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Nonlinear connections and the problem of metrizability

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Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

(1.1) A *nonlinear connection* in a vector bundle is nothing but a vector 1–form, which is a projector whose kernel is the vertical bundle. This paper is devoted to a study of some problems concerning such connections. Sections 2 and 3 are of preparatory nature: we briefly summarize the necessary tools of Frölicher–Nijenhuis formalism of vector valued forms and collect some fundamental facts concerning differential operators and their formal integrability ("Spencer-Quillen-Goldschmidt technique"). In section 4 we present some basic facts about nonlinear connections. We show that these can be characterized as *quasi-scalar* (nonlinear) differential operators and that their tension is in fact a generalization of the deflection known from Matsumoto's theory of Finsler connections. The homogeneity condition for nonlinear connections (which results in *linear* connections) is also expressed in a new, illuminating way. In section 5 we formulate the problem of metrizability and show that the question is the formal integrability of a first order linear differential operator arising from the connection. In section 6 we show that our operator is involutive, while in the concluding section 7 we demonstrate that in the flat case all of the conditions of the modern version of the Cartan–Kähler theorem are satisfied so that the metrizability problem always has a solution. Of course, the vanishing of the curvature is a very restrictive assumption, but we hope that this application of the Spencer-Quillen-Goldschmidt technique has enough interest and (possibly) the method used here may also be successful in more general and much more complicated cases.

(1.2) Before we embark on the actual topic of the paper, we wish to review briefly the basic conventions and notations that will be used in the sequel. — We shall always be working in the caterory of the finite dimensional, second countable, smooth manifolds. The tangent and cotangent bundle of a manifold M will be written as a triple $\tau_M = (TM, \tau, M)$ and $\tau_M^* = (T^*M, \tau^*, M)$, resp.; T is the tangent functor. $C^\infty(M)$ denotes the algebra of smooth functions on M; $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on M; $\wedge(M)$ and $S(M)$ are the exterior and the symmetric algebra over M. As usual, d is the operator of the exterior derivative, while $i_X : \wedge(M) \to \wedge(M)$ $(X \in \mathfrak{X}(M))$ is the substitution operator (or "interior product by X").

In coordinate calculations Einstein's summation convention will be used as appropriate.

2. Frölicher–Nijenhuis formalism

Let N be a manifold, $k \in \mathbb{N}$. A *vector valued k-form* (briefly a *vector k-form*) is a skew-symmetric $C^\infty(N)$-multilinear mapping $K : \mathfrak{X}^k(N) \to \mathfrak{X}(N)$, with the agreement that the vector 0-forms are the elements of $\mathfrak{X}(N)$. Frölicher-Nijenhuis theory associates to any vector k-form K two derivations of the exterior algebra $\wedge(N)$: a derivation of degree $k-1$ which is denoted by i_K and a derivation of degree k, denoted by d_K. i_K and d_K reduce to the above substitution operator i_X and the Lie derivative d_X in case of vector 0-forms (i.e. vector fields), while in general they can be described as follows:

(i) Derivations i_K annihilate $C^\infty(N)$ and are determined by their actions on the (scalar) 1-forms:

$$\begin{gathered} \forall \omega \in \wedge^1(N), \quad X_i \in \mathfrak{X}(N) \quad (1 \le i \le k): \\ i_K\omega(X_1, \ldots, X_k) = \omega\left(K\left(X_1, \ldots, X_k\right)\right). \end{gathered} \tag{1}$$

(ii) $d_K = [i_K, d] := i_K \circ d + (-1)^k d \circ i_K$, so — in particular —

$$\forall\ f \in C^\infty(N) : d_K f = i_K df; \tag{2}$$

this last relation determines K uniquely. (In (i) and (ii) $k := \deg K$ is the degree of K.)

The *bracket* of two derivations d_K, d_L is introduced thus:

$$[d_K, d_L] = d_K \circ d_L - (-1)^{k\ell} d_L \circ d_K \tag{3}$$

$(k = \deg K,\ \ell = \deg L)$. A fundamental result of the theory states that for each vector k-form K and vector ℓ-form L there exists a unique $k+\ell$-form $[K, L]$ such that

$$d_{[K,L]} = [d_K, d_L].$$

$[K, L]$ is called the *Frölicher-Nijenhuis bracket*, briefly the *bracket* of K and L. In case of vector 0-forms it reduces to the usual bracket of vector fields. In the general case, the evaluation of $[K, L]$ on vector fields is quite complicated, for an effective formula we refer to [10]. Fortunately, in this paper we shall have to evaluate brackets only in the following very simple situations:

(i) K is a vector 1-form, $L := Z \in \mathfrak{X}(N)$. Then

$$\forall X \in \mathfrak{X}(N) : [K, Z](X) = [K(X), Z] - K[X, Z]. \tag{4}$$

(ii) Let K be a vector 1-form again. By (3) we get that $[d_K, d_K] = 2d_K \circ d_K = d_{[K,K]}$, so

$$d_K^2 = d_{\frac{1}{2}[K,K]} \tag{5}$$

Here $N_K := \frac{1}{2}[K, K]$ is called the *Nijenhuis torsion* of K. It can be characterized by the formula $N_K(X, Y) = [K(X), K(Y)]+$ $K \circ K[X, Y] - K[K(X), Y] - K[X, K(Y)] \quad (X, Y \in \mathfrak{X}(N))$.

