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A characterization of the Janko group J_1 by an active fragment of its character table

By V. A. BELONOGOV (Ekaterinburg)

Abstract. The sporadic simple Janko group J_1 of order 175560 has a 15×15 character table X. In this paper it is given a characterization of J_1 by a 12×6 submatrix of X, which is an active fragment of X. The notion of the active fragment introduced in the author's book "Representations and characters in the theory of finite groups", Sverdlovsk, 1990 (in Russian).

1. Introduction

Let G be a finite group, X a character table of G, D the union of some conjugacy classes of G (a normal subset of G) and Φ a set of irreducible characters of G. Denote by $\mathrm{X}(\Phi, D)$ the submatrix of X consisting of the values of the functions from Φ on the elements from D. Let us say that $\mathrm{X}(\Phi, D)$ is an *active fragment* of X if D and Φ interact [1] (see Section 2 below).

The sporadic simple group J_1 of order 175560 has a 15×15 character table X [5]. It is easy to see that X has the following 12×6 active fragment:

$$A = \begin{array}{r} \\ 2 \rightarrow \\ 2 \rightarrow \\ 3 \rightarrow \\ 3 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 1 & \overset{\overset{3}{\downarrow}}{1} \\ 56 & 0 & 1 & -1 \\ 76 & -1 & -1 & 0 \\ 77 & 0 & 0 & 1 \\ 133 & 0 & 1 & 0 \\ 209 & -1 & 0 & 0 \end{bmatrix}.$$

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Here the matrix A is written in a compact form: a number with an arrow indicates the multiplicity of the corresponding row or column. (The full form of A is written in Lemma 2 below.)

The aim of this paper is to prove

Theorem. *Let G be a finite group. The following conditions are equivalent:*

(1) *the matrix A as above is an active fragment of a character table of G,*

(2) $G \cong J_1$.

Note that similar characterizations of the groups $PSL(2,q)$ and $Sz(q)$ are obtained in [2, Chapter 8], where some general methods of investigation of a group with a given active fragment of its character table are developed.

Let G be a group satisfying the condition (1). The proof of Theorem, as proofs of the characterizations above, includes the following steps to make clear the structure of G: the determination of the order of G, the extension of A to a larger fragment of a character table of G, the proof of simplicity of G. Besides the proof of Theorem contains a new step: the complete reconstruction of the centralizer of an involution of G. Finally, a result of Janko [5] completes the proof of Theorem.

2. Preliminaries

We use the standard notation and some definitions from [1], [2]. Let us recall something of these. Always G is a finite group, $\mathrm{Irr}(G)$ is the set of irreducible (complex) characters of G, $g^G := \{x^{-1}gx \mid x \in G\}$ and $o(g)$ is the order of g for $g \in G$, $k(G)$ is the number of the conjugacy classes of G, $k_G(D)$ is the number of the conjugacy classes of G contained in a normal subset D of G, $M \times N$ is the Cartesian product of sets M and N. If α is a class function of G and D is a normal subset of G then $\alpha|_D^0$ is a class function of G defined by the equation

$$\alpha|_D^0(g) = \begin{cases} 0 & \text{if } g \in G \setminus D, \\ \alpha(g) & \text{if } g \in D. \end{cases}$$

The function $\alpha|_D^0$ is called the *D-part* of α.

Suppose that D is a normal subset of G and $\Phi \subseteq \mathrm{Irr}(G)$. We say that D and Φ *interact* if the D-part $\phi|_D^0$ of every character ϕ in Φ is a linear combination of the characters in Φ: $\phi|_D^0 = \sum_{\psi\in\Phi} a_{\phi,\psi}\psi$, $a_{\phi,\psi}$ are complex numbers. (It is shown in [1], [2] that the classical concept of p-block of irreducible characters of a group may be defined in terms of interactions; note that every p-section of G interacts with every p-block of G.) We set

$$D^- = G \setminus D, \quad \Phi^- = \mathrm{Irr}(G) \setminus \Phi.$$

Proposition 1. *Let G be a finite group, D a normal subset of G and $\Phi \subseteq \mathrm{Irr}(G)$. The following conditions are equivalent:*

