Publ. Math. Debrecen
59 / 1-2 (2001), 161–173

Right Bol loops with a finite dimensional group of multiplications

By EKKEHART WINTERROTH (Erlangen)

Abstract. In this article we prove the existence of a large class of analytic right Bol loops having a Lie group as the group generated by all multiplications. To achieve this we define a particular type of analytic right Bol loop on the topological product of two Lie groups N and G. These loops arise from a certain modification of the construction of the semidirect product of G by N. Our main result is that the loops defined in this way have a Lie group as the group of multiplications if N is nilpotent.

1. Introduction

It is a commonly known fact that for a loop L multiplication with a fixed element is a permutation of L and hence can be used to define transformation groups: the group of left translations, generated by the multiplications from the left $\lambda_a(x) = a \cdot x$, the group of right translations, generated by the multiplications from the right $\rho_a(x) = x \cdot a$, and the group of multiplications, generated by all multiplications. The loop is then a homogeneous space of each of these groups having a section with certain properties, which make it possible to express the loop multiplication in terms of this section (see e.g. [1] p. 217–220 or [2]). A great deal of work has been done to study the relation between loops and their multiplication groups. Of special interest thereby is the question as to when – in the case of topological/analytic loops – these groups are Lie groups. Two of the most important general results in this direction are as follows: if in a loop the right (left) Bol identity holds, then the group of right (left)

Mathematics Subject Classification: 20N05.
Key words and phrases: analytical Bol loops, multiplication groups.
Partially supported by DAAD-grant 323 PPP Ungarn.

translations is a Lie group (see e.g. [4] p. 418–430) and if the right and the left Bol identity are satisfied[1] (see e.g. [3] Theorem 1.1), then even the group of multiplications is a Lie group. In previously known examples of loops, in which a Bol identity was satisified only on one side, the group of translations "on the other side", i.e. for a right Bol loop the group of left translations (and hence the group of multiplications), was always infinite dimensional. Thus, it seemed natural to conjecture that there is a very close connection between a loop satisfying certain weak forms of associativity and having a Lie group as one of the above groups. In the present paper, however, it will be shown that there exist (analytic) right Bol loops, which do not satisfy the left Bol identity, but have a Lie group as group of multiplications. Furthermore, these occur not only in a few pathological examples, but there exists a very large class of them.

Definition 1. A loop is a set L equipped with three binary operations $\cdot,\ /,\ \backslash : L \times L \mapsto L$ and a distinguished element 1, such that $1 \cdot a = a \cdot 1 = a$ holds and the equations $x \cdot a = b$ and $a \cdot x = b$ have the unique solutions b/a and $a\backslash b$, i.e. $b/a \cdot a = b$ and $a \cdot a\backslash b = b$.

L is called topological respectively analytic, if all three operations are continuous respectively analytic.

Definition 2. A loop L is a right Bol loop, if $[(x \cdot y)z]y = x[(y \cdot z)y]$ holds for all $x, y, z \in L$. A loop L is a left Bol loop, if $x[y(x \cdot z)] = [x(y \cdot x)]z$ holds for all $x, y, z \in L$.

To define the class of loops in question, we now adapt a construction from [5] to our purposes.

Let G and N be Lie groups and let $\beta : G \mapsto \operatorname{Aut} N$ be a homomorphism. We define a loop structure on the manifold $N \times G$: Let $(1, 1)$ be the distinguished element; for $(m, g), (n, h) \in N \times G$ we set

$$(m, g) \cdot (n, h) = (\beta(h)m \cdot n, gh)$$
$$(n, h)/(m, g) = (\beta(g)^{-1}(n \cdot m^{-1}), hg^{-1})$$
$$(m, g)\backslash(n, h) = (\beta(g^{-1}h)m^{-1} \cdot n, g^{-1}h).$$

A simple calculation shows that this is consistent with the above definition of a loop and it is obvious that the three operations are analytic. Hence,

[1] It is well known, that this is equivalent to the fact that a certain other identity, the Moufang identity, is satisfied. But we will make no use of this.