3. Differential operators and formal integrability

(3.1) Suppose that M is an n-dimensional manifold and consider a vector bundle $\xi = (E, p, M)$ of rank r over M. Let $\operatorname{Sec}\xi$ be the $C^\infty(M)$-module of sections of ξ and JE be the 1-jet space of ξ, i.e.

$$JE = \{j_x\sigma \mid x \in M, \sigma \in \operatorname{Sec}\xi\},$$

where $j_x\sigma$ is the 1-jet of σ at x. We denote by π the target projection

$$JE \to E, \quad j_x\sigma \mapsto \sigma(x).$$

It is well-known that the sequence

$$0 \to T^*M \otimes E \xrightarrow{\varepsilon} JE \xrightarrow{\pi} E \to 0,$$

where

$$\varepsilon : df(x) \otimes \sigma(x) \longmapsto j_x[(f - f(x))\sigma]$$

($f \in C^\infty(M)$, $\sigma \in \operatorname{Sec}\xi$, $x \in M$) is a short exact sequence of vector bundles. More generally (see e.g. [4], Ch. VI, §1), if J_kE is the space of k-jets ($k \in \mathbb{N}^+$),

$$\pi_k : J_kE \to J_{k-1}E, \quad j_k\sigma \longmapsto j_{k-1}\sigma$$

is the natural projection ($\pi_0 := \pi$, $j_0\sigma := \sigma$) and

$$\varepsilon : S^kTM \times E \to J_kE$$

is defined by

$$\varepsilon\left(df_1(x)\odot\cdots\odot df_k(x)\otimes\sigma(x)\right):=\left[j_k\left(f_1\ldots f_k\sigma\right)\right]_x$$

($f_i\in C^\infty(M)$, $f_i(x)=0$, $1\le i\le k$; $\odot$ is the symbol of symmetric product), then the sequence

$$0\to S^kTM\otimes E\to J_kE\xrightarrow{\varepsilon}J_{k-1}E\xrightarrow{\pi_k}0$$

is also a short exact sequence of vector bundles.

(3.2) Let $\eta=(F,\rho,M)$ be another vector bundle over M. — We recall that a mapping $P:\operatorname{Sec}\xi\to\operatorname{Sec}\eta$, $\sigma\mapsto P\sigma$ is said to be a *linear differential operator of order* k $(\in\mathbb{N}^+)$ if

$$\forall x\in M,\ \sigma\in\operatorname{Sec}\xi:(j_k\sigma)_x=0\implies P\sigma(x)=0.$$

Any differential operator of order k can be identified with an M-morphism $\overline{P}:J_kE\to F$ such that $P=\overline{P}_*\circ j_k$, where $\overline{P}_*:\operatorname{Sec}J_k\xi\to\operatorname{Sec}\eta$ is induced by $\overline{P}$ according to [6], p.63. We shall use this observation without any comment.

If $\operatorname{rank}\overline{P}$ is constant and $R_k:=\operatorname{Ker}\overline{P}$, then the sequence

$$0\to R_k\xrightarrow{\text{inc}}J_kE\xrightarrow{\overline{P}}F\to 0$$

is clearly exact (inc:= inclusion).

The *ℓ-th prolongation* $p_\ell(P)$ of P is defined as follows:

$$p_\ell(P):J_{k+\ell}E\to J_\ell F,\quad j_{k+\ell}(\sigma)\mapsto j_\ell(P\sigma).$$

Let $R_{k+\ell}:=\operatorname{Ker}p_\ell(P)$. The natural projection $\pi_{k+\ell}:J_{k+\ell}E\to J_kE$ induces a mapping $R_{k+\ell}\to R_k$ by restriction, it will be denoted also by $\pi_{k+\ell}$. — Keeping these notations, we can formulate the following fundamental

Definition 1. A k-th order differential operator is said to be *formally integrable* if $\forall\ell\in\mathbb{N}^+:\pi_{k+\ell}:R_{k+\ell}\to R_k$ is surjective.

Other key notions are explained in the next

Definition 2.

(i) The *symbol* of a differential operator $P:J_kE\to F$ is the mapping

$$s_0(P):=P\circ\varepsilon:S^kT^*M\otimes E\to F.$$

The symbol of the ℓ-th prolongation operator $p_\ell(P):J_{k+\ell}E\to J_\ell F$ is the mapping $s_\ell(P):S^{k+\ell}T^*M\otimes E\to S^\ell T^*M\otimes F$ introduced by the diagram

(ii) Suppose, in particular, that $P : \operatorname{Sec}\xi \to \operatorname{Sec}\eta$ is a first-order linear differential operator. Then

$$s_0(P) : T_x^*M \otimes E_x \to E_x, \quad (df)_x \otimes \sigma(x) \mapsto P(f\sigma)(x)$$

(where $f(x) = 0$). Fixing an element of T_xM, we get a partial mapping $E_x \to E_x$. — P is said to be *quasi-scalar* if each of these mappings means multiplication by a scalar (cf. [12], def. 19.33).

The third important preparatory step is summarized in

Definition 3. Consider a linear differential operator $P : J_kE \to F$. Let $\forall \ell \in \mathbb{N} : g_\ell := \operatorname{Ker} s_\ell(P)$. A basis $(e_i)_{1\le i\le n}$ of T_xM is said to be *quasi-regular* if

$$\dim (g_1)_x = \dim (g_0)_x + \sum_{j=1}^{n-1} \dim (g_0)_{(e_1,\dots,e_j)} ,$$

where

$$(g_0)_{(e_1,\dots,e_j)} := \left\{A \in g_0 \mid i_{e_1}A = \cdots = i_{e_j}A = 0\right\} \qquad (1 \le j \le n-1).$$

If for each $x \in M$ there exists a quasi-regular basis of T_xM then P is called *involutive*.

