$. Then l and $\gamma(P,l)$ are two different perpendiculars to l, which have the point Q in common. This contradicts A9. □

Thus, since $\gamma(A,l) \neq l$ for all A and l, $\psi(A,l)$ is well defined for all A and l.

Lemma 2. *There are four points, no three of which are collinear. For any line l there are at least three points on l.*

Proof. Let l be an arbitrary line, and let P be a point not on l and $Q := \psi(P,l)$. There must exist such a point P by A2. There must also exist a point R not on $\gamma(P,l)$. R is either on l, and thus different from Q, or is not on l. If it is not on l, then $\psi(R,l)$ is a point on l that is different from Q. Thus, there are always points P, R, such that P is not on l, R is on l, but different from $Q := \psi(P,l)$. In this situation $S := \psi(Q, \varphi(P,R))$ is different from both P and Q, for, by using the symmetry of the perpendicularity relation, if S were P, then $\varphi(R,P)$ and $\varphi(R,Q)$ would be two different perpendiculars from R to $\gamma(P,l)$, contradicting A9, if S were R, then $\varphi(P,R)$ and $\varphi(P,Q)$ would be two different perpendiculars from P to l. Thus:

(a) there are four points, no three of which are collinear. They are P, Q, S, $\psi(S,l)$, where l is an arbitrary line (a line does exist, since there are, by A2 two different points, so there is a line joining them by A1); and that

(b) every line contains at least three points, namely Q, R, $\psi(S,l)$. □

We now proceed to extend $\mathfrak{M}$ to a projective plane and to define the polarity π satisfying the desired conditions.

The set

$$P(l) := \{g \mid g \perp l\}$$

will be called an *exterior* point. Notice that $P(l)$ contains at least three elements, since there are three distinct points on l, and we may raise perpendiculars on each to l by A9.

If l_1, l_2 are such that l_1 does not intersect l_2 and such that no line is perpendicular on both l_1 and l_2, then

$$P(l_1, l_2) := \{g \mid g = l_1 \text{ or } g = l_2 \text{ or } \alpha(l_1, l_2, g)\}$$

will be called an *absolute* point. Note that different pairs (l_1, l_2) may define the same set of lines $P(l_1, l_2)$, and that this set contains at least three elements, for, by A11 there is a line g for which $\alpha(l_1, l_2, g)$ holds.

We introduce new lines by associating to every interior point Q an *exterior* line $\kappa(Q)$, and to every absolute point $P(l_1, l_2)$ an *absolute* line $\lambda(l_1, l_2)$.

We are now ready to define the incidence structure of the extended plane. The points and lines in $\mathfrak{M}$ will be called *interior.*

1. Interior points are not on absolute or exterior lines.
2. Absolute points are not on exterior lines.
3. An absolute point $P(l_1, l_2)$ is on an interior line l if and only if $\overline{\alpha}(l_1, l_2, l)$.
4. The absolute point $P(l_1, l_2)$ is on the absolute line $\lambda(g, h)$ if and only if $P(l_1, l_2) = P(g, h)$.
5. An exterior point $P(l)$ is on an interior line g if and only if $g \perp l$.
6. An exterior point $P(l)$ is on an exterior line $\kappa(Q)$ if and only if $Q \in l$ (thus $\kappa(Q) = \kappa(Q') \Leftrightarrow Q = Q'$).
7. An exterior point $P(l)$ is on an absolute line $\lambda(g, h)$ if and only if $l \in P(g, h)$ (i.e. if and only if $\overline{\alpha}(g, h, l)$).

Lemma 3. *The extended plane is a projective plane.*

Proof. For any two interior points there is exactly one line joining them by A1. An interior point A and an exterior point $P(l)$ have also exactly one line joinig them, namely $\gamma(A, l)$. Two exterior points $P(l)$ and $P(l')$ have exactly one line joining them, which is:

(α) an exterior line, $\kappa(Q)$, in case l and l' have a common (interior) point Q,
(β) an interior line k if k is perpendicular to l, l';
(γ) the absolute line $\lambda(l, l')$ if there is a line g such that $\alpha(l, l', g)$.