this construction defines a class of analytic loops. From now on we will refer to it as the "twisted semidirect product of G by N". Although this construction looks very similiar to the ordinary semidirect product, these loops are not in general groups. This can be seen from the following equations:

$$((m,1)(1,g))(1,h) = (\beta(g)m, g)(1,h) = (\beta(hg)m, gh)$$
$$(m,1)((1,g)(1,h)) = (m,1)(1,gh) = (\beta(gh)m, gh).$$

Now, if associativity holds, the left parts of both equations are equal and we get $\beta(gh) = \beta(hg)$. On the other hand, if we have $\beta(gh) = \beta(hg)$, then the right parts of both equations are equal and associativity holds. Thus, associativity holds, if and only if $\beta(gh) = \beta(hg)$ holds for all $g, h \in G$.

Proposition 1. *The twisted semidirect product of Lie groups is a right Bol loop. If it is also a left Bol loop, then it is a group.*

Proof. We will check the identities.

$$\begin{aligned}
(k,f)[[(m,g)(n,h)](m,g)] &= (k,f)[(\beta(h)m\cdot n, gh)(m,g)]\\
&= (k,f)(\beta(gh)m\cdot\beta(g)n\cdot m, ghg)\\
&= (\beta(ghg)k\cdot\beta(gh)m\cdot\beta(g)n\cdot m, fghg)\\
&= (\beta(hg)k\cdot\beta(h)m\cdot n, fgh)(m,g)\\
&= [(\beta(g)k\cdot m, fg)(n,h)](m,g)\\
&= [[(k,f)(m,g)](n,h)](m,g).
\end{aligned}$$

Hence the right Bol identity holds.

Now consider

$$\begin{aligned}
(1,g)[(m,1)[(1,g)(1,h)]] &= (1,g)[(m,1)(1,gh)]\\
&= (1,g)(\beta(gh)m, gh) = (\beta(gh)m, g^2h)
\end{aligned}$$

and

$$\begin{aligned}
[(1,g)[(m,1)(1,g)]](1,h) &= [(1,g)(\beta(g)m, g)](1,h)\\
&= (\beta(g)m, g^2)(1,h) = (\beta(hg)m, g^2h).
\end{aligned}$$

Thus, if the left Bol identity holds, we have $\beta(gh) = \beta(hg)$ and the twisted semidirect product is a group. □

2. The main results

From now on all Lie groups, Lie algebras and vector spaces are either real or complex.

Definition 3. Let L be a loop. The group of left translations of L is the transformation group of L generated by the left multiplications $\lambda_a : L \mapsto L$ $\lambda_a(x) = a \cdot x, a \in L$.

The group of right translations of L is the transformation group of L generated by the right multiplications $\rho_a : L \mapsto L$ $\rho_a(x) = x \cdot a, a \in L$. The group of multiplications of L is the group generated by all multiplications.

Now we will state our results.

Theorem 1. *Let N be a nilpotent and G be an arbitrary Lie group, then the group of left translations of any twisted semidirect product of G by N is itself a Lie group.*

Theorem 2. *Let N and G be as in Theorem 1, then the group of multiplications of any twisted semidirect product of G by N is a Lie group.*

In order to prove Theorem 1, we have to do some preparation. First we show that the group of left translations is isomorphic to a semidirect product of G by $\mathcal{L}$, where $\mathcal{L}$ is a group of mappings $G \mapsto N$ (Section 2.1). Thus, it is clear that it suffices to show that this group of mappings is a Lie group. The first step in this direction is to find a suitable criterion for when a Lie algebra of mappings from a vector space into a nilpotent Lie algebra is finite dimensional. By combining these results with the special property of nilpotent Lie algebras to carry also the structure of a "polynomial" Lie group, we can show that a certain group of mappings is a Lie group (Section 2.2). Thus, the proof of Theorem 1 is reduced to showing that $\mathcal{L}$ is isomorphic to a subgroup of a group of this type. Theorem 2 follows from Theorem 1, since it can be shown that for right Bol loops the group of multiplications is a factor group of a semidirect product of the group of right translations by the group of left translations (Proposition 7).