With the help of the terms introduced, the classical Cartan–Kähler theorem (see e.g. [3], Th. 18.13.8) can be translated into the following extremely compact and elegant form:

Cartan–Kähler–Goldschmidt theorem. *Let $P : J_kE \to F$ be a k-th order linear differential operator. If*
1° *P is involutive, and*
2° *$\pi_{k+1} : R_{k+1} \to R_k$ is surjective, then P is formally integrable.* □

For a comprehensive, but well–readable treatment of the subject we refer to [1] and [13]. A very good brief survey is presented in [5]; see also [2].

(3.3) To conclude this section, we take a look at the *nonlinear case.* — A first order nonlinear differential operator, briefly *differential operator* from ξ to η is simply a mapping $P : \operatorname{Sec}\xi \to \operatorname{Sec}\eta$ which has a factorisation of the form $P = \overline{P}_* \circ j$, where $\overline{P} : JE \to F$ is just fibre–preserving. Taking Gâteaux–derivative, nonlinear differential operators can be *linearized.* Namely, if $P : \operatorname{Sec}\xi \to \operatorname{Sec}\eta$ is a nonlinear differential operator, $\sigma \in \operatorname{Sec}\xi$ is fixed and

$$P'_\sigma(\rho) := \frac{d}{dt}P(\sigma + t\rho)\,|_{t=0},$$

then P'_σ is a linear differential operator which is called the *linearized operator* of P along σ.

We show a simple but very useful *example.* — Let $X \in \mathfrak{X}(E)$ be a projectable vector field:

$$X \overset{p}{\sim} \overline{X}, \quad \overline{X} \in \mathfrak{X}(M).$$

Then the mapping

$$P_X : \operatorname{Sec}\xi \to \operatorname{Sec}\xi,\ \sigma \mapsto \alpha \circ (T\sigma \circ \overline{X} - X \circ \sigma)$$

($\alpha : VE \to E$ is the well-known canonical surjection; see e.g. [6], p.291) is a nonlinear differential operator. Its linearized operator along a section σ acts as follows:

$$\forall \rho \in \operatorname{Sec}\xi : (P_X')_\sigma (\rho) = \alpha \circ [X, \rho^v] \circ \sigma,$$

where $\rho^v := \alpha^\sharp \rho$ is the vertical lift of ρ (see [18], Lemma 3). As for the *symbol* of P_X, an easy calculation yields the formula

$$s_0 \, (P_X)'_\sigma \, (df \otimes \rho)_x = (\overline{X}f)^v \sigma(x)\rho(x)$$
$$\left((\overline{X}f)^v = \overline{X}f \circ p\right),$$

from which it follows at first sight that $(P_x)'_\sigma$ —and consequently P_X— is *quasi-scalar.*

4. Nonlinear connections

(4.1) Let the above vector bundle $\xi = (E, p, M)$ be given. We shall write its vertical bundle as the triple $V\xi = (T^vE, p_v, E)$; $\mathfrak{X}^vE := \operatorname{Sec} V\xi$ is the module of *vertical vector fields.*

Definition 4. A *nonlinear connection*, briefly a *connection* in ξ is a vector 1-form $h : \mathfrak{X}(E) \to \mathfrak{X}(E)$, satisfying the following conditions:

1° $h^2 = h$, i.e. h is a *projector*;
2° $\operatorname{Ker} h = \mathfrak{X}^vE$.

Of course, this is only a possible one among the numerous definitions of a connection, which seems preferable to us only for practical reasons. — Let $\mathcal{H}(\xi)$ be the set of all connections in ξ. Then $\mathcal{H}(\xi)$ is an affine bundle ([14]), Def. 2.4.4) in a natural manner. Clearly, each $h \in \mathcal{H}(\xi)$ gives rise to direct decompositions $\mathfrak{X}(E) = \mathfrak{X}^vE \oplus \operatorname{Im} h$ (in the module sense) and $TE = T^vE \oplus H^vE$ (in the vector bundle sense), where $H\xi := \left(T^hE, p_h, E\right)$ is the so-called *horizontal subbundle* belonging to h, such that

$$\operatorname{Sec} H\xi = \operatorname{Im} h =: \mathfrak{X}^hE.$$

Elements of $\mathfrak{X}^hE$ are the *horizontal vector fields* of the connection. Finally, it is easy to show that there exists a unique mapping

$$\ell^h : \mathfrak{X}(M) \to \mathfrak{X}^hE, \quad X \mapsto X^h$$

characterized by the condition $X^h \overset{p}{\sim} X$; it is called *horizontal lifting.*

Proposition 1.

(a) *Let* $h \in \mathcal{H}(\xi)$, $X \in \mathfrak{X}(M)$, $\nabla_X := P_{X^h}$ (see (3.3)). *Then* ∇_X *is a quasi-scalar differential operator, said to be covariant derivative by* X *with respect to* h.

(b) *Consider the bundle* $V_M\xi := (T^vE, p \circ p_v, M)$ *and suppose that a mapping* $X \in \mathfrak{X}(M) \mapsto P_X$ *is given, such that*

(i) $\forall X \in \mathfrak{X}(M) : P_X : \operatorname{Sec}\xi \to \operatorname{Sec} V_M\xi$ *is a quasi-scalar (nonlinear) differential operator,*

(ii) $\forall \sigma \in \operatorname{Sec}\xi : p_v \circ P_X\sigma = \sigma$,

then there exists a unique connection $h \in \mathcal{H}(\xi)$ *for which* $\forall X \in \mathfrak{X}(M)$: $\nabla_X = \alpha \circ P_X$.

