Publ. Math. Debrecen
59 / 1-2 (2001), 187–193

On a problem of S. Mazur

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Abstract. In Problem 24 of "The Scottish Book" S. Mazur asked whether the Lebesgue measurability of an additive functional along every path in a Banach space implies its continuity. It is proved that this condition is equivalent to universal measurability.

Introduction

In about 1935 S. Mazur asked (Problem 24 of "The Scottish Book" [11]) the following question: An additive functional f in a Banach space X is given with the property

(P) for each path g in X, $f \circ g$ is Lebesgue measurable.

Is it true that f is continuous? This long-standing question was answered in 1984: Labuda and Mauldin [7] proved, that the answer is yes, more generally, if f is an additive operator mapping X into a Hausdorff topological vector space, and f has property (P), then it is continuous. The result was extended by Lipecki [8] to the case when X and Y are Abelian Hausdorff topological groups with X metrizable, connected, locally arcwise connected and complete. More general functional equations have also been considered. R. Ger [3] has obtained similar results concerning property (P) for Jensen convex functions, and L. Székelyhidi [12] for exponential polynomials.

Mathematics Subject Classification: Primary 28C15; Secondary 39B52.
Key words and phrases: universal measurability, additive functionals, functional equations, The Scottish Book.
This work is supported by OTKA T016846 grant.

The aim of this note is to prove that, on certain topological spaces, property (P) is equivalent to a more usual measurability condition, namely, universal measurability. This result makes it possible to use "measurability implies continuity" type results, that are known for more general functional equations, to get "property (P) implies continuity" type results. "Measurability implies continuity" type results exist for several functional equations and inequalities. See, for example, Gajda [2], Grosse–Erdman [4] and Járai [5]. In particular, using a result from the classic book of Schwartz [10], we obtain the result of Labuda and Mauldin.

Definition. We say that μ is a *measure over* X, if X is a set, μ maps 2^X into the set of nonnegative extended real numbers, and

$$\mu(A) \leq \sum_{B\in\mathcal{F}} \mu(B) \quad \text{whenever } \mathcal{F} \subset 2^X,\ \mathcal{F} \text{ is countable}, A \subset \cup\mathcal{F}.$$

We say that A is a μ *measurable set* if $A \subset X$ and $\mu(T) = \mu(T \cap A) + \mu(T \setminus A)$ whenever $T \subset X$. Let $\mathcal{M}_\mu$ denote the σ-algebra of all μ measurable sets. A measure is called *diffuse* if any set with only one element has measure 0.

If $Y \subset X$, the restriction of μ to 2^Y is a measure on Y denoted by μ_Y. Let $\mu\lfloor Y$ defined by $(\mu\lfloor Y)(A) = \mu(Y \cap A)$ for all $A \subset X$.

Every function $f : X \to Y$ induces a map $f_\#$ of measures over X defined by $(f_\#\mu)(B) = \mu(f^{-1}(B))$ for $B \subset Y$.

Under a *path* we mean a continuous function $g : I \to X$ of the *unit interval* $I = [0, 1]$ into the topological space X. The class of *Lebesgue measurable* subsets of I will be denoted by $\mathcal{L}$, and *Lebesgue measure* on the subsets of I by λ. A function with values in the topological space X is called *Lebesgue measurable* if it is $(\mathcal{L}, \mathcal{B}_X)$ measurable, where $\mathcal{B}_X$ is the class of Borel subsets of X, the σ-algebra generated by open sets. A measure μ over X is called a *Borel measure* if X is a topological space, all open sets are measurable, and for each set $A \subset X$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$. μ is a *Radon measure*, if X is a Hausdorff space, all open sets are measurable, μ is locally finite and

$$\mu(V) = \sup\{\mu(K) : K \text{ is compact},\ K \subset V\} \qquad \text{for any open set } V,$$

$$\mu(A) = \inf\{\mu(V) : V \text{ is open},\ A \subset V\} \qquad \text{for any subset } A \text{ of } X.$$

The *support* of a Borel measure is

$$\operatorname{spt}\mu = X \setminus \cup\{V : V \text{ is open},\ \mu(V) = 0\}.$$

If μ is a Radon measure, then $\mu(X \setminus \operatorname{spt} \mu) = 0$.

For a topological space X let

$$\mathcal{U}_X = \bigcap \{\mathcal{M}_\mu : \mu \text{ is a Borel measure on } X\},$$

the set of *universally measurable* subsets of X. Similarly, if X is a Hausdorff space, let

$$\mathcal{R}_X = \bigcap \{\mathcal{M}_\mu : \mu \text{ is a Radon measure on } X\}.$$

If in these definitions the expressions "Borel measure" and "Radon measure" are replaced by "diffuse probability Borel measure" and "diffuse probability Radon measure", respectively, then the classes $\mathcal{U}_X$ and $\mathcal{R}_X$ remain unchanged. In the case when X is a Polish space, $\mathcal{R}_X = \mathcal{U}_X$. More general results can be found in Schwartz [10, pp. 117–130]. See also Federer [1, 2.2.16].