2.1. The algebraic structure of the group of left translations

In this section, we show how the group of left translations decomposes as a semidirect product and that its normal subgroup (with respect to this decomposition) is isomorphic to a group of mappings from G to N. Though we use them only as an intermediate step in proving Theorem 1 & 2, the results of this section are, of course, of independent interest.

Before we proceed, we will introduce some notation. Denote by $\lambda_{(m,g)}$ the left multiplication with the element (m, g), i.e.

$$\lambda_{(m,g)}(x, y) = (m, g)(x, y) = (\beta(y)m \cdot x, g \cdot y).$$

Furthermore, let $\mathcal{G}$ be the group generated by the translations $\lambda_{(1,g)}$ and $\mathcal{L}$ the group generated by the elements $\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}$.

Theorem 3. *The group of left translations is a semidirect product of the subgroup $\mathcal{G}$ by the subgroup $\mathcal{L}$. The group $\mathcal{G}$ is isomorphic to G.*

Proof. In order to prove this theorem, three properties need to be explicated: that the group of left translations is isomorphic to $\mathcal{G}L$, i.e. that every element can be described as $\alpha\gamma$ with $\alpha \in \mathcal{G}$ and $\gamma \in \mathcal{L}$, that $\mathcal{L}$ is a normal subgroup and that $\mathcal{G} \cap \mathcal{L} = 1$.

We will begin with some algebraic properties of the multiplication of left translations.

Since $\lambda_{(m,g)}(x, y) = (m, g)(x, y) = (\beta(y)m \cdot x, g \cdot y)$, it follows from $\lambda_{(1,g)}(x, y) = (1, g)(x, y) = (x, g \cdot y)$, that $\lambda_{(1,g^{-1})} = \lambda^{-1}_{(1,g)}(I)$ and $\lambda_{(1,g)}\lambda_{(1,h)} = \lambda_{(1,gh)}(II)$.

And because we have $(1, g)(m, 1) = (m, g)$ and $\lambda^{-1}_{(m,1)} = \lambda_{(m^{-1},1)}$, it follows that $\lambda_{(1,g)}\lambda_{(m,1)} = \lambda_{(m,g)}(III)$ and $\lambda^{-1}_{(m,g)} = \lambda^{-1}_{(m,1)}\lambda^{-1}_{(1,g)} = \lambda_{(m^{-1},1)}\lambda_{(1,g^{-1})}(IV)$ hold.

To show that the group of left translations is isomorphic to $\mathcal{G}\mathcal{L}$ we will proceed by induction over word length. From (III) it is clear that for the generators there is nothing to show. But since $\lambda^{-1}_{(m,1)}\lambda^{-1}_{(1,g)} = \lambda_{(1,g^{-1})}\lambda_{(1,g)}\lambda_{(m^{-1},1)}\lambda_{(1,g^{-1})}$, from (IV) it follows, that $\lambda^{-1}_{(m,g)} = \alpha\gamma$ with $\alpha = \lambda_{(1,g^{-1})}$ and $\gamma = \lambda_{(1,g)}\lambda_{(m^{-1},1)}\lambda_{(1,g^{-1})}$.

Now let us assume that we can find suitable α and γ for any word of length k.

From (I) and (II) it is clear, that there is a $h \in G$ such that $\alpha = \lambda_{(1,h)}$. Then we have

$$\begin{aligned}\lambda_{(m,g)}\alpha\gamma &= \lambda_{(m,g)}\lambda_{(1,h)}\gamma = \lambda_{(1,g)}\lambda_{(m,1)}\lambda_{(1,h)}\gamma \\ &= \lambda_{(1,g)}\lambda_{(1,h)}\lambda_{(1,h^{-1})}\lambda_{(m,1)}\lambda_{(1,h)}\gamma,\end{aligned}$$

but $\lambda_{(1,g)}\lambda_{(1,h)} = \lambda_{(1,g)}\alpha$ lies in $\mathcal{G}$ and $\lambda_{(1,h^{-1})}\lambda_{(m,1)}\lambda_{(1,h)}\gamma$ lies in $\mathcal{L}$. Furthermore, we get