Proof. (a) follows immediately by (3.3). We sketch, how one can get the converse statement (b). — Observe first that condition (i) is meaningful because P_X can be identified with the operator $\alpha \circ P_X : \operatorname{Sec}\xi \to \operatorname{Sec}\xi$ canonically. To become conscious of this, let us define the mapping

$$\ell^h : \mathfrak{X}(M) \to \mathfrak{X}(E), \quad X \mapsto \ell^h(X) =: X^h$$

as follows:

$$X^h(z) := T_x[X(x)] - (P_X\sigma)(x), \quad \text{if} \ \ z = \sigma(x), \sigma \in \operatorname{Sec}\xi.$$

Applying condition (i), one can check by a direct calculation that $X^h(z)$ does not depend on the section for which $z = \sigma(x)$, so $X^h \in \mathfrak{X}(E)$ is a well-defined vector field. Since $(P_X\sigma)(x) \in T^V_{\sigma(x)}E := \operatorname{Ker} T_{\sigma(x)}P$, $X^h \overset{p}{\sim} X$. Finally, the introduced mapping ℓ^h determines uniquely a connection $h \in \mathcal{H}(\xi)$ such that $\operatorname{Im}\ell^h = \mathfrak{X}^hE$ and $\forall X \in \mathfrak{X}(M) : \nabla_X = \alpha \circ P_X$. □

Proposition 1 characterizes (nonlinear) connections as quasi-scalar (nonlinear) differential operators, extending the similar characterization of linear connections; cf. [11], §9.

(4.2) Let $C : E \to T^vE$ be the Liouville vector field on E.

Definition 5. (cf. [7]) The *tension* of a connection $h \in \mathcal{H}(\xi)$ is the vector 1-form $T := [h, C]$. In case of $T = 0$ we say that h satisfies the *homogeneity condition* (HC) and speak of a *linear connection.*

Note that by (4)

$$\begin{aligned} (HC) &\iff \forall X \in \mathfrak{X}(E) : [hX, C] - h[X, C] = 0 \\ &\iff \forall X \in \mathfrak{X}(M) : [X^h, C] = 0 \qquad (\text{since} \quad [X^h, C] \in \mathfrak{X}^vE). \end{aligned}$$

Another characterization of the linear connections is given by the

Proposition 2. $(HC) \iff \forall X \in \mathfrak{X}(M),\ \sigma \in \operatorname{Sec}\xi$:
$(\nabla_X \sigma)^v = \left[X^h, \sigma^v\right]$.

Proof.

(a) Observe first that the mapping

$$\mathfrak{X}(M) \times \operatorname{Sec}\xi \to \mathfrak{X}^v E, \quad (X, \sigma) \mapsto \left[X^h, \sigma^v\right]$$

has the following properties:

(i) it is $\mathbb{R}$-bilinear;

(ii) $\left[(fC)^h, \sigma^v\right] = (f \circ p)\left[X^h, \sigma^v\right]$;

(iii) $\left[X^h, (f\sigma)^v\right] = (f \circ p)\left[X^h, \sigma^v\right] + (Xf \circ p)\sigma^v$ $(f \in C^\infty(M))$.

In fact, (i) is obvious since the mappings $X \mapsto X^h$ and $\sigma \mapsto \sigma^v$ are $\mathbb{R}$-linear, while (ii) and (iii) can be obtained by a straightforward calculation:

$$\left[(fX)^h, \sigma^v\right] = \left[(f \circ p)X^h, \sigma^v\right] = (f \circ p)\left[X^h, \sigma^v\right] + \sigma^v(f \circ p)X^h = \\ = (f \circ p)\left[X^h, \sigma^v\right]$$

(because $Z \in \mathfrak{X}^v E \iff \forall g \in C^\infty(M) : Z(g \circ p) = 0$); the verification of (iii) is similar.

(b) Now we suppose that

$$\forall X \in \mathfrak{X}(M), \sigma \in \operatorname{Sec}\xi : (\nabla_X \sigma)^v = \left[X^h, \sigma^v\right].$$

Applying (ii) we get:

$$\forall f \in C^\infty(M) : \left[\nabla_X(f\sigma)\right]^v = \left[X^h, (f\sigma)^v\right] = (f \circ p)\left[X^h, \sigma^v\right] + \\ +(Xf \circ p)\sigma^v = (f \circ p)(\nabla_X \sigma)^v + (Xf \circ p)\sigma^v = \left[fV_X + (Xf)\sigma\right]^v,$$

consequently $\nabla_X f\sigma = f\nabla_X \sigma + (Xf)\sigma$.

It means that the operators ∇_X satisfy the well-known Koszul-axioms of the linear connections, from which (HC) easily follows (see e.g. [16]).

(c) To verify the converse statement, fix a fibered chart for ξ, e.g. the chart $\left(p^{-1}(U), \left(x^i\right)_{i=1}^n, \left(y^\kappa\right)_{\kappa=1}^r\right)$ described in [18], p.1168. Then there exist unique functions

$$\Gamma_i^\kappa : p^{-1}(U) \to \mathbb{R}; \quad 1 \le i \le n, \quad 1 \le \kappa \le r$$

such that

$$h\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \Gamma_i^\kappa \frac{\partial}{\partial y^\kappa}, \quad h\left(\frac{\partial}{\partial y^\kappa}\right) = 0;$$

these are the "connection parameters" of h. Over $p^{-1}(U)$, we get:

$$T = \left(y^\beta \frac{\partial \Gamma_i^\kappa}{\partial y^\beta} - \Gamma_1^\kappa \right) dx^i \otimes \frac{\partial}{\partial y^\kappa}.$$

On the other hand, an easy calculation shows that (locally)

$$\left[X^h, \sigma^h\right] = (\nabla_X \sigma)^v \iff \Gamma_i^\kappa = y^\beta \frac{\Gamma_i^\kappa}{\partial y^\beta},$$

so (HC) implies the condition in question. □

Comparing the criterion just obtained with the result of the linearization process sketched in (3.3), we get the following "natural interpretation" of (HC):