Results

Let

$$\mathcal{P}_X = \{A : A \subset X, g^{-1}(A) \in \mathcal{L} \text{ for each path } g \text{ in } X\}.$$

Trivially, $\mathcal{P}_X$ is a σ-algebra, and a function $f : X \to Y$ mapping X into a measurable space $(Y, \mathcal{S})$ is $(\mathcal{P}_X, \mathcal{S})$ measurable if and only if for each path g in X, the function $f \circ g$ is $(\mathcal{L}, \mathcal{S})$ measurable. So we have to investigate only the connection of $\mathcal{P}_X$ with other σ-algebras. Clearly, $\mathcal{B}_X \subset \mathcal{P}_X$. More generally, we have

Theorem. *If X is a Hausdorff topological space, then $\mathcal{R}_X \subset \mathcal{P}_X$.*

Proof. Suppose, that A is a subset of X for which $A \notin \mathcal{P}_X$. Then there exists a path g in X for which $g^{-1}(A) \notin \mathcal{L}$. Let $\mu = g_{\#}\lambda$. Using results 2.2.17, 2.1.5(4) from Federer [1], we get that μ is a Radon measure, and A cannot be μ measurable. □

The converse is not true in all Hausdorff spaces. Clearly, if X is totally disconnected, than $\mathcal{P}_X = 2^X$. Hence some connectedness properties are needed. To prove the converse for a wide class of spaces we need a lemma, which is more or less equivalent to a theorem of Marczewski published in Polish (see [7] and [8]).

Lemma. *Let μ be a diffuse probability Radon measure on a complete metric space X. Then for each $\varepsilon > 0$ there exists a compact subset C of I and a homeomorphism $g : C \to g(C) \subset X$ such that g is measure preserving between C and $g(C)$, that is, $g_{\#}(\lambda_C) = \mu\lfloor g(C)$, moreover $1 > \mu(g(C)) > 1 - \varepsilon$.*

Proof. Let $G(x, r)$ denote the open, $B(x, r)$ the closed ball with centre x and radius r, respectively. Let t_n be a stricly decreasing sequence tending to 1. Let us choose a compact set $K \subset X$ for which $1 > \mu(K) > 1 - \varepsilon/2$. We may suppose, that $\operatorname{spt}(\mu\lfloor K) = K$. For a fixed $x \in K$, the function $r \mapsto \mu(K \cap B(x, r))$ is positive for each positive r, monotone increasing, and tends to 0, as r tends to 0. Hence, we can choose a number $0 < r_x < 1/2$, for which the function is continuous at the point r_x, and has value less than $\frac{1}{2}\mu(K)$. Choosing a finite subcovering from the open covering $G(x, r_x)$, $x \in K$, we get points x_i and radii $r_i = r_{x_i}$, for which

$$\mu(K \cap B(x_i, r_i)) < \frac{1}{2}\mu(K) \qquad (i = 1, 2, \dots, n).$$

Moreover, we can choose radii $r_i' > r_i$, $i = 1, 2, \dots, n$, for which

$$\mu(K \cap G(x_i, r_i')) < \mu(K \cap B(x_i, r_i)) + \frac{\varepsilon}{2^2 n}.$$

Let $K_1 = K \cap B(x_1, r_1)$, and by induction let

$$K_i = (K \setminus (\cup_{j<i} G(x_j, r_j'))) \cap B(x_i, r_i) \quad \text{if } i = 2, 3, \dots, n.$$

If one of the sets K_i has measure zero, let us drop it, decreasing n. Finally, let

$$K^{(1)} = \bigcup_{i=1}^{n} K_i.$$

Clearly, the sets K_i are disjoint compact sets with diameter less then 1, for each i the inequality $0 < \mu(K_i) < 1/2$ holds, $K^{(1)} \subset K$, and $\mu(K \setminus K^{(1)}) < \varepsilon/2^2$. Let us select disjoint closed intervals C_i, $i = 1, 2, \dots, n$ in I, such that $\mu(K_i) < \lambda(C_i) < t_1 \mu(K_i)$, whenever $i = 1, 2, \dots, n$.

In the second step, let us repeat the above construction for each K_i instead of K. We get disjoint compact subsets

$$K_{i,1}, K_{i,2}, \dots, K_{i,n_i}, \qquad n_i > 1$$

of K_i, with diameter less than $1/2$, with measure positive but less then $1/4$, such that with the notation

$$K^{(2)} = \bigcup_{i_1=1}^{n} \bigcup_{i_2=1}^{n_{i_1}} K_{i_1,i_2},$$

$K^{(2)} \subset K^{(1)}$ and $\mu(K^{(1)} \setminus K^{(2)}) < \varepsilon/2^3$. We choose the corresponding closed subintervals

$$C_{i,1}, C_{i,2}, \dots , C_{i,n_i}$$

of C_i such that

$$\mu(K_{i_1,i_2}) < \lambda(C_{i_1,i_2}) < t_2\mu(K_{i_1,i_2})$$

is satisfied. Let

$$C^{(2)} = \bigcup_{i_1=1}^{n} \bigcup_{i_2=1}^{n_{i_1}} C_{i_1,i_2}.$$

Continuing this process, by induction we get the systems

$$K_{i_1,i_2,\dots,i_k} \quad \text{and} \quad C_{i_1,i_2,\dots,i_k}$$

and the sets $K^{(k)}$ and $C^{(k)}$. Let

$$K_\infty = \bigcap_{k=1}^{\infty} K^{(k)}, \qquad C = \bigcap_{k=1}^{\infty} C^{(k)}.$$