$$\begin{aligned}\lambda^{-1}_{(m,g)}\alpha\gamma &= \lambda_{(m^{-1},1)}\lambda_{(1,g^{-1})}\lambda_{(1,h)}\gamma\\ &= \lambda_{(1,g^{-1})}\lambda_{(1,h)}\lambda_{(1,h^{-1})}\lambda_{(1,g)}\lambda_{(m^{-1},1)}\lambda_{(1,g^{-1})}\lambda_{(1,h)}\gamma,\end{aligned}$$

with $\lambda_{(1,g^{-1})}\lambda_{(1,h)}\ \lambda_{(1,g^{-1})}\alpha$ in $\mathcal{G}$ and $\lambda_{(1,h^{-1})}\lambda_{(1,g)}\lambda_{(m^{-1},1)}\lambda_{(1,g^{-1})}\lambda_{(1,h)}\gamma$ in $\mathcal{L}$. Since any word of length $k+1$ can be generated by multiplying a word of length one from the left to a word of length k, the first part of the proof is finished.

To show that $\mathcal{L}$ is normal we have to show that $\alpha\gamma\alpha^{-1}$ lies in $\mathcal{L}$ for arbitrary elements $\alpha \in \mathcal{G}$ and $\gamma \in \mathcal{L}$.

Since conjugation with a group element is an automorphism, it suffices to show $\alpha\gamma\alpha^{-1} \in \mathcal{L}$ only for the generators of $\mathcal{L}$. From (I) and (II) it is clear that for any $\alpha \in \mathcal{G}$ there exists a $h \in G$, such that $\alpha = \lambda_{(1,h)}$. Thus, we have to show only that $\lambda_{(1,h)}\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}\lambda_{(1,h^{-1})}$ lies in $\mathcal{L}$. This is, however, obvious, since $\lambda_{(1,h)}\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}\lambda_{(1,h^{-1})} = \lambda_{(1,hg^{-1})}\lambda_{(m,1)}\lambda_{(1,gh^{-1})}$ by (II).

Thus, it remains to be shown that the neutral element is the only one both subgroups have in common. This is done by considering their action on $N \times G$. We have $\lambda_{(m,g)}(x,y) = (m,g)(x,y) = (\beta(y)m \cdot x, g \cdot y)$ by definition. So $\lambda_{(1,g)}(x,y) = (1,g)(x,y) = (x, g \cdot y)$ holds and by a simple calculation we get

$$\begin{aligned}\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}(x,y) &= ((1,g^{-1})((m,1)((1,g)(x,y))))\\ &= (\beta(gy)m \cdot x, y).\end{aligned}$$

But this means that $\mathcal{L}$ acts only on the first and $\mathcal{G}$ only on the second factor of the product $N \times G$, so their only common element is the neutral element. And from $\lambda_{(1,g)}(x,y) = (1,g)(x,y) = (x, g \cdot y)$ it is also obvious that $\mathcal{G}$ is isomorphic to G. □

Let X be a set, let U be a group and let $S_{X \mapsto U}$ be a set of mappings $X \mapsto U$. For two mappings $f, g \in S_{X \mapsto U}$ one can define their product mapping by taking the pointwise product, i.e. $fg(x) = f(x)g(x)$ and the multiplication defined in this way is obviously associative. Also each such mapping has a well defined multiplicative inverse $\iota(f)(x) = f(x)^{-1}$. Hence, any set of mappings into a group generates a group of mappings.

Proposition 2. *Let $\mathcal{L}(G,N)$ be the group generated by the set of mappings $\boldsymbol{Q} := \{f_{g,m} : G \mapsto N;\ f_{g,m}(y) = \beta(gy)m;\ g \in G,\ m \in N\}$; $\mathcal{L}(G,N)$ is isomorphic to $\mathcal{L}$.*

Proof. First we show that for every $\gamma \in \mathcal{L}$ there exists a mapping $f : G \mapsto N$, such that $\gamma(x,y) = (f(y) \cdot x, y)$. We do this again by induction over the word length.

