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# A generalization of the Hyers–Ulam–Rassias stability of the beta functional equation

By GWANG HUI KIM (Suwon)

**Abstract.** In this paper, we prove a generalization of the Hyers–Ulam–Rassias stability for the inverse form ($2'$) of the beta functional equation. As a consequence we obtain the Hyers–Ulam stability and the stability in the spirit of Găvruta for the gamma functional equation.

## 1. Introduction

In 1940, S. M. Ulam [16] raised the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, this problem was solved by D. H. Hyers [3]. Thereafter we usually say that the equation $E_1(h) = E_2(h)$ is stable in the Hyers–Ulam sense if for an approximate solution $f$ of this equation, i.e. for a function $f$ with $|E_1(f) - E_2(f)| \le \delta$ there exists a function $g$ such that $E_1(g) = E_2(g)$ and $|f(x) - g(x)| \le \epsilon$. In 1978 the Hyers–Ulam stability for additive mapping was generalized by Th. M. Rassias [12]. This result of Th. M. Rassias was again generalized by P. Găvruta [2] as follows:

If for an approximate solution $f$ of the equation $E_1(h) = E_2(h)$, i.e. for a function $f$ such that $|E_1(f) - E_2(f)| \le \phi$ holds with a given function $\phi$ there exists a function $g$ such that $E_1(g) = E_2(g)$ and $|g(x) - f(x)| \le \Phi(x)$ for some fixed function $\Phi$.

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The functional equation

$$f(x+1) = xf(x) \quad \text{for all } x > 0 \tag{1}$$

is called the gamma functional equation. It is well-known that the gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (x > 0)$$

is a solution of the gamma functional equation.

From the relation of gamma and beta function, that is,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x),$$

the functional equation

$$f(x+1, y+1) = \frac{xy}{(x+y)(x+y+1)} f(x,y) \quad \text{for all } x, y > 0 \tag{2}$$

will be called the beta functional equation. The beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

is a solution of the beta functional equation.

We consider the inverse functional equation of beta functional equation (2) as follows:

$$B(x+1, y+1)^{-1} = \frac{(x+y)(x+y+1)}{xy} B(x,y)^{-1}. \tag{2'}$$

In this paper, we shall investigate the modified Hyers–Ulam–Rassias stability of the functional equation (2′). Throughout this paper, we denote by $\mathbb{R}_+$ the set of all positive real numbers and $n_0$ is a nonnegative integer, and in particular the author will use a notation $x_i = x + i$ for the convenience of calculation and the appreciation of the reader. By using an idea of Găvruta [2] we can prove the following theorem:

**Theorem 1.** *Let $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a given mapping satisfying the inequality*

$$\Phi(x,y) := \sum_{j=0}^{\infty} \varphi(x_j, y_j) \prod_{i=0}^{j} \frac{x_i\, y_i}{(x_i + y_i)(x_i + y_i + 1)} < \infty \tag{3}$$

*for all $x, y \in \mathbb{R}_+$, and let $n_0$ be a given nonnegative integer.*

*Assume that a mapping $B : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the inequality*

$$\left| B(x+1, y+1)^{-1} - \frac{(x+y)(x+y+1)}{xy} B(x,y)^{-1} \right| \le \varphi(x,y) \tag{4}$$

*for all $x, y > n_0$. Then there exists a unique mapping $T : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the beta functional equation $(2')$ and the inequality*

$$\left| T(x,y)^{-1} - B(x,y)^{-1} \right| \le \Phi(x,y) \tag{5}$$

*for all $x, y > n_0$.*

## 2. Proof of the Theorem 1

For $x, y > n_0$, we use an induction on $n$ to prove

$$\begin{aligned} &\left| B(x_n,y_n)^{-1} - B(x,y)^{-1} \prod_{i=0}^{n-1} \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right| \\ &\qquad \le \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=1}^{n-1-j} \frac{(x_{i+j}+y_{i+j})(x_{i+j}+y_{i+j}+1)}{x_{i+j}\, y_{i+j}} \end{aligned} \tag{6}$$