(1) *D and Φ interact,*

(2) $\sum_{d\in D} \phi(d)\overline{\chi(d)} = 0$ *for all* $(\phi,\chi) \in \Phi \times \Phi^-$,

(3) $\sum_{\phi\in\Phi} \phi(d)\overline{\phi(g)} = 0$ *for all* $(d,g) \in D \times D^-$,

(4) $|G|^{-1}\sum_{d\in D}\sum_{\phi\in\Phi} \psi(d)\overline{\phi(d)}\phi(t) = \psi(t)$ *for all* $(\psi,t) \in \Phi \times D$.

Proof. See [1, Theorem 1]. □

Proposition 2. *Let X be a character table of a group G, D is a normal subset of G and $\Phi \subseteq \mathrm{Irr}(G)$. The following conditions are equivalent:*

(1) $\mathrm{rank}(\mathrm{X}(\Phi,D)) + \mathrm{rank}(\mathrm{X}(\Phi,D^-)) = |\Phi|$,

(2) $\mathrm{rank}(\mathrm{X}(\Phi,D)) + \mathrm{rank}(\mathrm{X}(\Phi^-,D)) = k_G(D)$,

(3) $\mathrm{rank}(\mathrm{X}(\Phi,D)) + \mathrm{rank}(\mathrm{X}(\Phi,D^-)) + \mathrm{rank}(\mathrm{X}(\Phi^-,D))+$
$\mathrm{rank}(\mathrm{X}(\Phi^-,D^-)) = k(G)$,

(4) *D and Φ interact.*

Proof. See [2, Theorem 8A6]. □

3. Proof of Theorem

Troughout this section, G is a finite group which satisfies the condition (1) of Theorem. Let a, b, c, d_1, d_2, d_3 be elements of G, corresponding to the first, second, ... , sixth column of A, respectively, and ϕ_i $(i = 1, \dots, 12)$

denotes the element of $\mathrm{Irr}(G)$, corresponding to i-th row of A. Let us set

$$D = a^G \cup b^G \cup c^G \cup d_1^G \cup d_2^G \cup d_3^G \quad \text{and} \quad \Phi = \{\phi_1, \dots, \phi_{12}\}.$$

Thus, $A = \mathrm{X}(\Phi, D)$ for a character table X of G.

Lemma 1. (1) $|G| = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.

(2) $|C_G(b)| = 7$, $|C_G(c)| = 11$.

(3) $|C_G(d_1)|^{-1} + |C_G(d_2)|^{-1} + |C_G(d_3)|^{-1} = 3/19$.

Proof. Let x^{-1}, y^{-1}, z^{-1}, t_1^{-1}, t_2^{-1}, t_3^{-1} denote the orders of the centralizers of a, b, c, d_1, d_2, d_3, respectively. By Proposition 1 ((1) $\Longrightarrow$ (4)), we have the equation

$$A \operatorname{diag}(x, y, z, t_1, t_2, t_3) A^* A = A.$$

Since $\operatorname{rank}(A) = 3$ and the 3×3 submatrix $A_{(2,3,4)}^{(6,9,12)}$ of A (superscripts are numbers of rows, subscripts are numbers of column) is non-singular, the equation is equivalent to

$$A^{(6,9,12)} \operatorname{diag}(x, y, z, t_1, t_2, t_3) A^* A_{(2,3,4)} = A_{(2,3,4)}^{(6,9,12)},$$

that is

$$\begin{bmatrix} 77x & 0 & 0 & t_1 & t_2 & t_3 \\ 133x & 0 & z & 0 & 0 & 0 \\ 209x & -y & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -360 & 360 & 120 \\ 4 & 3 & 1 \\ 3 & 8 & -1 \\ 1 & -1 & 6 \\ 1 & -1 & 6 \\ 1 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

From here we get $x = 1/175560$, $y = 1/7$, $z = 1/11$, $t_1 + t_2 + t_3 = 3/19$. Therefore the assertions (2) and (3) hold.