Corollary. *$h \in \mathcal{H}(\xi)$ satisfies (HC) iff the covariant derivatives belonging to h "essentially" (along sections and up to the canonical mapping α) coincide with their linearized operators.* □

(4.3) Suppose that $\nabla \in \mathcal{H}(V\xi)$ is a linear connection, interpreted as a mapping $\mathfrak{X}(E) \times \mathfrak{X}^v E \to \mathfrak{X}^v E$ satisfying the Koszul–axioms and let $h \in \mathcal{H}(\xi)$. Then the pair (∇, h) is called a *Matsumoto-pair* (see [17]). The *deflection* of (∇, h) is the vector 1-form

$$D(\nabla, h) : \mathfrak{X}(E) \to \mathfrak{X}(E), \quad X \mapsto \nabla_{hX} C,$$

where C is the Liouville vector field again. We see immediately that $D(\nabla, h)$ is semibasic. — In particular, choose by way of connection the *Berwald connection* ∇^B induced by h:

$$\nabla^B : \mathfrak{X}(E) \times \mathfrak{X}^v E \to \mathfrak{X}^v E, \quad (X, Y) \mapsto \nabla^i_{vX} Y + v[hX, Y],$$

where $v := id - h$ and $\nabla^i : \mathfrak{X}^v E \times \mathfrak{X}^v E \to \mathfrak{X}^v E$ is a *pseudoconnection* characterized by the following condition:

$$\forall \sigma \in \operatorname{Sec} \xi, \; Z \in \mathfrak{X}^v E : \nabla^i_Z \sigma^v = 0$$

([18]), Lemma 5). Then $\forall X \in \mathfrak{X}(E)$:

$$\begin{aligned} D\left(\nabla^B, h\right)(X) &= \nabla^B_{hX} C = v[hX, C] = \\ &= v([hX, C] - h[X, C]) = vT(X) = T(X) \end{aligned}$$

(since T is also semibasic). It means that in this case deflection and tension coincide. Let us emphasize: deflection plays an important role in Matsumoto's theory of Finsler–connections; see the monograph [9].

(4.4) To conclude our general view on connections, we recall a further basic concept.

Definition 6. The *curvature* of a connection $h \in \mathcal{H}(\xi)$ is the Nijenhuis torsion $R := N_h = \frac{1}{2}[h,h]$. h is called *flat* if $R = 0$.

It is easy to check that R is a semibasic 2-form:

$\mathrm{Im}R \subset \mathfrak{X}^v E$ and $R(X,Y) = 0$, if $X \in \mathfrak{X}^v E$ or $Y \in \mathfrak{X}^v E$.

5. The problem of metrizability

(5.1) Now and in the sequel we turn our attention to the tangent bundle case, i.e. we suppose that $\xi := \tau_M = (TM, \tau, M)$ and $h \in \mathcal{H}(\tau_M)$. Then the notion of *parallel vector fields* along curves has meaning ([7], Def. I.26) and we can raise the following "problem of metrizability":

(PM) Find a criterion for the existence of a (smooth) function $L : TM \to \mathbb{R}$ which (in Grifone's terminology) is *conserved by parallel transports*: for all smooth curve $c : I \to M$ and parallel vector field $X : I \to TM$, $L \circ X : I \to \mathbb{R}$ is *constant.*

As we have seen in section 2, the vector 1-form $h \in \mathcal{H}(\tau_M)$ determines a derivation d_h of $\wedge(M)$. The following observation is simple, but very useful:

Lemma 1. ([7]), Prop. I.28) *A function $L : TM \to \mathbb{R}$ is conserved by parallelism if*

$$d_h L = 0. \tag{6}$$

In coordinates (6) yields the system of partial differential equations

$$\frac{\partial L}{\partial x^i} - \Gamma_i^k \frac{\partial L}{\partial y^k} = 0, \quad 1 \leq i \leq n; \tag{7}$$

where the Γ_i^k-s are the connection parameters of h. So our problem is to find the integrability conditions of the system (7). Well now, to attack (PM), we propose the *study of formal integrability of the 1st order linear differential operator* d_h (we shall see soon that d_h really has such an interpretation, in a natural manner).

(5.2) By the way, we indicate that (PM) is essentially equivalent with the inverse problem of the calculus of variations. To formulate the latter in modern language, consider a semispray (i.e. a second order differential equation) $S : TM \to TTM$.

Definition 7. ([8]), p.189) S is said to be *variational* if there exists a (regular) function $L : TM \to \mathbb{R}$ such that

$$i_S dd_J L = d(L - CL),$$

where $J : TTM \to TTM$ is the canonical almost tangent structure (or vertical endomorphism).

By a fundamental result of J. Grifone ([7], Prop. I.41) $h := \frac{1}{2}(id + [J,S])$ is a connection in τ_M. Now it can be shown (see [15]) that

$$S \text{ is variational} \iff \begin{array}{l} \exists \text{ (regular) } L : TM \to \mathbb{R} \text{ such that} \\ d_h L = 0,\ h = \frac{1}{2}(id + [J,S]). \end{array}$$

From this point of view, an excellent survey and discussion of the problem can be found in the paper [8] of J. Klein.

6. The involutivity of d_h

First we give the promised interpretation of d_h as a suitable differential operator.
— Since $\forall X \in \mathfrak{X}(M)\ :\ d_h L(X) = L(hX) \in C^\infty(TM)$, d_h maps $C^\infty(TM)$ into $\operatorname{Sec}\tau^*_{TM}$. Here $C^\infty(TM)$ can be identified with

$$\operatorname{Sec}\xi, \quad \xi := (E, pr_2, TM), \quad E := \mathbb{R} \times TM.$$

So, in fact, d_h can be considered as a 1st order linear differential operator from $\operatorname{Sec}\xi$ into $\operatorname{Sec}\tau^*_{TM}$.