for all positive integers $n$, where we assume that $\prod_{i=1}^{0} \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} = 1$ conventionally. The inequality (6) immediately follows from (4) for the case of $n = 1$. If we assume that (6) holds true for some $n$, then we obtain for $n+1$

$$\begin{aligned} &\left| B(x_{n+1}, y_{n+1})^{-1} - B(x,y)^{-1} \prod_{i=0}^{n} \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right| \\ &\quad \le \left| B(x_{n+1}, y_{n+1})^{-1} - \frac{(x_n+y_n)(x_n+y_n+1)}{x_n\, y_n} B(x_n, y_n)^{-1} \right| \\ &\qquad + \frac{(x_n+y_n)(x_n+y_n+1)}{x_n\, y_n} \\ &\qquad \cdot \left| B(x_n, y_n)^{-1} - B(x,y)^{-1} \prod_{i=0}^{n-1} \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right| \\ &\quad \le \varphi(x_n, y_n) + \frac{(x_n+y_n)(x_n+y_n+1)}{x_n\, y_n} \end{aligned}$$

$$\cdot \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=1}^{n-1-j} \frac{(x_{i+j}+y_{i+j})(x_{i+j}+y_{i+j}+1)}{x_{i+j}\, y_{i+j}}$$
$$= \varphi(x_n, y_n) + \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=1}^{n-j} \frac{(x_{i+j}+y_{i+j})(x_{i+j}+y_{i+j}+1)}{x_{i+j}\, y_{i+j}}$$
$$= \sum_{j=0}^{n} \varphi(x_j, y_j) \prod_{i=1}^{n-j} \frac{(x_{i+j}+y_{i+j})(x_{i+j}+y_{i+j}+1)}{x_{i+j}\, y_{i+j}},$$

which completes the proof of (6). If we divide both sides in (6) by $\prod_{i=0}^{n-1} \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\; y_i}$, then we get

$$\left| B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1} - B(x,y)^{-1} \right| \le \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=0}^{j} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1} \tag{7}$$

for every $n \in \mathbb{N}$. By using (4) and(3) we have for $n > m > 0$

$$\Bigg| B(x_m, y_m)^{-1} \prod_{i=0}^{m-1} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1}$$
$$- B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1} \Bigg|$$
$$= \Bigg| B(x_m, y_m)^{-1} \prod_{i=0}^{m-1} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1}$$
$$- B(x_{m+1}, y_{m+1})^{-1} \prod_{i=0}^{m} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1}$$
$$+ - \cdots$$
$$+ B(x_{n-1}, y_{n-1})^{-1} \prod_{i=0}^{n-2} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1}$$
$$- B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1} \Bigg|$$

$$\leq \sum_{j=m}^{n-1} \left| B(x_j, y_j)^{-1} \frac{(x_j + y_j)(x_j + y_j + 1)}{x_j\, y_j} - B(x_{j+1}, y_{j+1})^{-1} \right| \prod_{i=0}^{j} \left( \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i\, y_i} \right)^{-1}$$

$$\leq \sum_{j=m}^{n-1} \varphi(x_j, y_j) \prod_{i=0}^{j} \left( \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i\, y_i} \right)^{-1} \quad \to 0,$$

as $m \to \infty$.

Therefore, the sequence

$$B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left( \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i\, y_i} \right)^{-1}$$

is a Cauchy sequence for $x, y > n_0$, and hence we can define a mapping $T_0 : (n_0, \infty) \times (n_0, \infty) \to \mathbb{R}_+$ by

$$T_0(x, y)^{-1} = \lim_{n\to\infty} B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left( \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i\, y_i} \right)^{-1} \tag{8}$$

for all $x, y > n_0$. Letting in (7) $n \to \infty$ and applying (8) and (3) we obtain (5).

