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A joint limit theorem for zeta-functions attached to certain cusp forms

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In honour of Professors Kálmán Győry and András Sarközy on the occasion of their 60th birthday

Abstract. In the paper a joint limit theorem in the sense of the weak convergence of probability measures for zeta-functions attached to certain cusp forms in the space of analytic function is obtained.

1. Introduction

Let $s=\sigma+it$ and z be complex variables, and let $F(z)$ be a holomorphic cusp form of weight κ for the full modular group $SL(2,\mathbb{Z})$. Moreover, suppose that $F(z)$ is a normalized eigenform. Then $F(z)$ has the Fourier series expansion

$$F(z)=\sum_{m=1}^{\infty}c(m)e^{2\pi imz},\qquad c(1)=1.$$

The Dirichlet series

$$\varphi(s,F)=\sum_{m=1}^{\infty}\frac{c(m)}{m^s}$$

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is called the zeta-function attached to the cusp form $F(z)$. It is well known that the latter series is absolutely convergent in the half-plane $\sigma > (\kappa+1)/2$, and can be continued analytically to an entire function. The zeta-function $\varphi(s,F)$ satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)\varphi(s,F) = (-1)^{\kappa/2}(2\pi)^{s-\kappa}\Gamma(\kappa-s)\varphi(\kappa-s,F),$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function. Consequently, the critical strip for $\varphi(s,F)$ is $(\kappa-1)/2 \leqslant \sigma \leqslant (\kappa+1)/2$.

The probabilistic study of the function $\varphi(s,F)$ was began in [7]. Here a limit theorem on the complex plane was obtained, and the limit measure was studied.

In [4] a limit theorem in the sense of the weak convergence of probability measures in the space of analytic functions for the function $\varphi(s,F)$ was obtained, and in [6] the latter theorem was applied to prove the universality of $\varphi(s,F)$.

Let γ stand for the unit circle on the complex plane $\mathbb{C}$, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. The infinite dimensional torus Ω is a compact topological Abelian group, therefore on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S, the probability Haar measure m_H exists. Thus we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ onto the coordinate space γ_p.

Moreover, for a region G on $\mathbb{C}$, let $H(G)$ denote the space of analytic on G functions equipped with the topology of uniform convergence on compacta.

The function $\varphi(s,F)$, for $\sigma > (\kappa+1)/2$, has the Euler product

$$\varphi(s,F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1}$$

with $c(p) = \alpha(p) + \beta(p)$. Let $D_0 = \{s \in \mathbb{C} : \sigma > \kappa/2\}$, and define on $(\Omega, \mathcal{B}(\Omega), m_H)$ the $H(D_0)$-valued random element $\varphi(s,\omega,F)$ by the formula

$$\varphi(s,\omega,F) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}, \qquad s \in D_0.$$

Let meas$\{A\}$ stand for the Lebesgue measure of the set A, and let, for $T > 0$,

$$\nu_T(\dots) = \frac{1}{T}\,\text{meas}\{\tau \in [0,T], \dots\},$$

where in place of dots a condition satisfied by τ is to be written. Then in [4] the following statement was proved.

Theorem A. *The probability measure*

$$\nu_T(\varphi(s+i\tau, F) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element $\varphi(s,\omega,F)$ *as* $T \to \infty$.

Now let $D = \{s \in \mathbb{C} : \kappa/2 < \sigma < (\kappa+1)/2\}$. In [6] Theorem A was used for the proof of the following universality theorem.

Theorem B. *Let* K *be a compact subset of* D *with connected complement, and let* $f(s)$ *be non-vanishing continuous function on* K *which is analytic in the interior of* K. *Then, for any* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}\bigl|\varphi(s+i\tau,F) - f(s)\bigr| < \varepsilon\right) > 0.$$

The aim of this paper is to prove a joint version of Theorem A for a collection of zeta-functions attached to cusp forms. Let $F_j(z)$ be a normalized eigenform of weight κ_j, and let

$$\varphi(s,F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)}{m^s}$$

be the corresponding zeta-function, $D_j = \{s \in \mathbb{C} : \sigma > \kappa_j/2\}$, $j = 1,\dots,n$. Moreover, let $H_n = H_n(D_1,\dots,D_n) = H(D_1)\times\dots\times H(D_n)$, and consider the probability measure

$$P_T(A) = \nu_T\left(\bigl(\varphi(s_1+i\tau,F_1),\dots,\varphi(s_n+i\tau,F_n)\bigr) \in A\right), \quad A \in \mathcal{B}(H_n),$$

where $s_1,\dots,s_n$ are complex variables. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the H_n-valued random element $\varphi(s_1,\dots,s_n,\omega; F_1,\dots,F_n)$ by the formula

$$\varphi(s_1,\dots,s_n,\omega;F_1,\dots,F_n) = \bigl(\varphi(s_1,\omega,F_1),\dots,\varphi(s_n,\omega,F_n)\bigr),$$

where

$$\varphi(s_j,\omega,F_j)=\sum_{m=1}^{\infty}\frac{c_j(m)\omega(m)}{m^{s_j}},\quad s_j\in D_j,\ j=1,\dots,n.$$