For words of length one it follows from

$$\begin{aligned}\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}(x,y) &= ((1,g^{-1})((m,1)((1,g)(x,y)))) \\ &= (\beta(gy)m \cdot x, y)\end{aligned}$$

and the fact that

$$(\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)})^{-1} = \lambda^{-1}_{(1,g)}\lambda^{-1}_{(m,1)}\lambda^{-1}_{(1,g^{-1})} = \lambda_{(1,g^{-1})}\lambda_{(m^{-1},1)}\lambda_{(1,g)}$$

holds.

Now assume we have found a suitable mapping f for every word of length k and let δ be a word of length k. Any word of length $k+1$ is of the form $\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)} \cdot \delta$. Then

$$\begin{aligned}\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)} \cdot \delta(x,y) &= \lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}(\delta(x,y)) \\ = \lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}(f(y) \cdot x, y) &= (\beta(gy)m \cdot f(y) \cdot x, y)\end{aligned}$$

holds. But since $y \mapsto \beta(gy)m \cdot f(y)$ again defines a mapping $G \mapsto N$, we have reached our goal.

Now for $\gamma \in \mathcal{L}$ let f_γ be the corresponding mapping. Since $\gamma \cdot \delta(x,y) = \gamma(f_\delta(y) \cdot x, y) = (f_\gamma(y) \cdot f_\delta(y) \cdot x, y)$, there exists a corresponding group of mappings and because

$$\begin{aligned}\lambda_{(1,g^{-1})}\lambda_{(m,1)}\lambda_{(1,g)}(x,y) &= ((1,g^{-1})((m,1)((1,g)(x,y)))) \\ &= (\beta(gy)m \cdot x, y)\end{aligned}$$

holds, it follows that it is generated by the set

$$\{f_{g,m} : G \mapsto N;\ f_{g,m}(y) = \beta(gy)m;\ g \in G,\ m \in N\} \qquad \square$$

2.2. Lie algebras of mappings

In this section we develop some results showing when a Lie group/Lie algebra of mappings from a vector space into a nilpotent Lie algebra is finite dimensional. This construction is the very heart of the proof of the main results. The question of how it can be applied to show that the group of mappings discovered in the previous section is a Lie group will be dealt with in the next section.

Let V be a finite dimensional vector space and let N be a finite dimensional Lie algebra. As is generally known, N^V, the set of all mappings $V \mapsto N$, carries (with addition and scalar multiplication defined pointwise) the structure of an infinite dimensional vector space. It also carries in a natural way (with the Lie bracket again defined pointwise) the structure of a nilpotent, infinite dimensional Lie algebra, which is nilpotent of the same degree as N. We will call this Lie algebra $M(V, N)$.

Yet, we are not interested in $M(V, N)$ as a whole, but in a subalgebra generated by a certain subspace. We want to show this Lie algebra to be finite dimensional.

Proposition 3. *Let W be a subspace of a (possibly infinite dimensional) Lie algebra H and let $l(W)$ be the subalgebra generated by W, then we have $l(W) = \sum_{i=1}^{\infty} W^i$, where $W^1 = W$ and $W^i = [W^{i-1}, W]$ for $i \geq 2$.*

Proof. By definition $l(W)$ consists of all linear combinations of elements of W and finitely iterated brackets with entries from W. Hence the proposition is true if $[W^i, W^j] \subseteq W^{i+j}$. But this is a generally known fact (see e.g. [6] p. 25, Proposition 5). □

Unlike the previous one, the next proposition is completely trivial, if the Lie algebra H is itself finite dimensional.

Proposition 4. *If W is finite dimensional and H nilpotent, then $l(W)$ is finite dimensional and nilpotent.*

Proof. Every subalgebra of a nilpotent Lie algebra is nilpotent. Hence, it remains only to show that $l(W)$ is finite dimensional.

Since H is nilpotent, there is a k such that $H^i = 0$ for $i \geq k$. Thus, also W^i vanishes for $i \geq k$. By the previous proposition it therefore suffices to show that W^i is finite dimensional for all i.