By (8) we can easily verify that $T_0$ satisfies (2′):

$$\begin{aligned} &T_0(x+1, y+1)^{-1} \\ &\quad = \lim_{n\to\infty} B(x_{n+1}, y_{n+1})^{-1} \prod_{i=0}^{n-1} \left( \frac{(x_{i+1} + y_{i+1})(x_{i+1} + y_{i+1} + 1)}{x_{i+1}\, y_{i+1}} \right)^{-1} \\ &\quad = \frac{(x+y)(x+y+1)}{xy} \lim_{n\to\infty} B(x_{n+1}, y_{n+1})^{-1} \prod_{i=0}^{n} \left( \frac{(x_i+y_i)(x_i+y_i+1)}{x_i\, y_i} \right)^{-1} \\ &\quad = \frac{(x+y)(x+y+1)}{xy} T_0(x, y)^{-1} \end{aligned}$$

for all $x, y > n_0$.

Now we assume that $G : (n_0, \infty) \times (n_0, \infty) \to \mathbb{R}_+$ is another mapping which satisfies (2′) as well as (5) for all $x, y > n_0$. By (2′), (5) and (3) we obtain

$$
\begin{aligned}
&\left|T_0(x,y)^{-1} - G(x,y)^{-1}\right| \\
&\qquad = \left|T_0(x_n,y_n)^{-1} - G(x_n,y_n)^{-1}\right| \prod_{i=0}^{n-1} \frac{x_i\, y_i}{(x_i+y_i)(x_i+y_i+1)} \\
&\qquad \le 2\Phi(x_n,y_n) \prod_{i=0}^{n-1} \frac{x_i\, y_i}{(x_i+y_i)(x_i+y_i+1)} \\
&\qquad = 2\sum_{j=0}^{\infty} \varphi(x_{n+j}, y_{n+j}) \prod_{i=0}^{n+j} \frac{x_i\, y_i}{(x_i+y_i)(x_i+y_i+1)} \\
&\qquad = 2\sum_{j=n}^{\infty} \varphi(x_j, y_j) \prod_{i=0}^{j} \frac{x_i\, y_i}{(x_i+y_i)(x_i+y_i+1)} \quad \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

for all $x, y > n_0$. This implies the uniqueness of $T_0$. Now we extend the function $T_0$ to $(0,\infty)\times(0,\infty)$. We define for $0 < x, y \le n_0$

$$
T(x,y) := \prod_{n=0}^{k-1} \frac{(x_n+y_n)(x_n+y_n+1)}{x_n\, y_n} \cdot T_0(x+k, y+k),
$$

where $k$ is the smallest natural number satisfying the inequalities $x+k > n_0$ and $y+k > n_0$. And also $T(x,y) = T_0(x,y)$ for all $x, y > n_0$. Then $T(x+1,y+1) = \frac{x\,y}{(x+y)(x+y+1)} T(x,y)$ for all $x, y > 0$.

Also the following inequality holds

$$
\left|T(x,y)^{-1} - B(x,y)^{-1}\right| < \Phi(x,y)
$$

for all $x, y > n_0$. Hence, the proof of the theorem is completed.

## 3. Applications to the gamma and beta functional equation

The following corollary that is called the Hyers–Ulam stability for the functional equation $(2')$ can be found in the author's papers ([7], [11]).

**Corollary 2.** *Assume that a mapping $B : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the inequality*

$$
\left| B(x+1,y+1)^{-1} - \frac{(x+y)(x+y+1)}{xy} B(x,y)^{-1} \right| \le \delta
$$

*for some* $\delta > 0$ *and for all* $x, y > n_0$. *Then there exists a unique mapping* $T : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *which satisfies the beta functional equation* $(2')$ *and the inequality*

$$|T(x,y)^{-1} - B(x,y)^{-1}| \le \delta$$

*for all* $x, y > n_0$.