To each point $x \in C$ there corresponds a unique sequence

$$i_1, i_2, \dots , i_k, \dots , \qquad 1 \le i_k \le n_{i_1,i_2,\dots,i_{k-1}}.$$

Similarly, to each $y \in K_\infty$ there corresponds a unique such sequence. Let $g(x) = y$ if the corresponding sequences are equal.

To prove that g is continuous, let $\eta > 0$. If $1/2^{k-1} < \eta$, and δ is the minimal distance between sets $C_{i_1,i_2,\dots,i_k}$, then $|x - x'| < \delta$ implies that $g(x)$ and $g(x')$ are in the same set $K_{i_1,i_2,\dots,i_k}$, hence have distance less than η.

If $\eta > 0$ and $t_k/2^k < \eta$, let δ be the minimal distance between sets $K_{i_1,i_2,\dots,i_k}$. If the distance between $g(x)$ and $g(x')$ less than δ, then x and x' are in the same set $C_{i_1,i_2,\dots,i_k}$, hence $|x-x'| < \eta$ and g^{-1} is continuous.

To prove that $g_\#(\lambda_C) = \mu\lfloor g(C)$, it is enough to prove that

$$\lambda(C \cap C_{i_1,\dots,i_k}) = \mu(K_\infty \cap K_{i_1,\dots,i_k}),$$

because every open subset of C and $g(C)$ can be written as a disjoint countable union of open subsets of C and $g(C)$ of this form, respectively, and the measures are Radon measures. But

$$\lambda(C \cap C_{i_1,\dots,i_k}) = \lim_{l\to\infty} \sum_{i_{k+1},\dots,i_l} \lambda(C_{i_1,\dots,i_k,i_{k+1},\dots,i_l}),$$

$$\mu(K_\infty \cap K_{i_1,\dots,i_k}) = \lim_{l\to\infty} \sum_{i_{k+1},\dots,i_l} \mu(K_{i_1,\dots,i_k,i_{k+1},\dots,i_l}),$$

and

$$\mu(K_{i_1,\dots,i_l}) < \lambda(C_{i_1,\dots,i_l}) < t_l \mu(K_{i_1,\dots,i_l}). \qquad \square$$

Theorem. *Let X be a complete, connected and locally arcwise connected metric space. Then $\mathcal{P}_X \subset \mathcal{R}_X$.*

Proof. Suppose, that A is a subset of X for which $A \notin \mathcal{R}_X$. We shall prove, that $A \notin \mathcal{P}_X$. Let us choose a diffuse probability Radon measure, for which $A \notin \mathcal{M}_\mu$. Let A' denote the complement of A. Then $\mu(A)+\mu(A') > \mu(X) = 1$. Let us choose an $\varepsilon > 0$ for which $\mu(A)+\mu(A') > 1+2\varepsilon$. By our lemma, there exist a compact subset C of I with λ measure greater then $1-\varepsilon$ and a measure preserving homeomorphism g between C and $K = g(C) \subset X$. By Theorem 5, §50 in Kuratowski [6] g can be extended to a path h mapping I into X. Clearly $h_\#(\lambda\lfloor C) = g_\#(\lambda_C) = \mu\lfloor K$. By the measurability of K, $\mu(A) = \mu(A \cap K) + \mu(A \setminus K)$, hence $\mu(A\cap K) > \mu(A)-\varepsilon$. Similarly, $\mu(A'\cap K) > \mu(A')-\varepsilon$. This proves, that A is not $\mu\lfloor K$ measurable. By 2.1.2 of Federer [1], $h^{-1}(A)$ is λ measurable if and only if A is $h_\#(\lambda\lfloor B)$ measurable for each subset B of X, which is not the case if $B = C$. Hence $A \notin \mathcal{P}_X$. $\square$

Corollary. *The answer to the problem of Mazur is affirmative, i.e. if for an additive functional f of a real Banach space X the function $f \circ g$ is Lebesgue measurable for all continuous function $g : [0,1] \to X$, then f is continuous.*

Proof. Using the notations of the definition, from the previous theorem we obtain that f is universally Radon measurable. Since for all

$x \in X$ the mapping $t \mapsto f(tx)$ is Lebesgue measurable and additive, by well-known results we obtain that f is homogeneous. By a theorem from the book of Schwartz [10] (p. 157), a linear and universally Radon measurable functional on a Banach space is continuous. It follows that f is continuous. □

Remark. The previous corollary can be extended to additive operators mapping a (real) ultrabornological topological vector space into a (real) locally convex Hausdorff topological vector space, if we use the ideas of the proof on p. 159 in the book of Schwartz [10].

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(Received September 22, 2000; file obtained March 28, 2001)