Theorem. *The probability measure P_T converges weakly to the distribution of the random element $\varphi(s_1,\dots,s_n,\omega;F_1,\dots,F_n)$ as $T\to\infty$.*

We recall that the distribution of $\varphi(s_1,\dots,s_n,\omega;F_1,\dots,F_n)$ is

$$m_H\big(\omega\in\Omega:\varphi(s_1,\dots,s_n,\omega;F_1,\dots,F_n)\in A\big),\quad A\in\mathcal{B}(H_n).$$

Note that the space H_n is a product of distinct spaces, and a joint theorem of a such kind is new in the theory of zeta-functions.

2. Lemmas

We recall that the family $\{P\}$ of probability measures on $(S,\mathcal{B}(S))$ is relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence, and $\{P\}$ is tight if for an arbitrary $\varepsilon>0$ there exists a compact set K such that $P(K)>1-\varepsilon$ for all measures P from $\{P\}$.

The relative compactness of a family of probability measures is related with the tightness of this family by Prokhorov's theorems. The statements of these theorems are contained in two following lemmas.

Lemma 1. *If the family of probability measures $\{P\}$ is tight, then it is relatively compact.*

Lemma 2. *Let S be a separable complete metric space. If the family of probability measures $\{P\}$ on $(S,\mathcal{B}(S))$ is relatively compact, then it is tight.*

Proof of Lemmas 1 and 2 can be found in [2].

Lemma 3. *The family of probability measures $\{P_T\}$ is relatively compact.*

PROOF. By Theorem A the probability measure

$$P_{jT}(A)=\nu_T\big(\varphi(s+i\tau,F_j)\in A\big),\quad A\in\mathcal{B}(H(D_j)),$$

converges weakly to the distribution of the random element $\varphi(s,\omega,F_j)$ as $T\to\infty$, $j=1,2,\dots,n$. Therefore the family $\{P_{jT}\}$ is relatively compact,

$j = 1, 2, \dots, n$. However, $H(D_j)$ is a complete separable space, hence by Lemma 2 the family $\{P_{jT}\}$ is tight, $j = 1, \dots, n$. Let $\varepsilon > 0$ be an arbitrary number. By the definition of tightness there exists a compact set $K_j \subset H(D_j)$ such that

$$P_{jT}\big(H(D_j) \setminus K_j\big) < \frac{\varepsilon}{n}, \quad j = 1, \dots, n, \tag{1}$$

for all $T > 0$. Let η_T be a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{P}(\eta_T \in A) = \frac{1}{T} \int_0^T I_A \, dt, \quad A \in \mathcal{B}(\mathbb{R}),$$

where I_A is the indicator function of the set A. Define the $H(D_j)$-valued random element $\varphi_{jT}(s)$ by

$$\varphi_{jT}(s) = \varphi(s + i\eta_T, F_j), \quad j = 1, \dots, n,$$

and take

$$\varphi_T(s_1, \dots, s_n) = \big(\varphi_{1T}(s_1), \dots, \varphi_{nT}(s_n)\big), \quad s_j \in D_j, \ j = 1, \dots, n.$$

Then by (1)

$$\mathbb{P}\big(\varphi_{jT}(s) \in H(D_j) \setminus K_j\big) < \frac{\varepsilon}{n}, \quad j = 1, \dots, n. \tag{2}$$

Now let $K = K_1 \times \dots \times K_n$. Then K is a compact set of the space H_n. Moreover, by (2) we obtain

$$\begin{aligned} P_T(H_n \setminus K) &= \mathbb{P}\big(\varphi_T(s_1, \dots, s_n) \in H_n \setminus K\big) \\ &= \mathbb{P}\Big(\bigcup_{j=1}^{n} \big(\varphi_{jT}(s) \in H(D_j) \setminus K_j\big)\Big) \\ &\leqslant \sum_{j=1}^{n} \mathbb{P}\big(\varphi_{jT}(s) \in H(D_j) \setminus K_j\big) < \varepsilon \end{aligned}$$

for all $T > 0$. This means that the family $\{P_T\}$ is tight. Hence, by Lemma 1, it is relatively compact. The lemma is proved.