Lemma 2.
(i) The symbol of d_h is $s_0 d_h = i_h$ (up to an obvious factor which will be omitted).
(ii) At each point $v \in TM$ the null space $(g_0)_v := \operatorname{Ker}(s_0 d_h)_v$ is n-dimensional.

Proof. (i) Consider an arbitrary function $f \in C^\infty(TM)$ with the condition $f(v) = 0$ and a section $\sigma \in \operatorname{Sec}\xi \cong C^\infty(TM)$. According to Definition 2,

$$(s_0 d_h)(df(v), \sigma(v)) = d_h(f\sigma)(v) = [(d_h f)\sigma + f d_h \sigma](v) = \\ = \sigma(v) d_h f(v) \overset{(2)}{=} \sigma(v) i_h df(v),$$

which gives our assertion.
(ii) Let $\omega \in \mathfrak{X}^* TM$ be a 1-form. By (i),

$$\omega \in g_0 = \operatorname{Ker} s_0 d_h \iff i_h \omega = 0 \overset{(1)}{\iff} \forall X \in \mathfrak{X}(TM) : \omega(hX) = 0.$$

It means that $\operatorname{Ker} s_0 d_h$ is nothing but the annullator of $\mathfrak{X}^h TM$, hence $\dim g_0 = \dim \tau_{TM} - \dim \operatorname{Im} h = 2n - n = n$. $\square$

Lemma 3.
(i) *The symbol of the first prolonged operator* $p_1(d_h)$ *is the mapping*

$$s_1 d_h : S^2 T^*TM \otimes E \to T^*TM \otimes T^*TM, \quad A \mapsto (s_1 d_h)\, A$$

given by

$$(s_1 d_h)\, A(X, Y) = A(X, hY); \quad X, Y \in \mathfrak{X}(TM).$$

(ii) $\forall v \in TM : \dim (g_1)_v = \frac{n(n+1)}{2}$.

Proof. (i) follows immediately by Definition 2 again, part (i) of the previous Lemma and by (1).
(ii) Let $\left(U, \left(u^i\right)\right)$ be a chart for M, and let $\left(\tau^{-1}(U); \left(x^i\right), \left(y^i\right)\right)$ be the induced chart for TM. Then

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \quad \frac{\partial}{\partial y^i} \quad (1 \le i \le n) \tag{8}$$

(the Γ_i^j-s are the parameters of $h \in \mathcal{H}(\tau_M)$) is a local basis for $\mathfrak{X}(TM)$. Since $g_1 := \operatorname{Ker}(s_1 d_h)$,

$$A \in g_1 \iff \forall X, Y \in \mathfrak{X}(TM) : A(X, hY) = 0.$$

Using the basis (5), over $\tau^{-1}(U)$

$$X = X^i \frac{\delta}{\delta x^i} + X^{n+i} \frac{\partial}{\partial y^i}, \quad hY = Y^i \frac{\delta}{\delta x^i};$$

so we get:

$$A \in g_1 \iff \begin{cases} A\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right) = 0 & (1 \le i, j \le n), \\ A\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^j}\right) = 0 & (1 \le i, j \le n), \end{cases} \tag{9}$$

Since A is symmetric, (9) gives $\frac{n(n+1)}{2} + n^2$ relations for its components, therefore the number of the independent components is

$$\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} - n^2 = \frac{n(n+1)}{2}.$$

Clearly, this is just the dimension of g_1. $\square$

Proposition 3. *The operator* d_h *is involutive.*

Proof. Let $\left(\tau^{-1}(U); \left(x^i\right), \left(y^i\right)\right)$ be an induced chart for TM.
First we note that

$$\delta y^i := \Gamma_j^i dx^j + dy^i, \quad dx^i \qquad (1 \le i \le n)$$

is the dual of the local basis (8), so (over $\tau^{-1}(U)$)

$$\forall\omega \in \mathfrak{X}^* TM : \omega = \omega_i \delta y^i + \omega_{n+i} dx^i.$$

In particular, if $\omega \in g_0$, then (as we have seen) ω annihilates $\mathfrak{X}^h TM$, hence

$$\forall\, 1 \leq i \leq n : \quad 0 = \omega\left(\frac{\delta}{\delta x^i}\right) = \omega_{n+j} dx^j \left(\frac{\delta}{\delta x^i}\right) = \omega_{n+i},$$

i.e.

$$\omega \in g_0 \iff \omega = \omega_i \delta y^i.$$

Now we are going to show that $\forall v \in \tau^{-1}(U)$:

$$\left(\left(\frac{\partial}{\partial y^i}\right)_v, \left(\frac{\delta}{\delta x^i}\right)_v\right) \quad 1 \leq i \leq n$$

is a quasi-regular basis. In fact, using the preceding remark,

$$(g_0)_{\frac{\partial}{\partial y^1}} \overset{\text{Def.3}}{=} \left\{\omega \in g_0 \mid i_{\frac{\partial}{\partial y^1}} \omega = 0\right\} = \left\{\omega = \omega_i \delta y^i \in g_0 \mid \omega_1 = 0\right\},$$

which means that

$$\dim (g_0)_{\frac{\partial}{\partial y^1}} = n - 1.$$

Similarly,

$$(g_0)_{\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right)} = \left\{\omega = \omega_i y^i \in g_0 \mid \omega_1 = \omega_2 = 0\right\} \Rightarrow$$

$$\dim (g_0)_{\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right)} = n - 2.$$