By A11 one and only one of (α), (β), (γ) must hold.

An interior point A and an absolute point $P(g,h)$ have a unique line joining them. Existence is implied by A14 and uniqueness by A5.

Two absolute points also have a unique line joining them. Existence is given by A13, and uniqueness follows from A6 and A3.

An exterior point $P(l)$ and an absolute point $P(g,h)$ can be joined by a unique line, which is the absolute line $\lambda(g,h)$ if $\overline{\alpha}(g,h,l)$, and which is the interior line m which satsifies $\overline{\alpha}(g,h,m)$ and is perpendicular to l (its existence and uniqueness follow from A15), otherwise.

Thus all pairs of distinct points have a unique joining line.

Two internal lines always intersect by A11. An internal line l and an external line $\kappa(A)$ meet in $P(\gamma(A,l))$. Two external lines $\kappa(A)$ and $\kappa(A')$, meet in $P(\varphi(A,A'))$. Two absolute lines $\lambda(a,b)$ and $\lambda(a',b')$ meet in $P(m)$, where m is the line joining $P(a,b)$ and $P(a',b')$. An absolute line $\lambda(a,b)$ and an exterior line $\kappa(Q)$ meet in $P(l)$, where l is the line joining Q and $P(a,b)$, which exists by A14. An absolute line $\lambda(a,b)$ and an interior line l meet either in $P(a,b)$, if $\overline{\alpha}(a,b,l)$, or in $P(g)$, where g is the line through $P(a,b)$ which is perpendicular to l (its existence and uniqueness being implied by A15). $\square$

The polarity π can now be defined on the extended plane constructed from $\mathfrak{M}$ as follows: For an interior line l, $\pi(l) := P(l)$; for an interior point A, $\pi(A) := \kappa(A)$; for an absolute point $P(g,h)$, we let $\pi(P(g,h)) := \lambda(g,h)$, and vice-versa $\pi(\lambda(g,h)) := P(g,h)$; for an exterior point $P(l)$, we let $\pi(P(l)) := l$, and for an exterior line $\kappa(Q)$ we let $\pi(\kappa(Q)) := Q$. We have made sure through our axioms and through the way we have defined the equality of the points and lines we have added to $\mathfrak{M}$ that these definitions do not depend on representatives. It is now plain that π does satsify the conditions (i) and (ii). We have thus proved

Theorem. Σ *is an axiom system for* BAER*'s generalized hyperbolic planes.*

Notice that, if we add three individual point-constants A, B, C to our language, we may rephrase Σ such that all axioms are universal-existential statements.

There are two natural questions that one would ask of the axiom system Σ. The first one is

Open Question 1. *Is every primitive notion of the language $\mathcal{L}$ independent of the others? In other words, is none of them definable from the others, the definition being valid in all models of Σ?*

We conjecture that the answer is yes, none of the primitive notions we have used is redundant. This is in stark contrast to plane elementary hyperbolic geometry, which is axiomatizable by means of incidence ($\in$) alone, as pointed out by K. Menger in [3] (a beautiful axiom system based on incidence alone was provided by H. L. Skala in [4]).

Another natural question, related to this possibility of axiomatizating hyperbolic geometry in terms of incidence alone, is

Open Question 2. *What is the incidence theory of* Baer*'s generalized hyperbolic planes, i.e. what is* $\mathrm{Cn}(\Sigma)\cap\mathcal{S}_\in$ *(where* Cn *stands for the set of all logical consequences and* $\mathcal{S}_\in$ *for the two-sorted first order language with* $\in$ *as its only non-logical primitive notion)?*

We conjecture that the answer was provided by the theorems proved by Baer about generalized hyperbolic planes. They state that there are infinitely many interior points as well as infinitely many exterior points on every interior line. If we denote by θ_n the statement that every line contains n different points, and by ω_n the statement that for any line l and any point P not on it there are n different lines through P not intersecting l, then we conjecture that $\mathrm{Cn}(\Sigma)\cap\mathcal{S}_\in = \mathrm{Cn}(\mathrm{A1},\ \mathrm{A2}, \{\theta_n, \omega_n \mid n \in \mathbb{N}\})$.

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