This is done by induction over i. W is finite dimensional, so for $i = 1$ the assumption is true. Now assume that W^i is finite dimensional, then it has a finite base e_μ. Since W is finite dimensional, it has a finite base f_ν. So the vector space W^{i+1} is generated by the set of all brackets $[e_\mu, f_\nu]$. But since both bases are finite, this is a finite set of vectors generating W^{i+1}. Hence, W^{i+1} is finite dimensional. □

$L(V, N)$, the space of all linear mappings $V \mapsto N$, is obviously a finite dimensional subspace of $M(V, N)$. By the previous proposition it generates a finite dimensional, nilpotent subalgebra $\mathcal{M}(V, N)$ of $M(V, N)$.

A particular property of finite dimensional, nilpotent Lie algebras is that they also carry the structure of nilpotent Lie groups: because of the nilpotency the Campbell–Taylor–Hausdorff series breaks off after finitely many steps, hence converges on the whole Lie algebra and defines, therefore, the structure of a nilpotent Lie group on it (see e.g. [7] p. 227, III.3.23). So $\mathcal{M}(V, N)$ can be viewed as a nilpotent Lie group.

On the other hand, this also applies to N, since it is a nilpotent Lie algebra itself. So it is possible to equipe $M(V, N)$ with the structure of a (abstract) nilpotent group: multiplication is defined pointwise by $(f \cdot g)(x) = f(x) *_N g(x) = f(x) + g(x) + \frac{1}{2}[f(x), g(x)] + \dots$ (Campbell–Taylor–Hausdorff multiplication in N), the inverse by $f^{-1}(x) = f(x)^{-1} = -f(x)$, the neutral element is the mapping, which is identical zero; the law of associativity holds obviously.

We now can consider the subgroup $\mathcal{L}(V, N)$ of $M(V, N)$, which is generated by $L(V, N)$. Our next aim is to show, that this group is a path connected subgroup of $\mathcal{M}(V, N)$. To do so, it is important to see that $\mathcal{M}(V, N)$ (with the group structure induced by the Campbell–Taylor–Hausdorff multiplication) is a subgroup of $M(V, N)$, with the group structure described above.

Proposition 5. *$\mathcal{M}(V, N)$ is a subgroup of $M(V, N)$.*

Proof. Let $*_{\mathcal{M}(V,N)}$ and $*_N$ be the Campbell–Taylor–Hausdorff multiplication in $\mathcal{M}(V, N)$ resp. N; let $[\,,\,]_{\mathcal{M}(V,N)}$ be the bracket in $\mathcal{M}(V, N)$ and $[,]$ the bracket in N; finally, let $\cdot$ be the multiplication in $M(V, N)$. Then $(f *_{\mathcal{M}(V,N)} g)(x) = f(x) + g(x) + \frac{1}{2}[f, g]_{\mathcal{M}(V,N)}(x) + \dots = f(x) + g(x) + \frac{1}{2}[f(x), g(x)] + \dots = f(x) *_N g(x) = (f \cdot g)(x)$ holds by the above considerations. □

Proposition 6. *$\mathcal{L}(V,N)$ is a path connected subgroup of $\mathcal{M}(V,N)$, hence a Lie group.*

PROOF. $\mathcal{L}(V,N)$ consists of all finite words $\prod_i f_i$ with $f_i \in L(V,N)$. To show that $\mathcal{L}(V,N)$ is a subgroup of $\mathcal{M}(V,N)$, it suffices by the previous proposition to show that all finite words are in $\mathcal{M}(V,N)$. To do so we proceed by induction over the word length. All words of length one are in $\mathcal{M}(V,N)$, since $\prod_{i=1}^1 f_i = f_1 \in L(V,N) \in \mathcal{M}(V,N)$. We assume now that all words of length k are in $\mathcal{M}(V,N)$, then

$$\begin{aligned}\left(\prod_{i=1}^{k+1} f_i\right)(x) &= \left(\prod_{i=1}^{k} f_i \cdot f_{k+1}\right)(x) = \prod_{i=1}^{k} f_i(x) *_N f_{k+1}(x)\\ &= \prod_{i=1}^{k} f_i(x) + f_{k+1}(x) + \frac{1}{2}\left[\prod_{i=1}^{k} f_i(x), f_{k+1}(x)\right] + \dots\\ &= \prod_{i=1}^{k} f_i(x) + f_{k+1}(x) + \frac{1}{2}\left[\prod_{i=1}^{k} f_i, f_{k+1}\right]_{\mathcal{M}(V,N)}(x) + \dots\\ &= \left(\prod_{i=1}^{k} f_i *_{\mathcal{M}(V,N)} f_{k+1}\right)(x)\end{aligned}$$