Let $s_1^{(j)}, \dots, s_k^{(j)}$ be arbitrary points on D_j, and we take

$$\sigma_1^{(j)} = \min_{1 \leqslant l \leqslant k} \Re s_l^{(j)}.$$

Then $\sigma_2^{(j)} = \kappa_j/2 - \sigma_1^{(j)} < 0$, and we set

$$\hat{D} = \left\{ s \in \mathbb{C} : \ \sigma > \max_{1 \leqslant j \leqslant n} \sigma_2^{(j)} \right\}.$$

Moreover, let u_{jl} be arbitrary complex numbers, and a function $h : H_n \to H(\hat{D})$ be defined by the formula

$$h(f_1, \dots, f_n; s) = \sum_{j=1}^{n} \sum_{l=1}^{k} u_{jl} f_j(s_l^{(j)} + s), \tag{3}$$

where $s \in \hat{D}$ and $f_j \in H(D_j)$, $j = 1, \dots, n$.

Now define

$$\varphi_h(s) = h\big(\varphi(s_1, F_1), \dots, \varphi(s_n, F_n); s\big), \tag{4}$$

and let, for brevity,

$$\varphi = \varphi(s_1, \dots, s_n) = \varphi(s_1, \dots, s_n, \omega; F_1, \dots, F_n).$$

Lemma 4. *The relation*

$$\varphi_h(s + \eta_T) \xrightarrow[T \to \infty]{\mathcal{D}} h(\varphi, s),$$

where $\xrightarrow[T \to \infty]{\mathcal{D}}$ *means the convergence in distribution, holds.*

Proof. The definition of the function h, for

$$\sigma > \max_{1 \leqslant j \leqslant n} \sigma_2^{(j)} + \frac{1}{2},$$

yield

$$\begin{aligned} \varphi_h(s) &= \sum_{j=1}^{n} \sum_{l=1}^{k} u_{jl} \sum_{m=1}^{\infty} \frac{c_j(m)}{m^{s_l^{(j)} + s}} \\ &= \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{j=1}^{n} \sum_{l=1}^{k} u_{jl} c_j(m) m^{-s_l^{(j)}} = \sum_{m=1}^{\infty} \frac{b(m)}{m^s}, \end{aligned}$$

where

$$b(m) = \sum_{j=1}^{n} \sum_{l=1}^{k} u_{jl} c_j(m) m^{-s_l^{(j)}}.$$

Since, by the Deligne theorem,

$$|c_j(m)| \leqslant m^{(\kappa_j - 1)/2} d(m),$$

where

$$d(m) = \sum_{d|m} 1,$$

the function $\varphi_h(s)$ in the half-plane

$$\sigma > \max_{1 \leqslant j \leqslant n} \sigma_2^{(j)} + \frac{1}{2}$$

is given by absolutely convergent Dirichlet series.

For $\sigma > (\kappa_j + 1)/2$ the function $\varphi(s, F_j)$ is given by absolutely convergent Dirichlet series and it is an entire function, $j = 1, \dots, n$. Moreover, for $|\sigma - \kappa_j/2| \leqslant 1/2$, the function $\varphi(s, F_j)$ satisfies the equality [3]

$$\varphi(s, F_j) = \sum_{m \leqslant (t/2\pi)^2} \frac{c(m)}{m^s} + (-1)^{\kappa_j/2} (2\pi)^{2s - \kappa_j} \frac{\Gamma(\kappa_j - s)}{\Gamma(s)} + R$$

with $R = O(t^{\kappa_j - 2\sigma} \log^2 t)$. Hence, in view of the Deligne estimate, we obtain that, for $\sigma > \kappa_j/2$, the function $\varphi(s, F_j)$ is of finite order. Therefore, in view of (3) and (4), for $s \in \hat{D}$, the function $\varphi_h(s)$ is of finite order.