Proceeding in the same way, finally we get:

$$\dim (g_0)_{\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{n-1}}\right)} = 1.$$

By these

$$\sum_{j=1}^{n-1} \dim (g_0)_{\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^j}\right)} = \sum_{j=1}^{n-1} (n - j) = \frac{n(n-1)}{2},$$

consequently

$$\dim (g_0) + \sum_{j=1}^{2n} \dim (g_0)_{(e_1, \dots, e_j)} \overset{\text{Lemma 2}}{=} n + \frac{n(n-1)}{2} =$$

$$= \frac{n(n+1)}{2} \overset{\text{Lemma 3}}{=} \dim (g_1)$$

$$\left(e_1 := \frac{\partial}{\partial y^i},\ e_{n+i} := \frac{\delta}{\delta x^i}; \quad 1 \leq i \leq n\right),$$

which means the involutivity of d_h. $\square$

7. The surjectivity of π_2

Keeping the previous notations and conventions, consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & S^2T^*TM\otimes E & \xrightarrow{s_1 d_h} & T^*TM\otimes T^*TM & \xrightarrow{\tau} & K & \longrightarrow & 0 \\
 & & \varepsilon\big\downarrow & & \varepsilon\big\downarrow\big\uparrow\nabla & & & & \\
R_2 & \longrightarrow & J_2E & \xrightarrow{p_1 d_h} & J(T^*TM) & & & & \\
\pi_2\big\downarrow & & \big\downarrow\pi_2 & & \pi_1\big\downarrow & & & & \\
R_1 & \longrightarrow & J_1E & \xrightarrow{d_h} & T^*TM & & & & \\
 & & \nwarrow\varepsilon & & \nearrow s_0 d_h & & & & \\
 & & & T^*TM\otimes E & & & & &
\end{array}
$$

Here $K := \dfrac{T^*TM \otimes T^*TM}{\operatorname{Im} s_1 d_h}$ is said to be the space of *obstructions*. By the "algorithm" sketched in [2], in order to verify the surjectivity of $\pi_2 : R_2 \to R_1$ one has to

(i) give a "good interpretation" for the space of obstructions and construct a mapping τ which makes the diagram commutative;

(ii) define a linear connection ∇ in τ^*_{TM} such that

$$\forall v \in TM,\ \sigma \in \operatorname{Sec}\xi : d_h\sigma(v) = 0 \Rightarrow \tau\left(\nabla d_h \sigma\right)_v = 0.$$

(To realize (ii), it is sufficient to give a suitable connection in τ_{TM} since it induces the desired connection in τ^*_{TM}.)

Of course, in our case the surjectivity of π_2 does not hold without further assumptions. In what follows, we are going to prove this only under the quite restrictive condition $N_h = 0$.

Lemma 4. *Let $v := id - h$, as above. The mapping $v : TTM \to TTM$ induces a pull-back map $v^* : \wedge^2 TM \to \wedge^2 TM$. — The space of obstructions is (pointwise) isomorphic to the space*

$$\operatorname{Ker} v^* \oplus \tau^0_2\left(T^vTM\right)$$

($\tau^0_2\left(T^vTM\right)$ is the space of (0,2) tensors $\mathfrak{X}^vTM \times \mathfrak{X}^vTM \to C^\infty(TM)$).

Proof. On the one hand, we have:

$$
\begin{aligned}
\dim K &= \dim\left(T^*TM \otimes T^*TM\right) - \operatorname{rank} s_1 d_h = \\
&= (2n)^2 - \left(\dim S^2T^*TM - \dim g_1\right) \overset{\text{Lemma 3}}{=} \\
&= 4n^2 - \frac{2n(2n+1)}{2} + \frac{n(n+1)}{2} = \left(n^2 + \frac{n(n-1)}{2}\right) + n^2.
\end{aligned}
$$

On the other hand,

$$\dim \operatorname{Ker} v^* = \dim \wedge^2 TM - \dim \operatorname{Im} v^* = \frac{2n(2n-1)}{2} - \frac{n(n-1)}{2} = \\ = n^2 + \frac{n(n-1)}{2},$$

while $\dim \tau_2^0 (T^v TM) = n^2$, from which the assertion follows. $\square$

Proposition 4. *Let the mappings*

$$\tau_1 : T^*TM \otimes T^*TM \to \operatorname{Ker} v^* \quad \textit{and}$$
$$\tau_2 : T^*TM \otimes T^*TM \to \tau_2^0 (T^v TM)$$

be defined as follows:

$$\forall B \in T^*TM \otimes T^*TM :$$
$$\tau_1 B(X,Y) := B(hX,Y) - B(hY,X) \quad (X,Y \in \mathfrak{X}(TM)),$$
$$\tau_2 B(U,V) := B(U,V) \quad (U,V \in \mathfrak{X}^v TM).$$

Then the sequence

$$S^2 T^*TM \xrightarrow{s_1 d_h} T^*TM \otimes T^*TM \xrightarrow{\tau := \tau_1 \oplus \tau_2} \operatorname{Ker} v^* \oplus \mathcal{T}_2^0 (T^v TM) \to 0$$

is exact

Proof. (a) We show that $\operatorname{Im} s_1 d_h \subset \operatorname{Ker} \tau$. In fact,

$$\forall B \in S^2 T^*TM; \quad \forall X,Y \in \mathfrak{X}(TM) :$$
$$[(\tau_1 \circ s_1 d_h) B](X,Y) = s_1 d_h(B)(hX,Y) - s_1 d_h(B)(hY,X) =$$
$$\overset{\text{Lemma 3, (i)}}{=} B(hX,hY) - B(hY,hX) = 0;$$
$$[(\tau_2 \circ s_1 d_h) B](vX,vY) = s_1 d_h(B)(vX,vY) = B(vX,h\, vY) = 0.$$