holds. So it remains to show that this subgroup is path connected. Yet, this is completely obvious, since it is generated by a path connected set. But path connected subgroups of Lie groups are analytic, according to the famous Yamabe Theorem (see e.g. [7] Theorem I.6.1 and Theorem I.9.15) and analytic subgroups of simply connected nilpotent Lie groups are simply connected and closed (see e.g. [7] Theorem III.3.31). □

2.3. Proof of Theorem 1 & 2

Now we have come to show how the results of the previous section can be used to prove that the group of mappings from Section 2.1 – and then as well the group of left translations as the group of multiplications – is a Lie group.

We will proceed in two steps: First we will prove the result for "polynomial" Lie groups. Then we will use this to show that it holds for N arbitrary nilpotent. Let N be a "polynomial" Lie group, i.e. a nilpotent

Lie algebra with the group structure induced by the Campbell–Taylor–Hausddorff multiplication. Since the group of left translations is isomorphic to the semidirect product of G and $\mathcal{L}$ (Theorem 3), we have to show only that $\mathcal{L}$ is a Lie group.

But $\mathcal{L}$ is isomorphic to $\mathcal{L}(G,N)$, a group of mappings from G into the polynomial Lie group N (Proposition 2). Since G itself is not a vectorspace, the results of the previous section cannot be applied directly. Thus, we have to find a suitable vector space V, such that $\mathcal{L}(G,N)$ can be considered a subgroup of the group generated by $L(V,N)$.

For this we will consider the algebra $\operatorname{End} N$, the algebra of all linear endomorphisms of N. Let V be the subalgebra of $\operatorname{End} N$ generated by $\operatorname{Aut} N$. Clearly, $\mathcal{L}(V,N)$, the group generated by $L(V,N)$, is finite dimensional, since it is according to Proposition 6 a path connected subgroup of $\mathcal{M}(V,N)$, which is finite dimensional according to Proposition 4. Thus, it suffices to find a subgroup of $\mathcal{L}(V,N)$ isomorphic to $\mathcal{L}(G,N)$.

Now consider mappings $h_{g,m} : V \mapsto N$ with $h_{g,m}(Y) = \beta(g)Ym$ for $g \in G$ and $m \in N$. Since V is not only a vector space, but also an algebra and because $\operatorname{Aut} N$ is contained in V, such mappings are well defined and, by the law of distributivity, they are linear. Thus, $\mathcal{P}$, the group of mappings generated by the set

$$\boldsymbol{P} := \{h_{g,m} : V \mapsto N;\ h_{g,m}(y) = \beta(g)Ym;\ g \in G,\ m \in N\},$$

is a subgroup of the group generated by $L(V,N)$. It is path connected, since its generating set is path connected. $\mathcal{L}(G,N)$ is generated by the set

$$\boldsymbol{Q} := \{f_{g,m} : G \mapsto N;\ f_{g,m}(y) = \beta(gy)m;\ g \in G,\ m \in N\}.$$

Obviously the map $j : \boldsymbol{Q} \mapsto \boldsymbol{P}$ with $j(\beta(gy)m) = \beta(g)Ym$ is a bijection; because

$$j\left(\prod_{i=1}^{k} \beta(g_i y)m_i\right) := \prod_{i=1}^{k} j(\beta(g_i y)m_i) = \prod_{i=1}^{k} \beta(g_i)Ym_i$$

holds, this extends to a map $\mathcal{L}(G,N) \mapsto \mathcal{P}$, which is clearly a homomorphism. Since it is a bijection on the generating sets, it is also an isomorphism. Thus, the polynomial case is done.