Finally, for $\sigma > \kappa_j/2$, the estimate

$$\int_0^T \left|\varphi(\sigma + it, F_j)\right|^2 dt = T \sum_{m=1}^{\infty} \frac{|c_j(m)|^2}{m^{2\sigma}} + o(T) = O(T)$$

is valid [7]. Therefore, by (3) and (4), the estimate

$$\int_0^T \left|\varphi_h(\sigma + it)\right|^2 dt = O(T),$$

for

$$\sigma > \max_{1\leqslant j\leqslant n} \sigma_2^{(j)},$$

is valid. Consequently, all conditions sufficient for the existence of the limit distribution in the space of analytic functions for the function $\varphi_h(s)$ are satisfied. To obtain a limit theorem of such kind limit theorems for Dirichlet polynomials, for absolutely convergent Dirichlet series as well as the approximation in mean by absolutely convergent Dirichlet series and some elements of the ergodic theory are used. The details can be found in [1], [5]. Thus, we have that the probability measure

$$\nu_T\big(\varphi_h(s+i\tau)\in A\big), \quad A\in\mathcal{B}(H(\hat{D})),$$

converges weakly to the distribution of the random element

$$\varphi_h(s,\omega)=\sum_{m=1}^{\infty}\frac{b(m)\omega(m)}{m^s}=\sum_{j=1}^{n}\sum_{l=1}^{k}u_{jl}\sum_{m=1}^{\infty}\frac{c_j(m)\omega(m)}{m^{s_l^{(j)}+s}}, \quad \omega\in\Omega,\ s\in\hat{D},$$

as $T\to\infty$. This is equivalent to the assertion of the lemma.

3. Proof of the Theorem

Since by Lemma 3 the family of probability measures $\{P_T\}$ is relatively compact, there exists a sequence $T_1\to\infty$ such that the measure P_{T_1} converges weakly to some probability measure P on $(H_n,\mathcal{B}(H_n))$. Therefore, there exists an H_n-valued random element

$$\tilde{\varphi}=\big(\tilde{\varphi}(s_1),\dots,\tilde{\varphi}(s_n)\big)$$

defined, say, on the probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and having the distribution P. Thus, we have

$$\varphi_{T_1} \xrightarrow[T_1\to\infty]{\mathcal{D}} \tilde{\varphi}. \tag{5}$$

From this, using the continuity of the function h, we obtain

$$h(\varphi_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\tilde{\varphi},s).$$

By the definition of $\varphi_h(s)$ hence we find that

$$\varphi_h(s+i\eta_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\tilde{\varphi}, s). \tag{6}$$

However, by Lemma 4,

$$\varphi_h(s+i\eta_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\varphi, s).$$

Consequently, in virtue of (6)

$$h(\varphi, s) \stackrel{\mathcal{D}}{=} h(\tilde{\varphi}, s). \tag{7}$$

Now consider the function $h_1 : H(\hat{D}) \to \mathbb{C}$ given by the formula

$$h_1(f) = f(0), \quad f \in H(\hat{D}).$$

Then the relation (7) implies

$$h_1\big(h(f)\big) \stackrel{\mathcal{D}}{=} h_1\big(h(\tilde{\varphi})\big),$$

or by the definition of h_1

$$h(\varphi)(0) \stackrel{\mathcal{D}}{=} h(\tilde{\varphi})(0).$$

This and the definition of the function h yield

$$\sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl}\varphi(s_l^{(j)}, \omega, F_j) \stackrel{\mathcal{D}}{=} \sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl}\tilde{\varphi}(s_l^{(j)}) \tag{8}$$

for arbitrary complex numbers u_{jl}. The family of parts in the space $\mathbb{C}^{nk}$ generated by all hyperplanes forms a determining class. Therefore, from (8) we obtain that the random elements $\varphi(s_l^{(j)}, \omega, F_j)$ and $\tilde{\varphi}(s_l^{(j)})$, $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant k$, have the same distribution.

Now let K_j be an arbitrary compact subset of D_j, $f_j \in H(D_j)$, and $\{s_l^{(j)}\}$, $1 \leqslant l < \infty$, be a sequence of points of K_j which is dense in K_j, $j = 1, \dots, n$. For an arbitrary $\varepsilon > 0$ we define

$$G = \Big\{(g_1, \dots, g_n) \in H_n : \sup_{s\in K_j} \big|g_j(s) - f_j(s)\big| \leqslant \varepsilon,\ j = 1, \dots, n\Big\}$$

and

$$G_k = \Big\{(g_1, \dots, g_n) \in H_n : \big|g_j\big(s_l^{(j)}\big) - f_j\big(s_l^{(j)}\big)\big| \leqslant \varepsilon, \\ j = 1, \dots, n,\ l = 1, \dots, k\Big\}.$$

Since the distributions of the random elements $\varphi(s_l^{(j)}, \omega, F_j)$ and $\tilde{\varphi}(s_l^{(j)})$, $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant k$, coincide, we have that

$$\begin{aligned} m_H\big(\omega \in \Omega) : \varphi(s_1, \dots, s_n, \omega; F_1, \dots, F_n) \in G_k\big) \\ = \tilde{\mathbb{P}}\big(\tilde{\varphi}(s_1), \dots, \tilde{\varphi}(s_n)) \in G_k\big). \end{aligned} \tag{9}$$