(b) $A \in \operatorname{Ker} \tau \iff A \in \operatorname{Ker} \tau_1 \wedge A \in \operatorname{Ker} \tau_2$. Condition $A \in \operatorname{Ker} \tau_1$ means that

$$A\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = A\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) \wedge A\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0 \quad (1 \le i,j \le n),$$

while $A \in \operatorname{Ker} \tau_2 \iff A\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0 \quad (1 \le i,j \le n)$. Hence the elements of $\operatorname{Ker} \tau$ can be characterized by $n^2 + \frac{n(n+1)}{2}$ independent components with respect to the local basis (8), consequently $\dim \operatorname{Ker} \tau = n^2 + \frac{n(n+1)}{2}$. But

$$\operatorname{rank} s_1 d_h = \dim S^2 T^*TM - \dim \operatorname{Ker} s_1 d_h \overset{\text{Lemma 3, (ii)}}{=} n^2 + \frac{n(n+1)}{2}.$$

Combining these with (a), we get: $\mathrm{Im} s_1 d_h = \mathrm{Ker}\,\tau$.

(c) Finally, since

$$\mathrm{rank}\,\tau = \dim\left(T^*TM \otimes T^*TM\right) - \dim \mathrm{Ker}\,\tau \overset{\text{(b)}}{=} (2n)^2 - \left(n^2 + \frac{n(n+1)}{2}\right) =$$
$$= 2n^2 + \frac{n(n-1)}{2} \overset{\text{Lemma 4}}{=} \dim K = \dim \mathrm{Ker}\, V + \dim \mathcal{T}_2^0 T^v TM,$$

τ is also surjective, which concludes the proof. □

Lemma 5. *Let ∇ be a linear connection in τ_{TM} with torsion T. Then for each 1-form $\omega \in \wedge^1 TM$, we have:*

$$\tau_1(\nabla\omega)(X,Y) = d_h\omega(X,Y) - \omega\big(\nabla_Y hX - \nabla_X hY + T(hX,Y) + \\ + T(X,hY) + h[X,Y]\big).$$

Proof. Using the rules of Frölicher–Nijenhuis calculus, the covariant differentiation and the definition of T, we get:

$$\begin{aligned} d_h\omega(X,Y) &= \left[(i_h \circ d - d \circ i_h)\,\omega\right](X,Y) = \left[(i_h d\omega)(X,Y) - d\,(i_h\omega)\right](X,Y) = \\ &= d\omega(hX,Y) + d\omega(X,hY) - X\,(i_h\omega)(Y) + \\ &\quad + Y\,(i_h\omega)(X) + \omega\,(h[X,Y]) = \\ &= hX\omega(Y) - hY\omega(X) - \omega([hX,Y] + [X,hY] - h[X,Y]) = \\ &= hX\omega(Y) - hY\omega(X) - \omega\,(\nabla_{hX}Y - \nabla_Y hX - T(hX,Y) + \\ &\quad + \nabla_X hX - \nabla_{hY}X - T(X,hY) - h[X,Y]) = \\ &= \nabla\omega(hX,Y) - \nabla\omega(hY,X) + \\ &\quad + \omega\,(\nabla_Y hX - \nabla_X hY + T(hX,Y) + T(X,hY) + h[X,Y])\,. \end{aligned}$$

Since $\nabla\omega(hX,Y) - \nabla\omega(hY,X) = [\tau_1(\nabla\omega)](X,Y)$, we have the desired formula. □

Corollary. *If $\omega_v = 0$, then $\forall v \in TM : \tau_1(\nabla\omega)_v = (d_h\omega)_v$.* □

Lemma 6. *Let ∇ be a linear connection in τ_{TM}. If $f \in C^\infty(TM)$ ($\cong \mathrm{Sec}\,\xi$) and $(d_h f)_x = 0$ ($x \in TM$), then $\tau_2\,(\nabla d_h f)_x = 0$.*

Proof. $\forall U, V \in \mathfrak{X}^V TM : \tau_2\,(\nabla d_h f)\,(U,V) := \nabla d_h f(U,V) =$
$= (\nabla_U d_h f)\,(V) = U\,[d_h f(V)] - d_h f\,(\nabla_U V)\,.$
Since $(d_h f)_x = 0$, it follows that

$$\left[\tau_2\,(d_h f)\right]_x (U_x, V_x) = \left[U\,(d_h f(V))\right]_x \overset{(2)}{=} U_x\,[i_h df(V)] = U_x[df(hV)].$$

But $V \in \mathfrak{X}^v TM \Rightarrow hV = 0$, so $\tau_2\,(d_h f)_x = 0$. □

Piecing together our previous findings, we arrive at the main result of the paper.

Theorem. *If $h \in \mathcal{H}(\tau_{TM})$ is a flat (nonlinear) connection, then the differential operator d_h is formally integrable.*

Proof. We have already known that d_h is involutive (Proposition 3). If ∇ is a linear connection in τ_{TM} and $(d_h f)_x = 0$, then in view of the Corollary of Lemma 5,

$$\tau_1 (d_h f)_x = (d_h d_h f)_x = \left(d_h^2 f\right)_x \overset{(5)}{=} 0,$$

since $[h, h] = 0$ by the flatness. Combining this with Lemma 6, we get that $\tau (d_h f)_x = 0$. By our above remarks this guarantees that condition 2° of the Cartan–Kähler–Goldschmidt theorem is also satisfied. □

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