Now let M be an arbitrary nilpotent Lie group. Then there exists a polynomial Lie group N and a surjective homomorphism $p : N \mapsto M$. Because of the surjectivity of p the set

$$\boldsymbol{Q}` := \{f_{g,p(m)} : G \mapsto M;\ f_{g,m}(y) = \beta(gy)p(m);\ g \in G,\ m \in N\}$$

generates $\mathcal{L}(G, N)$ and we get a surjection

$$p_* : \boldsymbol{Q} \mapsto \boldsymbol{Q}`$$

$$p_*(\beta(gy)m) := \beta(gy)p(m).$$

As above, this extends to a homomorphism $\mathcal{L}(G, N) \mapsto \mathcal{L}(G, M)$ and since it is surjective on the generating sets it is a surjection. Thus, we have proved Theorem 1.

Since the group of right translations of analytic right Bol loop is always a Lie group (see e.g. [4] p. 418–430), Theorem 2 follows now directly from the proposition below.

Proposition 7. *The group of multiplications of a right Bol loop is a factor group of a semidirect product of the group of right translations by the group of left translations.*

Proof. Denote the group of left translations by Λ, the group of right translations by Π and the group of multiplications by Θ. Let ρ_x denote the multiplication from the right and λ_x multiplication from the left with a given loop element x. It is possible to express the right Bol identity as $\rho_y\lambda_{xy} = \lambda_x\rho_y\lambda y$ (see Lemma 1 in [8]), but this implies that $\rho_y^{-1}\lambda_x\rho_y = \lambda_{xy}\lambda_y^{-1}$. Thus, we have a canonical action of Π as a group of automorphisms of Λ. Now consider the semidirect product of Π by Λ defined using this action. Then we get a mapping $\gamma : \Lambda \times \Pi \mapsto \Theta$ defined by $\gamma(\lambda, \rho) = \lambda\rho$. That this mapping is surjective follows now directly from the fact that $\rho\lambda = \rho\lambda\rho^{-1}\rho$. It is clear that it is a homomorphism, since $\gamma((\lambda_1, \rho_1)(\lambda_2, \rho_2)) = \gamma(\lambda_1\rho_1\lambda_2\rho_1^{-1}, \rho_1\rho_2) = \lambda_1\rho_1\lambda_2\rho_1^{-1}\rho_1\rho_2 = \lambda_1\rho_1\lambda_2\rho_2 = \gamma(\lambda_1, \rho_1)\gamma(\lambda_2, \rho_2)$ holds. □

Acknowledgements. The author wishes to thank P. Nagy and K. Strambach for many fruitful discussions while preparing the manuscript. Especially, he wishes to thank P. Nagy for the kind hospitality at the Institute of Mathematics of the University of Debrecen.

References

[1] K. H. Hoffmann and K. Strambach, Topological and analytic Loops, in: O. Chein et al.: Quasigroups and Loops – Theory and Applications, *Heldermann Verlag*, *Berlin*, 1990, 205–262.

[2] P. Nagy and K. Strambach, Loops, viewed from Groups and Lie Theory (*to appear*).

[3] P. Nagy and K. Strambach, Loops, their cores and symmetric spaces, *Israel Journal of Mathematics* **105** (1998), 285–322.

[4] P. O. Miheev and L. V. Sabinin, Quasigroups and differential Geometry, in: O. Chein et al.: Quasigroups and Loops – Theory and Applications, *Heldermann Verlag*, *Berlin*, 1990, 357–430.

[5] G. F. Birkemeier and S. Xiao, Loops which are Semidirect Products of Groups, *Communications in Algebra* **23** (1) (1995), 81–95.

[6] N. Jacobson, Lie Algebras, *John Wiley & Sons*, *New York*, 1962.

[7] J. Hilgert and K. H. Neeb, Lie-Gruppen und Lie-Algebren, *Vieweg*, *Braunschweig*, 1991.

[8] G. P. Nagy, Group Invariants of certain Burn Loop Classes, *Bull. Belg. Math. Soc.* **5** (1998), 403–415.

EKKEHART WINTERROTH
MATHEMATISCHES INSTITUT
UNIVERSITÄT ERLANGEN–NÜRNBERG
BISMARCKSTRASSE 1 1–2
D–91054 ERLANGEN
GERMANY

(Received July 5, 2000; revised February 12, 2001)