The sequence $\{s_l^{(j)}\}$ is dense in K_j. Therefore $G_1 \supset G_2 \supset \dots$ and $G_k \to G$ as $k \to \infty$. Hence and from (9) we obtain that

$$\begin{aligned} m_H\big(\omega \in \Omega :\ \varphi(s_1, \dots, s_n, \omega; F_1, \dots, F_n) \in G\big) \\ = \tilde{\mathbb{P}}\big((\tilde{\varphi}(s_1), \dots, \tilde{\varphi}(s_n)) \in G\big). \end{aligned} \tag{10}$$

Since the space H_n is separable, finite intersections of spheres in this space form a determining class [2]. Hence, in view of (10), we have that

$$\varphi \overset{\mathcal{D}}{=} \tilde{\varphi}.$$

Thus, by (5),

$$\varphi_{T_1} \xrightarrow[T_1 \to \infty]{\mathcal{D}} \varphi.$$

This shows that the probability measure P_{T_1} converges weakly to the distribution of the random element φ as $T_1 \to \infty$. However, by Lemma 3, the family $\{P_T\}$ is relatively compact, and the random element φ is independent on the choice of the sequence T_1. Since P_T converges weakly to P as $T \to \infty$ if and only if every subsequence $\{P_{T_1}\}$ contains another subsequence $\{P_{T_2}\}$ such that P_{T_2} converges weakly to P as $T \to \infty$, hence we obtain the assertion of the theorem.

In [4] it was obtained that, for almost all $\omega \in \Omega$,

$$\varphi(s_j, \omega, F_j) = \prod_p \left(1 - \frac{\alpha_j(p)\omega(p)}{p^{s_j}}\right)^{-1} \left(1 - \frac{\beta_j(p)\omega(p)}{p^{s_j}}\right)^{-1}, \qquad s_j \in D_j,$$

where $c_j(p) = \alpha_j(p) + \beta_j(p)$, $j = 1, \dots, n$. Let $a_p \in \gamma$, and

$$\underline{f}_p(s_1, \dots, s_n, a_p) = \Bigg(\log\left(1 - \frac{\alpha_1(p)a_p}{p^{s_1}}\right)^{-1}\left(1 - \frac{\beta_1(p)a_p}{p^{s_1}}\right)^{-1}, \dots, \log\left(1 - \frac{\alpha_n(p)a_p}{p^{s_n}}\right)^{-1}\left(1 - \frac{\beta_n(p)a_p}{p^{s_n}}\right)^{-1}\Bigg), \quad s_j \in D_j,$$

where

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots, \qquad |z| < 1.$$

To prove the joint universality of functions $\varphi(s_1, F_1), \dots, \varphi(s_n, F_n)$, we must show that the set of all convergent series

$$\sum_p \underline{f}_p(s_1, \dots, s_n, a_p)$$

is dense in H_n. This can be reduced to the following problem. Let $N > 0$, and $D_{jN} = \{s \in \mathbb{C} : \kappa_j/2 < \sigma < (\kappa_j + 1)/2, \, |t| < N\}$. Moreover, let μ_j be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_{jN}, $j = 1, \dots, n$, such that

$$\sum_p \left| \sum_{j=1}^n c_{jp} \int_{\mathbb{C}} p^{-s} \, d\mu_j h_j^{-1}(s) \right| < \infty.$$

Here $c_{jp} = c(p)p^{(1-\kappa_j)/2}$, and $h_j(s) = s - (\kappa_j - 1)/2$. Then we must show that

$$\int_{\mathbb{C}} s^r \, d\mu_j(s) = 0 \tag{11}$$

for all $r = 0, 1, 2, \dots,$ $j = 1, \dots, n$. The latter problem is sufficiently complicated, and we are not sure that there exists its positive answer. For example, even if

$$c_{jp} = c_p + b_{jp}, \qquad j = 1, \dots, n,$$

where, for $\sigma > 1/2$,

$$\sum_{j=1}^n \sum_p \frac{|b_{jp}|}{p^\sigma} < \infty,$$

we can obtain only that

$$\int_{\mathbb{C}} s^r \, d\mu(s) = 0, \quad r = 0, 1, 2, \dots,$$

with

$$\mu(A) = \sum_{j=1}^{n} \mu_j h_j^{-1}(A), \quad A \in \mathcal{B}(\mathbb{C}).$$

However, this is not sufficient to obtain (11). So, the existence of a joint version of Theorem B remains open.

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