Now we show that a=1. If not, $1 \in D^-$ and Proposition 1 ((1) $\Longrightarrow$ (3) for $(d, g) = (a, 1)$) implies a contradiction. Hence $a = 1$, $|G| = x^{-1}$ and (1) holds. □

Lemma 2. (1) *$k(G) = 15$ and the active fragment A may be extended to the following fragment of a character table of G:*

$\lvert C_G(g)\rvert$	$\lvert G\rvert$	7	11	19	19	19	
g	1	b	c	d_1	d_2	d_3	$g_1 \dots g_9$
ϕ_1	1	1	1	1	1	1	$1 \dots 1$
ϕ_2	56	0	1	−1	−1	−1	
ϕ_3	56	0	1	−1	−1	−1	
ϕ_4	76	−1	−1	0	0	0	
ϕ_5	76	−1	−1	0	0	0	
ϕ_6	77	0	0	1	1	1	
ϕ_7	77	0	0	1	1	1	
ϕ_8	77	0	0	1	1	1	
ϕ_9	133	0	1	0	0	0	
ϕ_{10}	133	0	1	0	0	0	
ϕ_{11}	133	0	1	0	0	0	
ϕ_{12}	209	−1	0	0	0	0	
χ_1	120	1	−1	a_{11}	a_{12}	a_{13}	$0 \dots 0$
χ_2	120	1	−1	a_{21}	a_{22}	a_{23}	$0 \dots 0$
χ_3	120	1	−1	a_{31}	a_{32}	a_{33}	$0 \dots 0$

(here a_{ij} are some complex numbers);

(2) $\sum_{i=1}^{3} a_{ij} = -1$ *for all* $j \in \{1, 2, 3\}$*;*

(3) $\lvert C_G(g_i)\rvert$ *divides* $8 \cdot 3 \cdot 5$ *for all* $i \in \{1, \dots, 9\}$*.*

Proof. Since $\operatorname{rank}(X(\Phi, D)) = 3$ and $k_G(D) = 6$, $\operatorname{rank}(X(\Phi^-, D)) = 6 - 3 = 3$ by Proposition 2. Therefore, $\lvert\Phi^-\rvert \geq 3$. By Proposition 1 ((1) $\Longrightarrow$ (2)), we have $\phi_{12}(1)\overline{\chi(1)} - \lvert b^G\rvert \phi_{12}(b)\overline{\chi(b)} = 0$ for $\chi \in \Phi^-$, whence $\chi(1) = 120\chi(b)$ and $\chi(b)$ is integer (for example, by [2, 2A6(3)]). Hence

$$\sum_{\chi\in\Phi^-} \chi(1)^2 = 120^2 \sum_{\chi\in\Phi^-} \chi(b)^2 \geq 120^2 \lvert\Phi^-\rvert \geq 120^2 \cdot 3.$$

But by the second ortogonality relation, $\sum_{\chi\in\Phi^-} \chi(1)^2 = \lvert G\rvert - \sum_{\phi\in\Phi} \phi(1)^2 = 120^2 \cdot 3$. Therefore,

$$\lvert\Phi^-\rvert = 3, \quad k(G) = 15,$$

$$\chi(1) = 120 \quad \text{and} \quad \chi(b) = 1 \quad \text{for all } \chi \in \Phi^-.$$

Thus we have filled two first columns of the character table X of G. By Proposition 1 ((1) $\implies$ (2)), $\phi_{11}(1)\overline{\chi(1)} + |c^G|\phi_{11}(c)\overline{\chi(c)} = 0$ for $\chi \in \Phi^-$. Therefore

$$\chi(c) = -1 \quad \text{for all } \chi \in \Phi^-$$

and hence we know third column of the character table.

At beginning the proof we noted that $\mathrm{rank}(\mathrm{X}(\Phi^-, D)) = |\Phi^-|$. Hence, by Proposition 2, $\mathrm{rank}(\mathrm{X}(\Phi^-, D^-)) = 0$ (obviously, Φ^- interacts also with D), that is

$$\chi_i(g_j) = 0 \quad \text{for all } i, j.$$

By the second ortogonality relation for the columns corresponding to 1 and d_j ($j \in \{1, 2, 3\}$), we have $1 - 56 \cdot 2 + 77 \cdot 3 + 120 \sum_{i=1}^{3} a_{ij} = 0$, where $a_{ij} = \chi_i(g_j)$. Whence the assertion (2) follows.