Proof. Apply Theorem 1 and condition (3) with $\varphi(x,y) = \delta$. Then we arrive

$$\begin{aligned}\Phi(x,y) &= \sum_{j=0}^{\infty} \delta \prod_{i=0}^{j} \frac{x_i\, y_i}{(x_i+y_i)(x_i+y_i+1)} = \delta \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{x_i\, y_i}{(x_i+y_i)(x_i+y_i+1)} \\ &< \delta\left(\frac{1}{2} + \frac{1}{2^2} + \cdots\right) = \delta. \qquad \square\end{aligned}$$

For the stability of the gamma functional equation we apply Theorem 1 to a single variable, and then we can get the following results. In the case $n_0 = 0$, Corollary 4 can be found in S.-M. Jung ([8], [9]), H. Alzer [1] and the author's [10].

**Theorem 3.** *Assume that a mapping* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies the inequality*

$$|g(x+1) - xg(x)| \le \varphi(x) \tag{9}$$

*for all* $x, y > n_0$. *Then there exists a unique mapping* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *which satisfies the the gamma functional equation* (1) *with*

$$|f(x) - g(x)| \le \Phi(x) \qquad \forall x > n_0,$$

*where* $\Phi(x) := \sum_{j=0}^{\infty} \varphi(x_j) \prod_{i=0}^{j} \frac{1}{x_i} < \infty$.

Proof. For any $x > n_0$ and for every positive integer $n$ we define

$$P_n(x) = g(x_n) \prod_{i=0}^{n-1} \frac{1}{x_i}.$$

By (9) we have

$$\begin{aligned}|P_{n+1}(x) - P_n(x)| &= |g(x_{n+1}) - x_n g(x_n)| \prod_{i=0}^{n} \frac{1}{x_i} \\ &\le \varphi(x_n) \prod_{i=0}^{n} \frac{1}{x_i} \qquad \text{for } x > n_0.\end{aligned} \tag{10}$$

Now we use induction on $n$ to prove

$$|P_n(x) - g(x)| \leq \sum_{j=0}^{n-1} \varphi(x_j) \prod_{i=0}^{j} \frac{1}{x_i} \tag{11}$$

for the fixed $x > n_0$ and for all positive integers $n$. For the case $n = 1$, the inequality (11) is an immediate consequence of (9). Assume that (11) holds true for some $n$. It then follows from (9) and (10)

$$|P_{n+1}(x) - g(x)| \leq |P_{n+1}(x) - P_n(x)| + |P_n(x) - g(x)| \leq \sum_{j=0}^{n} \varphi(x_j) \prod_{i=0}^{j} \frac{1}{x_i}.$$

which completes the proof of (11). Now let $m, n$ be positive integers with $n \geq m$. Suppose $x(> n_0)$ is given. By definition of $\Phi$, we have

$$\begin{aligned} |P_n(x) - P_m(x)| &\leq \sum_{j=m}^{n-1} |P_{j+1}(x) - P_j(x)| \\ &\leq \sum_{j=m}^{n-1} \varphi(x_j) \prod_{i=0}^{j} \frac{1}{x_i} \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

□

This implies that $\{P_n(x)\}$ is a Cauchy sequence for $x > n_0$. Next proceeding of the proof is very similiar to that of the Theorem 1.

**Corollary 4.** *Assume that a mapping $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the inequality*

$$|g(x + 1) - xg(x)| \leq \delta$$

*for some $\delta > 0$ and for all $x, y > n_0$. Then there exists a unique mapping $f : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the gamma functional equation* (1) *with*

$$|f(x) - g(x)| \leq \frac{e\delta}{x}$$

*for all $x > n_0$, where $e$ is the best possible constant.*

Proof. Apply $\delta = \varphi(x)$ in Theorem 3. We can find in [1] that $e$ is the best possible constant. □

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GWANG HUI KIM
DEPARTMENT OF MATHEMATICS
KANGNAM UNIVERSITY
SUWON, 449–702
KOREA

*E-mail*: ghkim @ kns.kangnam.ac.kr

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