Let $j \in \{1, 2, 3\}$. Then, by (2), there exists such $i \in \{1, 2, 3\}$ that $\chi_i(d_j) \neq 0$. Since $\chi_i(1) = 120 = 8 \cdot 3 \cdot 5$ is the product of the orders of some Sylow subgroups of G, then as known (see, for example, [2, 5B8]) $\chi_i(g) = 0$ provided $(o(g), 120) \neq 1$. Hence $(o(d_j), 8 \cdot 3 \cdot 5) = 1$. Similarly, from $\phi_6(d_j) \neq 0$ it follows that $(o(d_j), 7 \cdot 11) = 1$. Therefore, $o(d_1) = o(d_2) = o(d_3) = 19$. This and the statement (3) of Lemma 1 imply

$$|C(d_1)| = |C(d_2)| = |C(d_3)| = 19.$$

So, the assertion (1) holds.

Let $j \in \{1, \ldots, 9\}$. Suppose that $o(g_j) = 19$. Then $\langle g_j \rangle = \langle d_1 \rangle$ and, as known (for example, by [2, 2A15]), $\chi_i(g_j)^\alpha = a_{ij}$ ($i = 1, 2, 3$) for an automorphism α of the field $\mathbb{Q}(e^{2\pi i/19})$. But it is impossible by (2). Therefore $(o(g_j), 19) = 1$. Similarly, $(o(g_j), 7) = (o(g_j), 11) = 1$. These equations and (1) (see the row "$|C_G(g)|$") imply (3). $\square$

Lemma 3. *The group G is simple.*

PROOF. Suppose that N is a normal subgroup of G and $\{1\} \neq N \neq G$. Then $N = \bigcap_{\chi \in \Gamma} \mathrm{Ker}\,\chi$, where $\Gamma \subseteq \mathrm{Irr}(G) \setminus \{\phi_1\}$ and $\mathrm{Ker}\,\chi = \{g \in G \mid \chi(g) = \chi(1)\}$. Lemma 2 implies that the elements b, c, d_1, d_2, d_3 cannot belong to $\mathrm{Ker}\,\chi$ for all $\chi \in \mathrm{Irr}(G)\setminus\{\phi_1\}$. Therefore $|N|$ divides $8 \cdot 3 \cdot 5$. Let P be a Sylow subgroup of N. By the Frattini argument, $N_G(P) \supseteq \langle d \rangle$, where $|\langle d \rangle| = 19$. Since $|P| < 19$, $P\langle d \rangle = P \times \langle d \rangle$. But this is a contradiction, since $C_G(d) = \langle d \rangle$ by Lemma 2. $\square$

Lemma 4. (1) *All involutions of G are conjugate.*

(2) *G has an element of order 6.*

(3) *G has at least two conjugacy classes of elements of order 5.*

(4) *G has at least two conjugacy classes of elements of order 10.*

(5) *If G has an element of order 15, the sequence $o(g_1), \dots, o(g_9)$ (in the notation of Lemma 2) within ranking is 2, 3, 5, 5, 6, 10, 10, 15, 15.*

Proof. (1) Let T be a Sylow 2-subgroup of G and M be a maximal subgroup of T. Since G is simple (by Lemma 3), it follows from [4, IV.1.7] that every involution of T is conjugate with an involution of M. Therefore, assertion (1) will be proved if we shall find a maximal subgroup M in T such that all involutions of M are conjugate in G. We may suppose that T has not cyclic maximal subgroups and hence T is elementary abelian. Then $\mathrm{Aut}(T) \cong GL(3,2)$ is a group of order $8 \cdot 3 \cdot 7$ without subgroups of order 21. Hence $|N_G(T)/C_G(T)| = n \in \{1, 3, 7\}$. By the Burnside theorem [4, IV.2.6] and simplicity of G, $n \neq 1$. In the case $n = 7$ all involutions of T are conjugate in $N_G(T)$. In the case $n = 3$, T has a maximal subgroup M all involutions of which are conjugate in $N_G(T)$. Thus, (1) holds.

(2) Let g be an element of order 7 of G. Then $\mathrm{Aut}(\langle g\rangle)$ is a cyclic group of order 6 and so $|N_G(\langle g\rangle)/C_G(\langle g\rangle)| = n$ must be a divisor of 6. Since all elements of order 7 in G are conjugate by Lemma 2, all elements of $\langle g\rangle \setminus \{1\}$ are conjugate in $N_G(\langle g\rangle)$ by [4, IV.2.5]. Therefore $n = 6$ and so $N_G(\langle g\rangle)$ has an element s of order 6 ($N_G(\langle g\rangle) = \langle g\rangle\langle s\rangle$ is the Frobenius group of order 42).

(3) Let g be an element of order 5. Then $N_G(\langle g\rangle)/C_G(g)$ is a cyclic group of order $n \in \{2, 4\}$. In the same way as in the proof of (2), we get (considering the element c) that G has an element of order 10. Hence $|C_G(g)|$ is even. If $n = 4$, T has a cyclic quotient group of order 4; therefore, T is abelian and G has a normal 2-complement by [4, IV.2.7]. This contradicts to Lemma 3. Therefore, $n = 2$, whence, by [4, IV.2.5], (3) holds.

(4) This follows easily from (3).

(5) Suppose that G has an element of order 15. It follows from (3) that G has at least two classes of elements of order 15. From this and (1)–(4) it follows that $G \setminus D$ consists of 9 conjugacy classes of G, and the orders of their elements are 2, 3, 5, 5, 6, 10, 10, 15, 15. □

Lemma 5. *Let t be an involution of G. Then $C_G(t) = \langle t \rangle \times F$, where $F \cong A_5$.*

Proof. Since G has only one conjugacy class of involutions and G has elements of order 6 and 10 (by Lemma 4), $|C_G(t)| = 8 \cdot 3 \cdot 5$ by Lemma 2(3).

Assume that $C_G(t)$ is solvable. Then it has a Hall subgroup of order $3 \cdot 5$ which, clearly, is cyclic. Consequently, G has an element x of order 15 and the element xt of order 30. But this contradicts to the assertion (5) of Lemma 4.

Thus $C_G(t)$ is nonsolvable. Set $C = C_G(t)$. Then $C/\langle t \rangle \cong A_5$ and all involutions of $C/\langle t \rangle$ are conjugate in $C/\langle t \rangle$. Therefore, all (three) subgroups of order 4 of C contaning t are conjugate in C. Hence T is elementary abelian or quaternion. If T is elementary abelian then, by [4, IV.2.2], $T \cap Z(C) \cap C' = 1$, whence $t \notin C'$ and $C = \langle t \rangle \times C' \cong \langle t \rangle \times A_5$. The case where T is the quaternion group (and then $C \cong SL(2,5)$) may be exclude by the result of [3]. □

Now *the proof of Theorem* follows immediately from Lemmas 3, 4(1), 5 and the theorem of Janko [5].

References

[1] V. A. Belonogov, D-blocks of characters of a finite group, *Amer. Math. Soc. Transl.* (2) **143** (1989), 103–128.

[2] V. A. Belonogov, Representations and characters in the theory of finite groups, *UrO AN SSSR*, *Sverdlovsk*, 1990. (in *Russian*)

[3] R. Brauer and M. Suzuki, On finite groups of even order whose 2-Sylow subgroup is a quaternion group, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 1757–1759.

[4] B. Huppert, Endliche Gruppen I, *Springer-Verlag*, *Berlin*, 1967.

[5] Z. Janko, A new finite simple group whith abelian Sylow 2-subgroups and its characterization, *J. Algebra* **3** (1966), 147–186.

V. A. BELONOGOV
INSTITUTE OF MATHEMATICS AND MECHANICS
URAL BRANCH OF RUSSIAN ACADEMY OF SCIENCES
S. KOVALEVSKAYA ST. 16
EKATERINBURG, 620219
RUSSIA

E-mail: belonogov@imm.uran.ru

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